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Consistent Coordinate Transformation for Relativistic Circular Motion and Speeds of Light

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A world system is composed of the world lines of the rest observers in the system. We present a relativistic coordinate transformation, termed the transformation under constant light speed with the same angle (TCL-SA), between a rotating world system and the isotropic system. In TCL-SA, the constancy of the two-way speed of light holds, and the angles of rotation before and after the transformation are the same. Additionally a transformation for inertial world systems is derived from it through the limit operation of circular motion to linear motion. The generalized Sagnac effect involves linear motion, as well as circular motion. We deal with the generalized effect via TCL-SA and via the framework of Mansouri and Sexl (MS), analyzing the speeds of light. Their analysis results correspond to each other and are in agreement with the experimental results. Within the framework of special and general relativity (SGR), traditionally the Sagnac effect has been dealt with by using the Galilean transformation (GT) in cylindrical coordinates together with the invariant line element. Applying the same traditional methods to an inertial frame in place of the rotating one, we show that the speed of light with respect to proper time is anisotropic in the inertial frame, even if the Lorentz transformation, instead of GT, is employed. The local speeds of light obtained via the traditional methods within SGR correspond to those derived from TCL-SA and from the MS framework.

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I. INTRODUCTION

Traditionally, the Galilean transformation (GT) between the rotating and the inertial frames has been exploited in cylindrical coordinates to handle circular motions within the framework of special and general relativity (SGR) [1–7]. The traditional approach based on the non-relativistic GT and the constancy of the speed of light under the standard synchrony cannot comprehensively deal with relativistic circular motions. Some inconsistencies, such as the problem of time gap, arise [2, 3]. In this paper, a relativistic transformation between the rotating world system $\tilde{\mathsf{S}}'$ and the isotropic system S, which is termed the transformation under constant light speed with the same angle (TCL-SA), is derived based on the Lorentz transformation (LT). The world system consists of the world lines of rest observers in the system. The world lines of rotating observers compose $\tilde{\mathsf{S}}'$ and they are obtained by using the LT. In S, the speed of light is a constant c regardless of the propagation direction. The TCL-SA holds the constancy of the two-way speed of light in $\tilde{\mathsf{S}}'$. Circular motion can be regarded as locally inertial. Accordingly, a coordinate transformation between S and an inertial world system is derived from the TCL-SA.

A curvilinear motion can be described as an infinite number of linear motions. The framework of Mansouri and Sexl (MS) [8], handling these linear motions, can deal with arbitrary motions including circular motion. In the derivation of the coordinate transformation between $\tilde{\mathsf{S}}'$ and S, given an unprimed rotational angle ϕ in S, it is necessary to find the corresponding primed one ϕ'. Using the MS framework, we can obtain ϕ', which is shown to be the same as ϕ. The TCL-SA is consistent with the MS framework because ϕ' is derived from it. In contrast, the primed rotational angle in the existing transformation under constant light speed with a different angle (TCL-DA) [9] is other than the one from the MS framework.

The generalized Sagnac effect [10–13], which involves linear motion as well as circular motion, indicates that the speed of light is anisotropic not only in rotating frames but also in inertial frames. We investigate the generalized Sagnac effect via TCL-SA, analyzing the speeds of light in the rotating and the inertial frames. The analysis via TCL-SA takes no account of the motion

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of the laboratory frame. Considering its motion, the generalized Sagnac effect is also analyzed by using the MS framework. These analyses are in agreement with the experimental results [10,11]. Moreover, the local speeds of light derived from TCL-SA are shown to be the same as those derived from the MS framework.

It seems to have been known to most physicists that SGR can resolve the Sagnac effect without contradictions, although it cannot provide consistent explanations on the speed of light in the rotating frame [2,3,14–18]. We show that the difference between the travel times of two light beams traversing a circumference in opposition directions can be exactly obtained via the traditional methods utilizing GT and the invariant line element. However, this does not mean that SGR can consistently explain the Sagnac effect. Applying the same traditional methods to an inertial frame in place of the rotating one, we show that the Sagnac effect takes place also in linear motions, as actually observed in the experiment of the generalized Sagnac effect. It implies that the speed of light is anisotropic in inertial frames. Even if LT, rather than GT, is used for the transformation between inertial frames, the local speed of light with respect to proper time is shown to be anisotropic. The speeds of light calculated through the traditional methods correspond to those derived from TCL-SA and from the MS framework. As clearly proven in Ref. 18, the equivalence of inertial frames under light speed constancy is mathematically infeasible, which one can readily see from the relationships of relative velocity between four arbitrary inertial frames.

The rest of this paper is organized as follows. In Sec. II, the TCL-SA is derived. Section III presents the MS framework, showing that the rotational angles ϕ and ϕ' are equal in the coordinate transformation between $\tilde{S}'$ and S. Section IV investigates the Sagnac effect and the speeds of light via TCL-SA and via the MS framework. In Sec. V, traditional approaches within the framework of SGR are employed to deal with the Sagnac effect and to find the speeds of light in inertial frames. Finally, Sec. VI presents conclusions, together with a brief discussion on the usefulness of LT.

II. RELATIVISTIC TRANSFORMATION FOR CIRCULAR MOTION

In this section, we derive the TCL-SA based on the relativistic circular approach presented in Refs. 9 and 19. The circular approach employs the unprimed and the primed coordinate systems S, $\tilde{S}$, $\tilde{S}'$, and S' for single observers in the complex Euclidean space, as illustrated in Fig. 1, where time is represented as an imaginary number. The speed of light is assumed to be a constant c irrespective of the propagation direction in S and its time coordinate is expressed as $\tau = ict$ where $i = (-1)^{1/2}$ and t denotes time. The coordinate time $\tilde{\tau}$ of $\tilde{S}$ is the same as τ. The z-components do not change by coordinate transformations and are omitted if not necessary. We denote the coordinate vectors of S and $\tilde{S}$ by $\boldsymbol{p} = [\tau, x, y]^T$ and $\tilde{\boldsymbol{p}} = [\tilde{\tau},\ \tilde{x},\ \tilde{y}]^T$, respectively, where T stands for the transpose.

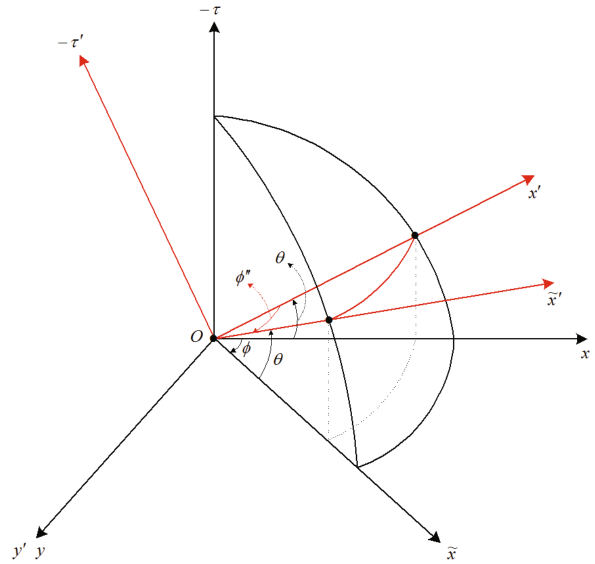

Fig. 1. (Color online) Unprimed coordinate systems S and $\tilde{S}$ and primed ones S' and $\tilde{S}'$ corresponding to S and $\tilde{S}$, respectively.

