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The Effect of the Frequency Inhomogeneity on the Synchronous States in Systems of Coupled Oscillators

Junhyeok Kim and Seunghwan Kim*

Nonlinear and Complex Systems Laboratory, Department of Physics, Pohang University of Science and Technology, Pohang 37673, Korea

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Despite the ubiquity of systems with heterogeneous connectivity, the role of inhomogeneity on network synchronization has not been fully explored. We studied the effect of multiple inhomogeneities in coupling strengths and natural frequencies on the phase locking behaviors of coupled oscillators. In particular, we found that different Gaussian distributions in natural frequencies lead to diverse synchronization states of the coupled oscillator systems. With the stability analysis and numerical simulations, we analysed the transitions between synchronization states and its breakup in detail.

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I. INTRODUCTION

The collective behaviors of complex systems such as flashing firefies, neurons in a human brain, and power grids have been widely explored via models of coupled oscillator systems. In 1984, Kuramoto introduced a simple, yet generic model which describes salient features of phase dynamics in such systems [1]. The coupled oscillator systems exhibit diverse synchronous behaviors, where their stability is determined by several factors including the inhomogeneity of several intrinsic characteristics [1–5] and the coupling between oscillators [5–7].In particular, the inhomogeneity of intrinsic frequencies has been studied as the main factor for leading to different synchronization states [8,9]. It is shown that this inhomogeneity can induce the various bifurcations [10–12], the synchronization transition [9,12–14] and the hysteresis [12–14] in coupled oscillator systems.

Our study focuses on multiple inhomogeneities both in the coupling strengths and the natural frequencies in the coupled oscillator systems. We want to explore the synergetic effect of various inhomogeneities on the phase dynamics, in particular, diverse states of synchrony and its break-up. We introduce an extended Kuramoto model for the coupled oscillators and derive a condition for the existence of the fully locked state analytically. With the help of numerical simulations, various transitions due to inhomogeneities from full synchrony to other partially locked and drifting states are explored. We end with the conclusion.

*E-mail: swan@postech.ac.kr

II. MODEL AND ANALYSIS

In this paper, we consider a mean-field model for the phase-reduced system of weakly coupled identical limit-cycle oscillators [15,16]:

$$\dot{\theta}_i = \omega_i + \frac{K_i}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i - \beta), \qquad i = 1, 2, ..., N,\ \ \beta \in [0, \pi/2) \quad (1)$$

where $\theta_i(t)$ is the phase of an oscillator i at time t, N the total number of oscillators, ω_i the intrinsic frequency of the oscillators and the set $\{\omega\}$ is taken to be the Gaussian distribution. The interacting term on the right hand side comes from the coupling between oscillators. We consider a set of inhomogeneous coupling strengths, $\{K_i\}$, which are chosen randomly from the Gaussian distribution. Note that K_i represents the effective coupling strength between the oscillator i and others, which is positive for all oscillators.

Using a phase reduction method, we can obtain the coupling function $H(\theta) = \sin(\theta - \beta)$ as a first order approximation to general coupling functions [3,4]. In the interacting term, the phase delay β determines the shape of the coupling function H, affecting the synchronous behaviors of system [1,2]. It can be considered as time delays in interactions between oscillators [17].

In a stationary state, the populational oscillation can be described by an order parameter $Re^{i\Theta} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$. Here R measures the degree of synchrony and coherence of the oscillations of the system. Therefore, we can obtain a mean field frequency, Ω. With this definition, Eq. (1) becomes

$$\dot{\phi}_i = \omega_i - \Omega + K_i R \sin(\Phi - \phi_i - \beta), \tag{2}$$

where $\phi_i \equiv \theta_i - \Omega t$ and $\Phi \equiv \Theta - \Omega t$. When the system reaches a stationary state, R and Φ do not depend on time.