The coordinate system $\tilde{S}$ is rotated by ϕ with respect to S and their coordinate vectors $\boldsymbol{p}$ and $\tilde{\boldsymbol{p}}$ are related by

$$\tilde{\boldsymbol{p}} = \boldsymbol{A}(\phi)\boldsymbol{p}, \tag{1}$$

where $\boldsymbol{A}(\phi)$ is a rotation matrix:

$$\boldsymbol{A}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}. \tag{2}$$

Obviously $\boldsymbol{A}^{-1}(\phi) = \boldsymbol{A}(-\phi)$, so $\boldsymbol{p} = \boldsymbol{A}(-\phi)\tilde{\boldsymbol{p}}$. The coordinate systems S' and $\tilde{S}'$ are the primed ones corresponding to S and $\tilde{S}$, respectively. Their coordinate vectors are denoted as $\boldsymbol{p}' = [\tau', x', y']^T$ and $\tilde{\boldsymbol{p}}' = [\tilde{\tau}', \tilde{x}', \tilde{y}']^T$ where $\tilde{\tau}' = \tau' = ict'$. The $\tilde{\tau}$- and $\tilde{x}$-axes of $\tilde{S}$ and the $\tilde{\tau}'$- and $\tilde{x}'$-axes of $\tilde{S}'$ lie on an identical plane, as can be seen from Fig. 1. An observer $\tilde{O}'$ at rest in $\tilde{S}'$ is in motion in the direction of the $\tilde{x}$-axis at a normalized speed $\beta = \nu/c$, as seen in S. In Fig. 1, the angles from the $\tilde{x}$-axis to the $\tilde{x}'$-axis and from the x-axis to the x'-axis are both θ, which is a complex number, and the trigonometric functions $\cos\theta$ and $\sin\theta$ are given as $\cos\theta = 1/(1-\beta^2)^{1/2}$ and $\sin\theta = -i\beta/(1-\beta^2)^{1/2}$. The normalized speed is written in terms of θ as

$$\beta = i\tan\theta. \tag{3}$$

One can see that if $\beta = 0$, $\theta = 0$. In Sec. II.1, considering $\tilde{O}'$ as if it is in rectilinear motion, we obtain the relationships between the rotational angles ϕ and ϕ'. In Sec. II.2 and II.3, considering $\tilde{O}'$ in circular motion, we derive the coordinate transformation between $\tilde{\mathsf{S}}'$ and S.

1. Angle of rotation

Let the coordinate vector of $\tilde{O}'$ be described as $\tilde{\boldsymbol{p}} = [\tilde{\tau}, \tilde{x}, \tilde{y}]^T$ in $\tilde{S}$. Performing the LT for $\tilde{\boldsymbol{p}}$ in the $\tilde{x}$-axis direction, we obtain its coordinate vector $\tilde{\boldsymbol{p}}' = [\tilde{\tau}', \tilde{x}', \tilde{y}']^T$ in $\tilde{S}'$:

$$\tilde{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)\tilde{\boldsymbol{p}}, \tag{4}$$

where $\boldsymbol{T}_L(\theta)$ is the LT matrix

$$\boldsymbol{T}_L(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

One can easily see that when $\beta = 0$, $\boldsymbol{T}_L(\theta)$ is reduced to an identity matrix. Generally a conversion matrix of coordinates between S' and $\tilde{S}'$ can be represented as $\boldsymbol{A}'(\phi')$. Using Eqs. (1) and (4) and $\boldsymbol{p}' = \boldsymbol{A}'^{-1}(\phi')\tilde{\boldsymbol{p}}'$, we have

$$\boldsymbol{p}' = \boldsymbol{T}_{LR}(\theta, \phi)\boldsymbol{p}, \tag{6}$$

where

$$\boldsymbol{T}_{LR}(\theta, \phi) = \boldsymbol{A}'^{-1}(\phi')\boldsymbol{T}_L(\theta)\boldsymbol{A}(\phi). \tag{7}$$

To solve Eq. (6), we need to find the unknown $\boldsymbol{A}'(\phi')$. The rotational angle ϕ'' in Fig. 1, which is the angle between the x'- and $\tilde{x}'$-axes, is expressed as [19]

$$\phi'' = \phi\cos\theta_R, \tag{8}$$

where $\cos\theta_R = 1/(1+\beta^2)^{1/2}$. The $\tilde{y}'$-axis lies on the x-y plane, and the locus of the $\tilde{x}'$-axis forms a cone as ϕ increases from zero to 2π. When ϕ varies from zero to 2π, the $\tilde{x}$-axis spans the x-y plane of S while the $\tilde{x}'$-axis spans the lateral surface of the cone. The period of ϕ is 2π. In contrast, the period of ϕ'' is less than 2π, which implies that the space spanned by the $\tilde{x}'$-axis is curved. A representation in S' of the coordinate vector $\tilde{\boldsymbol{p}}'$ may be obtained by spatially rotating the $\tilde{x}'$- and $\tilde{y}'$-axes by $-\phi''$ so that they correspond to the x'- and y'-axes. In the rotation, the time components of $\boldsymbol{p}'$ and $\tilde{\boldsymbol{p}}'$ are identical. Then $\boldsymbol{A}'(\phi')$ can be written as

$$\boldsymbol{A}'(\phi') = \boldsymbol{A}(\phi''). \tag{9}$$

However, the time axes of S' and $\tilde{S}'$ are oriented differently, as seen in Fig. 2. Hence, first, it will be necessary to make a rotation such that they have the same orientation. To this end, the τ'-x' and the $\tilde{\tau}'$-$\tilde{x}'$ planes are rotated by θ downward, as in Fig. 2, so that the circular sector OP_0P_1 and both τ'- and $\tilde{\tau}'$- axes correspond to,

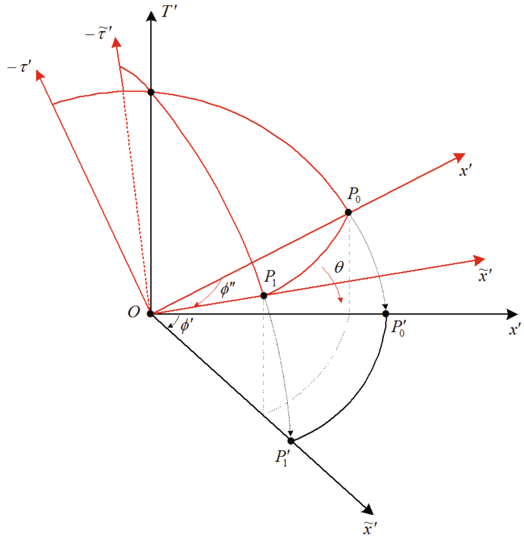

Fig. 2. (Color online) Rotation of the τ'-x' and $\tilde{\tau}'$-$\tilde{x}'$ planes by θ.

respectively, the sector $OP_0'P_1'$ and the T'-axis orthogonal to it. The resultant arc $P_0'P_1'$ becomes larger than the arc P_0P_1 and the angle ϕ' between the x'- and $\tilde{x}'$-axes becomes

$$\phi' = \phi. \tag{10}$$

In Eq. (10), ϕ' is defined in S'. If it is defined in $\tilde{S}'$, its sign is changed so that $\phi' = -\phi$. The conversion matrix is given by

$$\boldsymbol{A}'(\phi') = \boldsymbol{A}(\phi) \tag{11}$$

with ϕ' defined in S'.

We must determine which one of Eqs. (9) and (11) will be used for the conversion in the primed. Equation (6) represents a generalized LT which can be applied irrespective of the direction of motion in the x-y plane. The MS framework, into which clock synchronizations can be incorporated, provides a general transformation between inertial frames. In Sec. III, we derive $\boldsymbol{A}'(\phi')$ by using the MS framework and Eq. (7). The derived result corresponds to Eq. (11).

2. Approach to circular motion

With ϕ' obtained as Eq. (10), the TCL-SA can be discovered through the same circular approach used for the derivation of the TCL-DA [9]. This subsection briefly introduces the circular approach. The coordinate system $\tilde{S}$ is rotating with an angular velocity of ω relative to S, and its rotational angle ϕ at an instant τ is

$$\phi = \omega_c\tau, \tag{12}$$

where $\omega_c = \omega/ic$. Observers O and $\tilde{O}$ are located at radius r in S and $\tilde{S}$, respectively. The coordinate vector in S of O is written as $\boldsymbol{p} = [\tau, 0, -r]^T$ and the coordinate vector in $\tilde{S}$ of $\tilde{O}$ as $\tilde{\boldsymbol{p}} = [\tau, 0, -r]^T$. The observers O' and $\tilde{O}'$ are the primed ones corresponding to the unprimed O and $\tilde{O}$, respectively. The transformation (6) is valid when ϕ is constant. In the event that ϕ is time varying, differential coordinate vectors should be used [19]. For constant ϕ, the transformation equations from $\tilde{S}$ to S' are Eq. (4) and $\boldsymbol{p}' = \boldsymbol{A}(-\phi')\tilde{\boldsymbol{p}}'$. For time varying ϕ, the transformation of the coordinates in $\tilde{S}$ of $\tilde{O}$ to S' is given, using differential vectors, by

$$d\overline{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)d\tilde{\boldsymbol{p}}, \tag{13a}$$

$$d\boldsymbol{p}' = \boldsymbol{A}(-\phi')d\overline{\boldsymbol{p}}', \tag{13b}$$

where $\beta = r\omega/c$. Equation (13b) is valid irrespective of whether ϕ' is defined in S' or in $\tilde{S}'$ [19]. Depending on the coordinate system selected to define ϕ', only the point of view on the relative motion between them is different.