When $\dot{\phi}_i = 0$, the oscillators approach a stable fixed point ${\phi_i}^*$ asymptotically. Then the oscillators become phase-locked in the rotating frame with a frequency Ω, so that we get

$$\omega_i - \Omega = K_i R \sin\left({\phi_i}^* - \Phi + \beta\right). \tag{3}$$

For Eq. (3) to be valid, $K_i R$ should be larger than $|\omega_i - \Omega|$. Therefore, we obtain a domain of the locked subpopulation, $K_i \in \mathcal{D}_l \equiv \{K_i : K_i R > |\omega_i - \Omega|\ \}$. For the stability of the fixed point,

$$\cos\left({\phi_i}^* - \Phi + \beta\right) > 0. \tag{4}$$

From Eqs. (3) and (4), we obtain the fixed points:

$${\phi_i}^* = \sin^{-1}\left(\frac{\omega_i - \Omega}{K_i R}\right) + \Phi - \beta. \tag{5}$$

Following Ko *et al.*, we can obtain the self-consistency equation for Ω and R using the invariant probability density and the order parameter contribution from two subpopulations [16]. In this study, we use the inhomogeneous natural frequencies, so that we need to consider their domain as well for the self-consistent calculation. Especially, we can obtain Ω and R easily if the probability density function is even.

Now, we consider the regime of the full synchrony, where $\dot{\phi}_i = 0$ for all i. Let θ_{mid} the phase of the oscillator whose coupling strength is the same, $\bar{K}$. If the natural frequencies and the coupling strengths are strongly correlated and the probability density function, $g(\omega)$, is even, then $\omega_{\mathrm{mid}} = \bar{\omega} = 0$ and ${\phi_{\mathrm{mid}}}^* - \Phi = 0$. Therefore, Eq. (3) becomes

$$\Omega = -\bar{K} R \sin\left({\phi_{\mathrm{mid}}}^* - \Phi + \beta\right) = -\bar{K}\sin\beta. \tag{6}$$

From the condition for the existence of the fully locked state, we can obtain the range of σ_w and σ_K for full synchrony ($R \approx 1$), so that we obtain

$$\begin{cases} K_{\mathrm{max}} R \approx K_{\mathrm{max}} > |\omega_{\mathrm{max}} - \Omega| \\ K_{\mathrm{min}} R \approx K_{\mathrm{min}} > |\omega_{\mathrm{min}} - \Omega|. \end{cases} \tag{7}$$

With a probability distribution function, $f(x)$, it is assumed $x_{\mathrm{max}} = \bar{x} + 3\sigma_x$ and $x_{\mathrm{min}} = \bar{x} - 3\sigma_x$ statistically. Then Eq. (7) become

$$\begin{cases} \bar{K} + 3\sigma_K > |3\sigma_\omega + \bar{K}\sin\beta| \\ \bar{K} - 3\sigma_K > |-3\sigma_\omega + \bar{K}\sin\beta|. \end{cases} \tag{8}$$

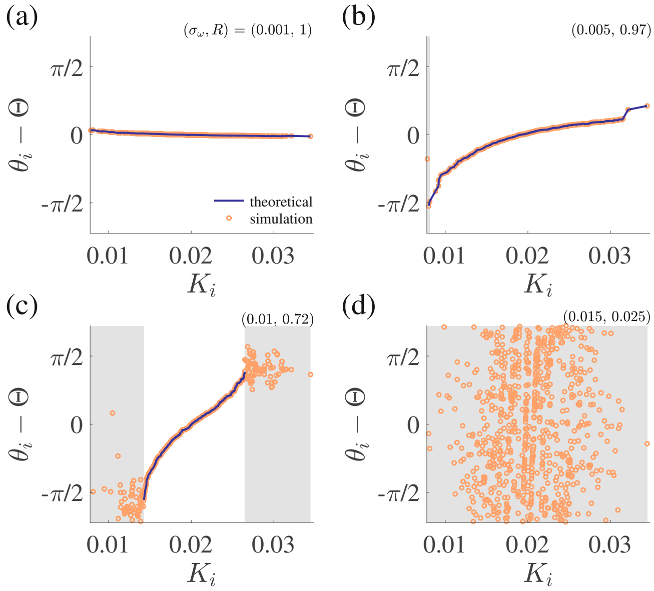

Fig. 1. (Color online) The phase distributions obtained from numerical simulations of the system with a fixed β, 0.1π. (a) - (d) The coupling strengths are randomly selected from a Gaussian distribution with a mean of 20×10^{-3} and a standard deviation of 4.4×10^{-3}. For the natural frequencies, σ_ω are 0.001, 0.005, 0.01, 0.015, respectively. When they are correlated with each other, the system has three synchronous states according to the phase dynamics; (a) S_l, fully locked state (b), (c) S_p, partially synchrony states (d) S_d, fully drifting state. In the figures, solid lines correspond to theoretical curves for locked phases from Eq. (5), and the shaded regions are for K_i values for the drifting subpopulations obtained theoretically from Eq. (5).