Motions are relatively described in the LT. Hence, in the transformation from S to $\tilde{S}'$, the latter can be viewed as fixed while the former as rotating. Accordingly the spatial components of a differential vector $d\boldsymbol{p}$ of $\boldsymbol{p}$ are divided into the $\tilde{x}$- and $\tilde{y}$-components and then the LT is made in the $\tilde{x}$- direction:

$$d\overline{\boldsymbol{p}} = \boldsymbol{A}(\phi)d\boldsymbol{p}, \tag{14a}$$

$$d\tilde{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)d\overline{\boldsymbol{p}}, \tag{14b}$$

where ϕ and θ are defined in S. When motions are described with the point of view that $\tilde{S}$ (or equivalently $\tilde{S}'$) is rotating with respect to S, similarly a transformation from $\tilde{S}'$ to S can be obtained [19]. However, the experimental results of circular motion are in agreement with Eq. (14), but in disagreement with the resultant one from the other viewpoint [9].

The coordinate systems $\tilde{S}$ and S' have been introduced in the process to discover the transformation between the coordinates of O and $\tilde{O}'$ who are real observers. The observer $\tilde{O}(O')$ can be considered to be $\tilde{O}'(O)$ seen in $S(\tilde{S}')$.

3. Coordinate transformation

The coordinate systems S, $\tilde{S}$, $\tilde{S}'$, and S' are the ones for single observers who see the world through the Lorentz lens. On the other hand, a world system consists of a collection of world lines, which can be obtained from Eq. (14). A rotating world system in the unprimed is denoted by $\tilde{\mathsf{S}}$, which is rotating at an angular velocity ω in S. We assume without loss of generality that S is identical to the isotropic system S, which means that each event in S is the same as the event at the same coordinates in S. Then $\tilde{\mathsf{S}}$ becomes equal to $\tilde{S}$. The world systems S' and $\tilde{\mathsf{S}}'$ are the primed ones corresponding to S and $\tilde{\mathsf{S}}$, respectively. We use a subscript 's' to represent the spatial vector from a space-time coordinate vector. For example, the spatial vector of $\tilde{\boldsymbol{p}}$ is expressed as $\tilde{\boldsymbol{p}}_s(= [\tilde{x}, \tilde{y}]^T)$.

It is the transformation between S and $\tilde{\mathsf{S}}'$ that we seek. Recall the coordinate vector in $\tilde{S}$ of $\tilde{O}$, who is equivalent to $\tilde{O}'$ seen in S, is $\tilde{\boldsymbol{p}} = [\tau, 0, -r]^T$. The spatial vector in S of $\tilde{O}$ is written as $\boldsymbol{p}_s = r[\sin\phi, -\cos\phi]^T$. When $\tau = 0$, the spatial vector of $\tilde{O}$ is $\boldsymbol{p}_s = [0, -r]^T$ and $\tilde{O}$ meets O who is at rest in S. The differential vector of $\boldsymbol{p}$ is $d\boldsymbol{p} = [d\tau, d\boldsymbol{p}_s^T]^T$ where $d\boldsymbol{p}_s = rd\phi[\cos\phi, \sin\phi]^T$. Using Eq. (14) and $rd\phi = d\tau\tan\theta$ from Eqs. (12) and (3), we have

$$d\tau' = \frac{d\tau}{\cos\theta}, \tag{15a}$$

$$d\overline{\boldsymbol{p}}_s' = 0. \tag{15b}$$

The direction of motion of $\tilde{O}$ is orthogonal to the radial direction so that the LT has no effect on the radial component of $\tilde{\boldsymbol{p}}_s$, as can be seen from Eq. (15b). Equation (15b) means that $\tilde{O}'$ is at rest in $\tilde{S}'$. Hence $\tilde{O}'$ is expected to rotate at an angular velocity with respect to S'. Unfortunately, we cannot proceed further to find the spatial coordinates of $\tilde{O}'$ as $d\tilde{\boldsymbol{p}}_s' = \boldsymbol{0}$. However, fortunately, we can derive the radius r' of $\tilde{O}'$ by using Eq. (13) [9]. In the derivation of the radius, the ϕ' defined in $\tilde{S}'$ is employed and thus $\phi' = -\phi$. Because $\tilde{\boldsymbol{p}}_s = [0, -r]^T$, $\tilde{\boldsymbol{p}} = [d\tau, 0, 0]^T$. Substituting the $d\tilde{\boldsymbol{p}}$ into Eq. (13a) yields

$$d\overline{\boldsymbol{p}}' = d\tau[\cos\theta, -\sin\theta, 0]^T. \tag{16}$$

From Eqs. (13b) and (16), $d\boldsymbol{p}_s'$ can be written as $d\boldsymbol{p}_s' = r'd\phi'[\cos\phi', \sin\phi']^T$ with

$$r' = r\cos\theta, \tag{17}$$

where we used the relationships of $d\tau = -d\phi'/\omega_c$ and $\sin\theta/\omega_c = \sin\theta/(\tan\theta/r) = r\cos\theta$.

Equations (10), (15a), and (17) lead to the transformation between S' and S:

$$t' = \frac{t}{\cos\theta}, \quad r' = r\cos\theta,$$
$$\phi' = \phi, \qquad z' = z, \tag{18}$$

where ϕ' is defined in S'. The angular frequency of $\tilde{O}'$ when seen in S' is written as

$$\omega' = \frac{d\phi'}{dt'} = \omega\cos\theta. \tag{19}$$

As $\tilde{\mathsf{S}}'$ rotates at the angular frequency ω' with respect to S', the azimuthal angle $\tilde{\phi}'$ is related to ϕ' by $\tilde{\phi}' = \phi' - \omega't'$, which is expressed as $\tilde{\phi}' = \phi - \omega t$. The coordinate transformation between $\tilde{\mathsf{S}}'$ and S is written as

$$t' = \frac{t}{\cos\theta}, \qquad r' = r\cos\theta,$$
$$\tilde{\phi}' = \phi - \omega t, \quad z' = z. \tag{20}$$

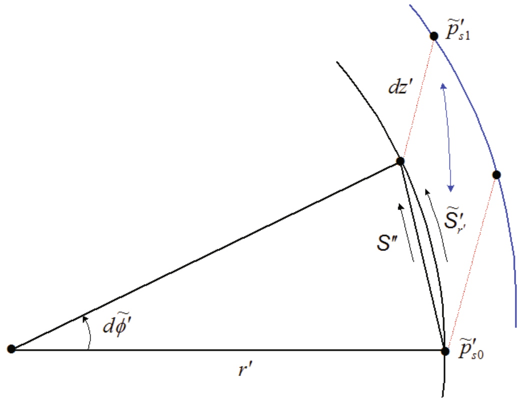

Fig. 3. (Color online) World systems $\tilde{\mathsf{S}}'_{r'}$ and S''.

The rotating system can be regarded as locally inertial though it is in accelerated motion. Therefore a coordinate transformation between an inertial world system and the isotropic system S can be derived from Eq. (20). To this end, we introduce a differential arc $\tilde{\mathsf{S}}'_{r'}$ of radius r' subtended by a differential angle $d\tilde{\phi}'$ in $\tilde{\mathsf{S}}'$, as can be seen from Fig. 3 where an inertial world system S'' moves at constant speed β relative to S. If r' is so large or $d\tilde{\phi}'$ is so small that the differential arc can be approximated as a line segment which lies in S'', then $r'd\tilde{\phi}'$ and $rd\phi$ approach dx'' and dx, respectively, where $\boldsymbol{p}'' = [\tau'', x'']^T$ is the coordinate vector of S''. The quantity $r'd\tilde{\phi}'$ is given from Eq. (20) by

$$dx'' = r'd\tilde{\phi}' = r\cos\theta d\phi - r\omega\cos\theta dt. \tag{21}$$

Using $r\omega_c = \tan\theta$, we have

$$dx'' = \cos\theta dx - \sin\theta d\tau. \tag{22}$$

The time component of the coordinate vector of S'' is equal to that of $\tilde{\mathsf{S}}'_{r'}$. The transformation between S'' and S is written as

$$d\boldsymbol{p}'' = \boldsymbol{T}_I(\theta)d\boldsymbol{p}, \tag{23}$$

where

$$\boldsymbol{T}_I(\theta) = \begin{bmatrix} 1/\cos\theta & 0 \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{24}$$

The inertial transformation of Eq. (23), which shows absolute simultaneity, has attracted some interest for the coordinate transformation between inertial frames [8,14, 20–22]. When the inertial transformation and the LT are compared, their spatial components are, as can be seen from Eqs. (5) and (24), identical and the time components are different. We have obtained Eqs. (18) and (20) based on the LT for the instantaneous transformation for inertial frames, as in Eqs. (13a) and (14b). However, the transformation for inertial frames is given as Eq. (23), not the Lorentz one. It may be pointed out that if Eq. (23) is correct, Eqs. (18) and (20) would not be accurate because they have been derived from LT. We have used LT to find world lines, which are independent of synchronization procedures. The world line is the same irrespective of the absolute and the standard synchronizations.

III. MANSOURI-SEXL FRAMEWORK

An inertial frame S_k is in uniform linear motion at a normalized velocity $\boldsymbol{\beta}_k$ relative to the isotropic frame S and its coordinate vector is denoted as $\boldsymbol{p}_{(k)} = [\tau_{(k)}, \boldsymbol{p}^T_{(k)s}]^T$ where $\boldsymbol{p}_{(k)s} = [x_{(k)}, y_{(k)}, z_{(k)}]^T$. For a vector $\boldsymbol{q}$, we denote its normalized vector by $\hat{\boldsymbol{q}}$ and its magnitude by q. For example, $\hat{\boldsymbol{\beta}}_k = \boldsymbol{\beta}_k/|\boldsymbol{\beta}_k|$ and $\beta_k = |\boldsymbol{\beta}_k|$ where $|\cdot|$ designates the Euclidean norm. In the MS framework, the coordinates of S are transformed into S_k as follows [8,12,18]:

$$\boldsymbol{p}_k = \boldsymbol{T}_G(\boldsymbol{\beta}_k)\boldsymbol{p}, \tag{25}$$

where $\boldsymbol{T}_G(\boldsymbol{\beta}_k)$ can be expressed as

$$\boldsymbol{T}_G(\boldsymbol{\beta}_k) = \begin{bmatrix} g_k & i\boldsymbol{\rho}^T_k \\ ib_k\boldsymbol{\beta}_k & \boldsymbol{M}(\boldsymbol{\beta}_k) \end{bmatrix} \tag{26}$$

with

$$g_k = a_k - b_k\boldsymbol{\varepsilon}^T_k\boldsymbol{\beta}_k, \tag{27}$$

$$\boldsymbol{\rho}_k = (b_k - d_k)(\boldsymbol{\varepsilon}^T_k\hat{\boldsymbol{\beta}}_k)\hat{\boldsymbol{\beta}}_k + d_k\boldsymbol{\varepsilon}_k, \tag{28}$$

$$\boldsymbol{M}(\boldsymbol{\beta}_k) = (b_k - d_k)\hat{\boldsymbol{\beta}}_k\hat{\boldsymbol{\beta}}^T_k + d_k\boldsymbol{I}, \tag{29}$$

and $\boldsymbol{I}$ denoting an identity matrix. The transformation coefficients a_k and b_k are associated with time dilation and length contraction, respectively, and the synchronization vector $\boldsymbol{\varepsilon}_k$ is determined by a synchronization scheme in S_k.

Because $\boldsymbol{p} = \boldsymbol{T}^{-1}_G(\boldsymbol{\beta}_i)\boldsymbol{p}_i$, the transformation from one inertial frame S_i to another S_j is expressed as

$$\boldsymbol{p}_{(j)} = \boldsymbol{T}_{(G)}(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)\boldsymbol{p}_{(i)}, \tag{30}$$

where

$$\boldsymbol{T}_G(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i) = \boldsymbol{T}_G(\boldsymbol{\beta}_j)\boldsymbol{T}^{-1}_G(\boldsymbol{\beta}_i). \tag{31}$$

According to Eq. (30), the spatial vector appears to depend on the synchronization vector, but it does not, as seen in Eq. (25). The transformation coefficients in Eq. (25) can be given, in accordance with special relativity, as

$$a_k = \gamma^{-1}_k, \quad b_k = \gamma_k, \quad d_k = 1, \tag{32}$$

where $\gamma_k = (1 - \beta^2_k)^{-1/2}(= \cos\theta_k)$. If $\boldsymbol{\varepsilon}_k = \boldsymbol{0}$, $\boldsymbol{T}_G(\boldsymbol{\beta})$ $= \boldsymbol{T}_I(\theta)$ when $\boldsymbol{\beta}$ has the same direction as the x-axis. We adopt the standard synchronization so that $\boldsymbol{\varepsilon}_k =$

$-\boldsymbol{\beta}_k$. Then $\boldsymbol{T}_G(\boldsymbol{\beta})$ becomes equal to the LT matrix, and thus $\boldsymbol{T}_G(\boldsymbol{\beta}) = \boldsymbol{T}_L(\theta)$ for $\boldsymbol{\beta}$ parallel to the x-axis. Clearly $\boldsymbol{T}_G^{-1}(\boldsymbol{\beta}_k) = \boldsymbol{T}_G^T(\boldsymbol{\beta}_k)$, which leads to $\boldsymbol{T}_G^{-1}(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i) = \boldsymbol{T}_G^T(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)$.

An object O_k, $k = i, j$, is placed at the origin of S_k. The normalized velocity $\boldsymbol{\beta}_{ji}$ of O_j as seen in S_i is calculated as [12,18,23]

$$\boldsymbol{\beta}_{ji} = \gamma_j\gamma_{ij}^{-1}[\gamma_i(\hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i^T\boldsymbol{\beta}_j)-\boldsymbol{\beta}_i)+\boldsymbol{\beta}_j-\hat{\boldsymbol{\beta}}_i(\boldsymbol{\beta}_i^T\boldsymbol{\beta}_j)]. \quad (33)$$

The inverse of $\boldsymbol{T}_G(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)$ is $\boldsymbol{T}_G(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)$. In Eq. (33), the γ_{ij} corresponds to the (1, 1)-entry of $\boldsymbol{T}_G(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j)$, which is given by [12,18]

$$\gamma_{ij} = (1-\beta_{ji}^2)^{-1/2}, \quad (34)$$

and represents the time dilation factor. Given $\boldsymbol{\beta}_i$ and $\boldsymbol{\beta}_j$, it can be obtained from them. As $\boldsymbol{T}_G(\boldsymbol{\beta}_i, \boldsymbol{\beta}_j) = \boldsymbol{T}_G^T(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)$, $\gamma_{ij} = \gamma_{ji}$, which leads to the equality $\beta_{ji} = \beta_{ij}$. Though the magnitudes of $\boldsymbol{\beta}_{ji}$ and $\boldsymbol{\beta}_{ij}$ are equal, in general $\boldsymbol{\beta}_{ij} \neq -\boldsymbol{\beta}_{ji}$.

It is well known that proper time (PT) is independent of synchronization schemes and can be discovered in any inertial frame if relative velocity is known. We use a subscript '$\circ$' in PT, say $\tau_{k\circ}$, to distinguish it from the adjusted time (AT) through the synchronization of clocks. The PT interval is measured at the same place while the AT interval is between different places. The PT of O_j can be expressed as

$$\tau_{(j)\circ} = \tau_{(i)}/\gamma_{ji} \quad (35a)$$
$$= \tau/\gamma_j. \quad (35b)$$

Note that Eq. (35a) is valid even if i and j are interchanged. The first row of $\boldsymbol{T}_G(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)$ is given by [12, 18]

$$\boldsymbol{T}_G(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)|_{1r} = \gamma_{ji}[1, -i\boldsymbol{\beta}_{ji}^T]. \quad (36)$$

The motion of O_j is described as $\boldsymbol{p}_{(i)} = \tau_{(i)}[1, -i\boldsymbol{\beta}_{ji}^T]^T$ in S_i. Recall that Eq. (25) with $k = j$ and Eq. (30) are the representations for the same $\boldsymbol{p}_{(j)}$. Equation (35a) is obtained by substituting the $\boldsymbol{p}_{(i)}$ into Eq. (30), and Eq. (35b) by substituting $\boldsymbol{p} = \tau[1, -i\boldsymbol{\beta}_j^T]^T$ into Eq. (25) with $k = j$. Equation (35) shows that the PT of O_j is the same for any S_i regardless of $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}_j$.

Now, we are ready to deal with the primed rotation angle ϕ' remaining unsolved. In Eq. (6), $\tilde{S}$ is rotated by a constant ϕ in the x-y plane with respect to S. The z-components do not vary with the transformation and are dropped. It is seen by comparing Eqs. (6) and (25) that $\boldsymbol{T}_{LR}(\theta, \phi)$ should be equal to $\boldsymbol{T}_G(\boldsymbol{\beta})$ when $\boldsymbol{\beta} = \beta\boldsymbol{A}_s(-\phi)\hat{\boldsymbol{x}}$ where $\hat{\boldsymbol{x}} = [1, 0]^T$ and $\boldsymbol{A}_s(\phi)$ is a spatial rotation matrix given by

$$\boldsymbol{A}_s(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}. \quad (37)$$

The conversion matrix $\boldsymbol{A}'(\phi')$, which is unknown, is expressed from the equality as

$$\boldsymbol{A}'^{-1}(\phi') = \boldsymbol{T}_G(\boldsymbol{\beta})\boldsymbol{A}^{-1}(\phi)\boldsymbol{T}_L^{-1}(\theta). \quad (38)$$

Recalling Eq. (3) and substituting $\boldsymbol{\beta} = \beta\boldsymbol{A}_s(-\phi)\hat{\boldsymbol{x}}$, $b = \cos\theta$, $a = b^{-1}$, and $d = 1$ into $\boldsymbol{T}_G(\boldsymbol{\beta})$, we have

$$\boldsymbol{T}_G(\boldsymbol{\beta}) = \begin{bmatrix} \cos\theta & \sin\theta(\boldsymbol{A}_s(-\phi)\hat{\boldsymbol{x}})^T \\ -\sin\theta\boldsymbol{A}_s(-\phi)\hat{\boldsymbol{x}} & (\cos\theta-1)\boldsymbol{A}_s(-\phi)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^T\boldsymbol{A}_s^T(-\phi)+\boldsymbol{I} \end{bmatrix}. \quad (39)$$

The $\boldsymbol{T}_G(\boldsymbol{\beta})$ is a generalized LT matrix. It is straight forward to confirm by direct computation that $\boldsymbol{T}_G^T(\boldsymbol{\beta})\boldsymbol{T}_G(\boldsymbol{\beta}) = \boldsymbol{I}$. Right-multiplying both sides of Eq. (39) by $\boldsymbol{A}^{-1}(\phi)$ ($= \boldsymbol{A}(-\phi)$) leads to

$$\boldsymbol{T}_G(\boldsymbol{\beta})\boldsymbol{A}(-\phi) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix}. \quad (40)$$

Inserting Eq. (40) in Eq. (38) yields $\boldsymbol{A}'^{-1}(\phi') = \boldsymbol{A}(-\phi)(= \boldsymbol{A}^{-1}(\phi))$, which substantiates Eq. (11) together with ϕ' given as Eq. (10).

IV. SPEEDS OF LIGHT UNDER THE UNIQUE ISOTROPIC FRAME

Unless the isotropic frame is unique, physical quantities such as PT and Doppler shifted frequency, which are independent of synchronization procedures, are not uniquely determined [18]. The speeds of light are analyzed via the TCL-SA that transforms the coordinates between the unique isotropic system S and the rotating world system $\tilde{\mathsf{S}}'$. The laboratory frame will be different from the isotropic system. Considering the motion of the laboratory frame relative to S, we also make the analysis based on the transformation (30) in the standard synchronization. These analyses are consistent with the experimental results such as the Saganc effect.

1. Two-way speed of light in TCL-SA

If r' is fixed, so is r, and vice versa. Here, the two-way speed of light in TCL-SA is investigated when radius r is fixed. For the investigation, we extend the world system $\tilde{\mathsf{S}}'_{r'}$, which represented a differential arc in Sec. II, such that it includes the surface of a cylinder of radius r'. As illustrated in Fig. 3, a photon b takes a roundtrip between two spatial points $\tilde{p}'_{s0}$ and $\tilde{p}'_{s1}$ located at the same radius r' in $\tilde{\mathsf{S}}'$ where $\tilde{p}'_{s0} = (r', 0, 0)$ and $\tilde{p}'_{s1} = (r', d\tilde{\phi}', dz')$. The squared distance between the spatial points is written as

$$d\tilde{l'}^2 = (r'd\tilde{\phi}')^2 + dz'^2. \tag{41}$$

When a photon moves in S' by $r'd\tilde{\phi}'$ and dz' in the azimuthal and the z'-axis directions, respectively, it does in S by $rd\phi$ and dz in the respective directions. Because the speed of light is c in S, it follows that

$$(cdt)^2 - (rd\phi)^2 - dz^2 = 0. \tag{42}$$

Substituting $d\phi = d\tilde{\phi} + \omega dt$ into Eq. (42) and solving the quadratic equation of cdt, we have

$$cdt = [\beta rd\tilde{\phi} + ((rd\tilde{\phi})^2(1-\beta^2)dz^2)^{1/2}]\cos^2\theta, \tag{43}$$

where $cdt > 0$ irrespective of the sign of $d\tilde{\phi}$. From Eqs. (43), (41), and (20), one can easily see that cdt is expressed as

$$cdt = \beta rd\tilde{\phi}\cos^2\theta + d\tilde{l'}\cos\theta. \tag{44}$$

The travel times cdt_+ and cdt_- for $\tilde{p}'_{s0} \to \tilde{p}'_{s1}$ and $\tilde{p}'_{s1} \to \tilde{p}'_{s0}$, respectively, are given by

$$cdt_\pm = \pm\beta rd\tilde{\phi}\cos^2\theta + d\tilde{l'}\cos\theta \tag{45}$$

where $d\tilde{\phi} > 0$. The time elapsed during the round trip is

$$cdt_\updownarrow = cdt_+ + cdt_- = 2d\tilde{l'}\cos\theta. \tag{46}$$

The roundtrip time in $\tilde{\mathsf{S}}'_{r'}$ is related to $dt_\updownarrow$ by $d\tilde{t}'_\updownarrow = dt_\updownarrow/\cos\theta$ according to the first equation of the transformation (20) and then

$$\tilde{c}'_\updownarrow = \frac{2d\tilde{l'}}{d\tilde{t}'_\updownarrow} = c. \tag{47}$$

Equation (47) indicates that the roundtrip speed of light is constant irrespective of the propagation direction in the rotating world system $\tilde{\mathsf{S}}'_{r'}$ with radius fixed. The TCL-SA holds the constancy of the two-way speed of light, which remains the same also in the inertial transformation (23) derived from Eq. (20).

2. Local speeds of light and the Sagnac effect

Recalling $\tilde{\phi} = \tilde{\phi}'$ and $r' = r\cos\theta$ and using Eq. (44), we can find the one-way speed of light in $\tilde{\mathsf{S}}'_{r'}$. Equation (44) is rewritten as

$$cdt = d\tilde{l'}(1 + \beta\cos\xi')\cos\theta, \tag{48}$$

where $\cos\xi' = r'd\tilde{\phi}'/d\tilde{l'}$. The speed of light is given by

$$\tilde{c}' = \frac{d\tilde{l'}}{dt'}\left(= \frac{d\tilde{l'}}{dt}\frac{dt}{dt'}\right) \tag{49a}$$

$$= \frac{c}{1 + \beta\cos\xi'}. \tag{49b}$$

The one-way speed of light is dependent on the propagation angle ξ' and is anisotropic in $\tilde{\mathsf{S}}'_{r'}$, though the two-way speed is isotropic. The ξ' indicates the angle from the direction of motion of the primed frame to the propagation direction of light and Eq. (49b) is also valid for rectilinear motion. In other words, when a light signal propagates in an inertial world system S'' with a propagation angle ξ' with respect to the direction of motion of S'' its speed is also given by Eq. (49b). Of course, the same speed can be obtained from Eq. (23) [14].

Once the local speed of light is known, it is an easy task to solve the Sagnac effect. Suppose that two counter-propagating light beams traverse a circumference of radius r on a circular plate which, as seen in the laboratory frame, rotates around its center with an angular velocity of ω. The laboratory frame is assumed to be isotropic and is represented by S. Its motion will be considered later. Because $dz' = 0$ in the Sagnac experiment, $d\tilde{l'} = r'|d\tilde{\phi}'|$. For the counter-propagating light beams $b_\pm$, where b_+ and b_- denotes the co-rotating and counter-rotating ones, respectively, $\cos\xi' = \pm 1$ and their speeds are given by

$$\tilde{c}'_\pm = \frac{c}{1 \pm \beta}. \tag{50}$$

Then the elapsed times of $b_\pm$ traversing the respective paths are calculated as

$$t'_\pm = \int_{path}\frac{d\tilde{l'}}{\tilde{c}'_\pm} = \frac{l'_p(1 \pm \beta)}{c}, \tag{51}$$

where $l'_p = 2\pi r'$ is the rest length of the circumference. The time difference is given by

$$\Delta t'_D = t'_+ - t'_- = \frac{2l'_p\beta}{c}. \tag{52}$$

The analysis result is in agreement with the experimental results of the Sagnac effect. The generalized Sagnac effect involves linear motion as well as circular motion. Though we considered only circular motion for simplicity, the analysis for the generalized Sagnac effect can be easily made. Because Eq. (49) is valid for linear motion, so is Eq. (50). In the experiment of the generalized Sagnac effect [10,11], two light beams $b_\pm$ traverse an optical fiber loop in opposite directions. The integration in Eq. (51) is performed over the path of the optical loop. Then their travel time difference is identical with Eq. (52) where l'_p is the rest length of the loop.

The laboratory frame will be other than the isotropic S. Let us take account of its motion in the generalized Sagnac effect. Though the Earth rotates, it can be considered to belong to an inertial frame S_i for a very short time of the test. The standard synchronization is introduced into S_i. The optical fiber loop has an arbitrary shape. A curved line can be approximated by many straight lines. The fiber loop is divided into n line segments such that its motion can be handled by linear motions. As n tends to infinity, the linearized shape approaches the original one. Each segment adopts the standard synchronization. The jth segment of the linearized shape is in linear motion momentarily at a velocity of $\boldsymbol{\beta}_{ji}$ relative to S_i, where $\beta_{ji} = \beta$, and it belongs to a standard-synchronized S_j. The coordinate vectors of S_i and S_j are related by Eq. (30) and the relationship between their differential vectors is given by

$$d\boldsymbol{p}_{(j)} = \boldsymbol{T}_G(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i) d\boldsymbol{p}_{(i)}. \tag{53}$$

In Eq. (53) $\boldsymbol{\beta}_k$, $k = i, j$, is the velocity of S_k with respect to S and then $\boldsymbol{\beta}_{ji}$ is expressed as Eq. (33).

In the standard-synchronized S_i, $d\tau_{(i)}^2 + |d\boldsymbol{p}_{(i)s}|^2 = 0$ for a light beam. Using this fact and Eq. (53), the rest length of the jth segment is calculated as [12]

$$dl_j = |d\boldsymbol{p}_{(j)s}| = -d\tau_{(i)}\gamma_{ji}(i + \boldsymbol{\beta}_{ji}^T \boldsymbol{c}_{i\tau}), \tag{54}$$

where $\boldsymbol{c}_{i\tau} = d\boldsymbol{p}_{(i)s}/d\tau_{(i)}$, which has a unit magnitude, is a velocity of light with respect to AT. The dl_j represents the travel length in S_j of a photon traversing a segment $d\boldsymbol{p}_{(i)s}$ for $d\tau_{(i)}$ in S_i. Equation (54) has been obtained from Eq. (53) under the standard synchronization and dl_j appears to depend on the synchronization vector $\boldsymbol{\varepsilon}_i$. The same differential vector is expressed according to Eq. (25) as $d\boldsymbol{p}_{(j)} = \boldsymbol{T}_G(\boldsymbol{\beta}_j) d\boldsymbol{p}$ and then dl_j is written as

$$dl_j = -d\tau\gamma_j(i + \boldsymbol{\beta}_j^T \boldsymbol{c}_\tau), \tag{55}$$

where $\boldsymbol{c}_\tau = d\boldsymbol{p}_s/d\tau$. Cleary dl_j is irrelevant to clock synchronization. The speed of light with respect to PT in S_j is given from Eq. (54) by

$$c_j = \frac{dl_j}{dt_{(j)\circ}} = \gamma_{ji}^2 c(1 - \boldsymbol{\beta}_{ji}^T \hat{\boldsymbol{c}}_i). \tag{56}$$

For a light beam, $d\boldsymbol{p}_{(i)s} = d\tau_{(i)}\boldsymbol{c}_{i\tau} (= cdt_{(i)}\boldsymbol{c}_i)$. Substituting this relationship into Eq. (53) and recalling Eq. (36), one can see that $d\tau_{(j)} = d\tau_{(i)}\gamma_{ji}(1 - i\boldsymbol{\beta}_{ji}^T \boldsymbol{c}_{i\tau})$. It is confirmed from this expression for $d\tau_{(j)}$ and Eq. (54) that the speed of light with respect to AT is written as $c'_j = icdl_j/d\tau_{(j)} = c$ in the standard-synchronized S_j.

Setting $\boldsymbol{\beta}_i = \mathbf{0}$ in Eq. (56), we can find the speed of light in terms of absolute velocity:

$$c' = \gamma^2 c(1 - \boldsymbol{\beta}^T \hat{\boldsymbol{c}}). \tag{57}$$

The c' becomes equal to the c_j when $\boldsymbol{\beta} = \boldsymbol{\beta}_j$ and $\boldsymbol{\beta}_i = \mathbf{0}$. Equation (49b) also represents the speed of light as a function of the absolute velocity. In Appendix, Eqs. (49b) and (57) have been proven to be identical.

Using Eq. (56), one can readily obtain the difference between the elapsed times during the travel of $b_\pm$ in S_j, which is calculated as

$$\Delta t_j = \frac{2dl_j \boldsymbol{\beta}_{ji}^T \hat{\boldsymbol{c}}_{i+}}{\gamma_{ji}^2 c(1 - (\boldsymbol{\beta}_{ji}^T \hat{\boldsymbol{c}}_{i+})^2)}, \tag{58}$$

where $\boldsymbol{c}_{i+}$ denotes the velocity in S_i of the co-propagating light beam b_+. The overall time difference $\Delta t'_D$ is obtained by integrating Eq. (58) over the loop.

In case the directions of $\boldsymbol{c}_{i+}$ and $\boldsymbol{\beta}_{ji}$ are the same, the denominator of Eq. (58) is reduced to c and $\Delta t'_D$ is given by Eq. (52) where $l'_p = \lim_{n\to\infty} \sum_{j=1}^n dl_j$. Meanwhile, if the angle between $\boldsymbol{c}_{i+}$ and $\boldsymbol{\beta}_{ji}$ is ξ so that $\hat{\boldsymbol{\beta}}_{ji}^T \hat{\boldsymbol{c}}_{i+} = \cos\xi$, Eq. (58) can be approximated as $\Delta t_j = 2dl_j\beta_{ji}\cos\xi/c$. Then the time difference is given by

$$\Delta t'_D = \frac{2l'_p \beta\cos\xi}{c}. \tag{59}$$

As far as the time difference is concerned, the effective length of the optical loop is reduced by $\cos\xi$ times, which has been experimentally observed [10]. The generalized Sagnac effect shows that the speed of light is anisotropic not only in rotating frames but also in inertial frames.

V. TRADITIONAL APPROACHES

Traditional approaches [1–7], within the framework of SGR, to circular motion usually exploit the GT between the inertial and the rotating frames, representing the latter in the cylindrical coordinate system. Each infinitely small region in the rotating frame can be regarded as inertial, which allows the application of LT so that the line element becomes invariant. It seems to have been recognized that the Sagnac effect can be, without contradictions, explained within SGR. Relying on the GT and the fundamental principles of SGR, one can exactly find the difference between the travel times of the counter-rotating light beams in the experiment of the Sagnac effect. Following the same traditional methods, however, we can show that the speed of light with respect to PT is anisotropic in the inertial frame, even though LT instead of GT is employed.

1. Sagnac effect in rotating frames

In SGR, inertial frames are equivalent according to the principle of relativity. The laboratory frame can be considered to be isotropic so that it is represented here as S. The time difference can be calculated by employing the line element, which is written in S as

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\phi^2 + dz^2. \quad (60)$$

The coordinates of S and $\tilde{S}$ are related by the GT:

$$\tilde{t} = t, \quad \tilde{r} = r, \quad \tilde{\phi} = \phi - \omega\tilde{t}, \quad \tilde{z} = z. \quad (61)$$

Substituting Eq. (61) into Eq. (60) yields

$$ds^2 = -\gamma^{-2}(cd\tilde{t})^2 + 2\beta\tilde{r}d\tilde{\phi}(cd\tilde{t}) + d\tilde{l}^2, \quad (62)$$

where

$$d\tilde{l}^2 = d\tilde{r}^2 + \tilde{r}^2d\tilde{\phi}^2 + d\tilde{z}^2. \quad (63)$$

Recall $d\tilde{r} = d\tilde{z} = 0$ in the Sagnac experiment, and then $d\tilde{l} = \tilde{r}|d\tilde{\phi}|$. Setting $ds = 0$ to find the elapsed time for the travel of light, we have

$$cd\tilde{t} = \gamma^2(\beta\tilde{r}d\tilde{\phi} + d\tilde{l}). \quad (64)$$

When the counter-rotating light beams $b_\pm$ traverse a circumference of radius r as seen in S, it is easy, using Eq. (64), to obtain the travel time difference for $b_\pm$, which is given by

$$\Delta\tilde{t}_D = \frac{2\gamma^2\tilde{l}_c\beta}{c}, \quad (65)$$

where $\tilde{l}_c = 2\pi\tilde{r}(= 2\pi r)$.

As one can see from the non-relativistic transformation (61), the quantities $\Delta\tilde{t}_D$ and $\tilde{l}_c$ are not relativistic values. It is necessary to find proper ones, which can be done by exploiting metric tensors. In an arbitrary coordinate system, the coordinates of which are represented as (x^0, x^1, x^2, x^3), the line element can generally be written as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -(dx^0_\bullet)^2 + dl^2_\circ, \quad (66)$$

where

$$dx^0_\bullet = |g_{0i}|^{1/2}(dx^0 + g^{-1}_{00}g_{0i}dx^i), \quad (67)$$
$$dl_\circ = (dl^2 - g^{-1}_{00}g_{0i}g_{0j}dx^i dx^j)^{1/2} \quad (68)$$

with $dl = (g_{ij}dx^i dx^j)^{1/2}$. In the above equations, repeated Greek and Latin indices are summed over 0 through 3 and over 1 through 3, respectively. The quantity $dl_\circ$ is the proper distance [4], and the PT is given by

$$dx^0_\circ = |g_{00}|^{1/2}dx^0. \quad (69)$$

The proper distance and time are calculated as Eqs. (68) and (69) because infinitely small regions are regarded as inertial so that the LT can be applied. The nonzero tensor elements of $\tilde{S}$ are written from Eq. (62) as

$$g_{00} = -\gamma^{-2}, \qquad g_{02} = g_{20} = \tilde{\beta}\tilde{r},$$
$$g_{11} = g_{33} = 1, \qquad g_{22} = \tilde{r}^2. \quad (70)$$

Using Eqs. (68) and (70), we have the proper length of the circle

$$\tilde{l}_{c\circ} = \gamma\tilde{l}_c. \quad (71)$$

If the actual radii in S and in the rotating frame are related by the second equation of Eq. (20), the $\tilde{l}_{c\circ}$ corresponds to the rest length of the circle. From Eqs. (69) and (70), $d\tilde{t}_\circ = d\tilde{t}/\gamma$. Dividing both sides of Eq. (65) by γ, we have

$$\Delta\tilde{t}_{D\circ} = \frac{2\tilde{l}_{c\circ}\beta}{c}. \quad (72)$$

The time difference (72) is exactly identical to Eq. (52) for the circular path.

2. Speeds of light in inertial frames

As shown above, the exact time difference can be obtained through the traditional methods based on SGR, which does not imply its consistency, though. Traditional approaches face self-contradictions when discovering the velocities of light in the rotating frame [2, 3, 14]. The rotating frame can be regarded as locally inertial. Thus the local speed of light is considered the constant c because inertial frames are isotropic according to the postulates of special relativity. If the speed of light is really c at every point on the circumference, however, the travel times of $b_\pm$ are the same and the Sagnac effect cannot take place. Invoking the conventionality of simultaneity to escape this dilemma, many relativists have often claimed that different synchronizations should be introduced for the local and the global speeds of light [15–17]. The Sagnac effect is a global event and the time difference is due to the difference in the global, *i.e.*, average, speeds, to which different synchronizations, not the standard synchrony, are applied. The travel times correspond to the PTs of the light detector, which are independent of the synchronization of clocks. Hence, the gauge freedom of synchronization can be applied to the analysis of the Sagnac effect. As a result, the exact travel times are obtained under the standard synchrony, as explained above. However, we can show by applying the same traditional method that the Sagnac effect takes place in inertial frames as well, which has been empirically observed [10,11]. The conventionality of simultaneity cannot save the light speed constancy in inertial frames.

Suppose that S' moves at a constant velocity β in the x-axis direction relative to the isotropic frame S. A co-propagating light beam b_+ travels from x'_0 to x'_1 and the counter-propagating b_- does from x'_1 to x'_0 on the x'-axis where $x'_1 > x'_0$. Let us first employ the GT between S' and S:

$$t' = t, \quad x' = x - \beta ct, \quad y' = y, \quad z' = z. \tag{73}$$

The line element is expressed in S' as

$$ds^2 = -\gamma^2(cdt')^2 + 2\beta dx'(cdt) + dl'^2, \tag{74}$$

where $dl' = (dx'^2 + dy'^2 + dz'^2)^{1/2}$. Note that Eq. (62) becomes equal to Eq. (74) if tildes and $\tilde{r}d\tilde{\phi}$ are replaced by primes and dx'. The non-zero tensor elements are given by

$$g_{00} = -\gamma^{-2}, \qquad g_{01} = g_{10} = \beta,$$
$$g_{11} = g_{22} = g_{33} = 1. \tag{75}$$

Then $dt'_\circ = dt/\gamma$ and $dl'_\circ = (\gamma^2 dx'^2 + dw'^2)^{1/2}$ where $dw' = (dy'^2 + dz'^2)^{1/2}$. From $ds = 0$.

$$cdt'(= cdt) = \gamma^2(\beta dx' + |dx'/\cos\xi'|), \tag{76}$$

where $\tan\xi' = dw'/\gamma dx'$. For $b_\pm$, $\cos\xi' = \pm 1$. As $dw' = 0$, $dl'_\circ = \gamma|dx'|$. The times taken during the travels of $b_\pm$ are calculated as

$$t'_\pm = \frac{\gamma^2 l'(1 \pm \beta)}{c}, \tag{77}$$

where $l' = x'_1 - x'_0$. The travel time difference is given in terms of PT by

$$\Delta t'_{D\circ} = \frac{2l'_\circ\beta}{c}, \tag{78}$$

where $l'_\circ = \gamma l'$ is the rest length of the segment in S'. Equation (78) is consistent with Eq. (52) and with the experimental result of the generalized Sagnac effect.

Even if LT is exploited for the transformation, the same time difference is obtained. According to LT, the coordinates of S' are related to those of S by

$$ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct),$$
$$y' = y, \qquad z' = z, \tag{79}$$

and then its metric tensor is given by

$$g_{00} = -1, \quad g_{11} = g_{22} = g_{33} = 1. \tag{80}$$

Recall that in Eqs. (61) and (73), the transformed time is the same as that in S. In the traditional methods, the travel time differences have first been found, as Eq. (65), in terms of time in S and then they are converted to the PT differences, as Eq. (72), observed by the light detector. Though LT is employed, we utilize Eq. (76) to obtain elapsed times in S. Equation (76) is rewritten as

$$cdt = \gamma^2(\beta dx'' + |dx''/\cos\xi'|), \tag{81}$$

where $x'' = x - \beta ct$ and $\tan\xi' = dw'/\gamma dx''$ with $dw' = (dy'^2 + dz'^2)^{1/2}$. In the GT, $dx'' = dx$. On the contrary, the length contraction occurs in LT. From the second equation of the transformation (79) with $dt = 0$, $dx' = \gamma dx (= \gamma dx'')$, which leads to $\tan\xi' = dw'/dx'$. The ξ' indicates the propagation angle of light with respect to the x'-axis in S'. From Eqs. (69) and (80), $dt'_\circ = dt/\gamma$. Multiplying both sides of Eq. (81) by $1/\gamma$ yields

$$cdt'_\circ = \beta dx' + |dx'/\cos\xi'|. \tag{82}$$

The differential distance is $dl' = (dx'^2 + dy'^2 + dz'^2)^{1/2}$ $(= dl'_\circ)$ in S' and $\cos\xi' = dx'/dl'$. It is obvious that $|dx'/\cos\xi'| = dx'/\cos\xi'$ $(= dl')$. The speed of light with respect to PT is expressed as

$$c' = \frac{dl'_\circ}{dt'_\circ} \tag{83a}$$
$$= \frac{c}{1 + \beta\cos\xi'}. \tag{83b}$$

Note that Eq. (83b) is equal to Eq. (49b). For $b_\pm$, $\cos\xi'$ $= \pm 1$. The difference between their travel times becomes identical to Eq. (78) where $l'_\circ = x'_1 - x'_0$.

The same speed as Eq. (83b) can be obtained directly from LT. Substituting $dx' = \gamma(dx - \beta cdt)$ into Eq. (68) and recalling that $dy' = dy$ and $dz' = dz$, we have

$$dl'^2_\circ(= dl'^2) = (\gamma\beta)^2(cdt)^2 - 2\gamma^2\beta dx(cdt) + (\gamma\beta)^2 dx^2 + dl^2, \tag{84}$$

where $dl = (dx^2 + dy^2 + dz^2)^{1/2}$. For a light signal, $dl = cdt$. Equation (84) is rewritten as

$$dl'^2_\circ = \gamma^2(cdt - \beta dx)^2. \tag{85}$$

The distance that a light signal travels in S' is given by

$$dl'_\circ = \gamma(cdt - \beta dx). \tag{86}$$

The propagation direction of light is ξ with respect to the x-axis so that $\cos\xi = dx/dl(= dx/cdt)$. Substituting Eq. (86) into Eq. (83a) results in the same speed as Eq. (57). As proven in Appendix, Eqs. (57) and (83b) are identical. The $dl'_\circ$ is equal to $cdt'(= \gamma(cdt - \beta dx))$ and the speed of light with respect to AT is c in S', as expected.

PT is irrelevant to the synchronization of clocks. In virtue of the irrelevance, elapsed times in terms of PT can be exactly discovered regardless of synchronization procedures. However, the exact calculation of elapsed times does not always imply the consistency of coordinate transformations. We can find exact travel times even if GT is employed for the coordinate transformation between inertial frames. The spatial vector is also independent of the synchronization of clocks. As a result, exact speeds, with respect to PT, can be obtained regardless of clock synchronizations. The speeds (57) and (83b) with respect to PT are the local speeds of light, which show anisotropy in inertial frames even if LT in place of GT is used for the coordinate transformation.

VI. CONCLUSION

We have presented a relativistic coordinate transformation, TCL-SA, between a rotating world system $\tilde{\mathsf{S}}'$ and the isotropic system S. In TCL-SA, the angle of rotation in the primed is identical with that in the unprimed so that the period of angle is 2π in both, in contrast to TCL-DA in which the period is not equal to 2π in the primed. As in TCL-DA, TCL-SA shows the constancy of the two-way speed of light and enables us to find the inertial transformation between S and an inertial world system through the limit operation of circular motion to linear motion. Additionally TCL-SA is consistent with the MS framework as the conversion matrix $\boldsymbol{A}'(\phi')$ is derived from it, as shown in Sec. III. According to Ref. 1, "For uniform rotation in the case of the Sagnac effect one would expect on intuitive grounds that a Galilean rotation (absolute time) might give the correct choice of space-time coordinate transformation. In consideration, however, of well-known experiences with electromagnetic theory in the realm of uniform translations where the Galilean translation (absolute time) is not an adequate substitute for a Lorentz translation, it is useful to give special attention to the question of selecting the right transformation for uniform rotations." The TCL-SA has been, not "selected", derived for "the right transformation for uniform rotations" based on the LT. The derived coordinate transformation is not only compatible with the MS framework but also consistent with the null result of the Michelson-Morley experiment [24], the transverse Doppler effect in circular motion [25], the time differences in the Hafele-Keating experiment [26], and, of course, the Sagnac effect.

Under the uniqueness of the isotropic frame [18], the speeds of light have been investigated in the rotating and the inertial world systems via TCL-SA and via the MS framework. The analysis results are in agreement with the experimental results for the Sagnac effect including the generalized one that involves linear motions as well. As described in Sec. V, the difference between the travel times of light in the experiment of the Sagnac effect can be exactly found through the traditional approach, which does not imply that the local speed of light is the constant c. Applying the same traditional methods, we have shown that the travel times of the two light beams traversing a line segment in opposite directions are different and that the speed of light is anisotropic also in inertial frames. Even though inertial frames are not isotropic, exploiting the LT that requires only relative velocities without the need for absolute velocities, we can exactly obtain some physical quantities such as PT, Doppler shift, and spatial length independent of clock synchronization [18]. The predictions of the LT associated with these quantities have been shown to be very accurate through numerous experiments [15, 23, 27–30]. It may have led to the firm belief that special relativity has been experimentally verified. When the relative velocity is $\boldsymbol{\beta}_{ji}$ the first row of the LT matrix is equal to the right side of Eq. (36). The reason why LT can provide exact quantities is because the first rows of $\boldsymbol{T}_G(\boldsymbol{\beta}_j, \boldsymbol{\beta}_i)$ and $\boldsymbol{T}_G(\boldsymbol{\beta}_{ji})$ under the standard synchronization are the same. Adopting the standard synchrony, despite the anisotropy of the light speed, but not being subject to the postulates of special relativity, the useful LT that requires relative velocities only can be utilized. It must be a very effective method to approach physics problems. Even the local speed of light can be exactly discovered from the LT, as shown in Sec. V. Moreover it can also solve the generalized Sagnac effect [18].

It is an easy task to see the mathematical infeasibility of the postulates. To this end, I raise a question: given four inertial frames S_i with relative velocities $\boldsymbol{\beta}_{ji}$, $i, j = 1, \cdots, 4$, what are the relationships between their coordinate vectors? I believe any physicists, if they are unable to give consistent answers to this easy question, would not think that the equivalence of inertial frames under light speed constancy is mathematically feasible, unless they are blind believers in the sacred tenet of the postulates. The speed of light is anisotropic in inertial frames. Nature itself reveals the uniqueness of the isotropic frame. TCL and the inertial transformation (23) are consistent with the unique isotropic frame.

APPENDIX

Without loss of generality, the direction of $\boldsymbol{\beta}$ is assumed to be parallel to the x-axis. Recall $\cos\theta = \gamma$ and $\tan\theta = \beta/i$. Including the z-components, the differential coordinate vectors of $\tilde{\mathsf{S}}'_{r'}$ and S are related by

$$d\tilde{\boldsymbol{p}}' = \boldsymbol{T}_I(\beta) d\boldsymbol{p}, \tag{A1}$$

where $d\tilde{\boldsymbol{p}}' = [d\tau', r' d\tilde{\phi}', dz']^T$, $d\boldsymbol{p} = [d\tau, dx, dz]^T$, and

$$\boldsymbol{T}_I(\beta) = \begin{bmatrix} 1/\gamma & 0 & 0 \\ i\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A2}$$

As illustrated in Fig. 3, when a photon traverses $d\tilde{\boldsymbol{p}}'_s$ in $\tilde{\mathsf{S}}'_{r'}$, it does $d\boldsymbol{p}_s$ in S. From $|d\boldsymbol{p}| = 0$

$$dl = -i d\tau, \tag{A3}$$

where $dl (= |d\boldsymbol{p}_s|) = (dx^2 + dz^2)^{1/2}$. Let the propagation angle, the angle from $\boldsymbol{\beta}$ to $\hat{\boldsymbol{c}}$, be ξ. Then

$$\tan\xi = \frac{dz}{dx}, \tag{A4a}$$

$$\cos\xi = \frac{dx}{dl} = \hat{\boldsymbol{\beta}}^T \hat{\boldsymbol{c}}. \tag{A4b}$$

Using Eqs. (A2)–(A4), we have

$$r' d\tilde{\phi}' = \gamma dl(-\beta + \cos\xi). \tag{A5}$$

From Eq. (A4), $\sin\xi = dz/dl$. The squared differential distance in $\tilde{\mathsf{S}}'_{r'}$ is calculated as

$$\begin{aligned} d\tilde{l}'^2 &= (r'd\tilde{\phi}')^2 + dz'^2 (= |d\tilde{\boldsymbol{p}}'_s|^2) \\ &= (\gamma dl)^2(-\beta + \cos\xi)^2 + (dl\sin\xi)^2 \\ &= [\gamma dl(1-\beta\cos\xi)]^2 \end{aligned} \quad \text{(A6)}$$

and the distance is given by

$$d\tilde{l}' = \gamma dl(1-\beta\cos\xi). \quad \text{(A7)}$$

Recall $dl/dt = c$ and $dt/dt' = \gamma$. Substituting Eqs. (A7) and (A4b) into Eq. (49a) yields Eq. (57).

Alternatively we can show the equality by exploiting the relationship between ξ' and ξ known in special relativity. The propagation angle of light is ξ' (with respect to the direction of motion of S'' in Fig. 3). The spatial vector is independent of synchronization vectors, as can be seen from Eqs. (25) and (A1), and so is the direction. It is well known in special relativity that $\cos\xi'$ is represented as

$$\cos\xi' = \frac{\cos\xi - \beta}{1-\beta\cos\xi}. \quad \text{(A8)}$$

Substituting Eqs. (A8) and (A4b) into Eq. (49b) yields Eq. (57).

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