From these equations, we can derive a condition for the fully locked state,

$$\begin{aligned} \min\left[\frac{\bar{K}}{3}\left(1 - \sin\beta\right) + \sigma_K, \frac{\bar{K}}{3}\left(1 + \sin\beta\right) - \sigma_K\right] \\ > \sigma_\omega > \max\left[-\frac{\bar{K}}{3}\left(1 - \sin\beta\right) + \sigma_K, 0\right] \end{aligned} \tag{9}$$

Notice that this condition for the fully locked state depends on both σ_K and σ_ω, respectively.

III. NUMERICAL SIMULATIONS

We simulate the model in Eq. (1) numerically, using a fourth order Runge-Kutta method with a total number of oscillators $N = 1{,}000$ and the time step $\Delta t = 0.01$. The set of intrinsic frequencies $\{\omega_i\}$ is chosen randomly with the Gaussian distribution with a density $\mathcal{N}(\bar{\omega} = 0, \sigma_\omega^2)$. The mean of natural frequencies, $\bar{\omega}$, is set to zero in order to make $g(\omega)$ an even function without a loss of generality. In addition, the natural frequencies and the coupling strengths are correlated with each other.

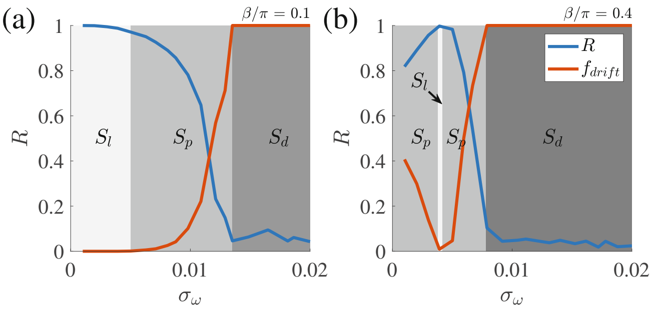

Fig. 2. (Color online) The distribution of the order parameter, R, and the fraction of the drifting oscillators, $f_{\rm drift}$, from numerical simulations as a function of σ_ω, where (a) $\beta = 0.1\pi$ and (b) 0.4π. They allow us to distinguish the phase state of the system and they have the opposite values due to their physical meaning. The mean of natural frequencies is zero. A set of different synchronous dynamics, S_l, S_p and S_d, are illustrated in Fig. 1.

We find that our system shows different synchronous states depending on the standard deviation of the natural frequencies, σ_ω. In Fig. 1(a) - (c), theoretical predictions for the phases from Eq. (5) fit well with the simulations. The system can have three possible types of dynamical states; S_l(fully locked), S_p(partial synchrony) and S_d(fully drifting). Furthermore, Fig. 2 shows the distribution of the order parameter, R, and the fraction of the drifting oscillators, $f_{\rm drift}$, depending on σ_ω for a given β. When the system is S_l, the values of (R, $f_{\rm drift}$) become (1, 0). On the other hand, they approach (0, 1) in case of S_d. According to their physical meanings, they have complementary values between 0 and 1.

In most cases, the order parameter of the system, R, decreases as the inhomogeneity in the natural frequencies increases. When σ_ω is sufficiently large, in addition, the system can become asynchronous regardless of β. However, in this cases R may increase as shown in the left region of Fig. 2(b). In this case, a range of the fully locked state in the parameter space becomes very narrow. For larger σ_ω, R eventually begins to decrease.

In Fig. 3(a), we show the phase diagrams for the system with Gaussian natural frequencies. When the phase delay β is smaller than 0.1π, all oscillators are locked with a lower values of σ_ω. However, if β is higher than 0.1π, a drifting region may appear for the weakly coupled oscillators. If σ_ω satisfies Eq. (9), full synchrony occurs inside the region bounded by the red dotted curves(**- -**). In this fully-locked domain, the order parameter, R become almost 1 from the simulation results, which can be seen in the gradation in color. The boundaries between S_l and S_p are determined analytically from Eq. (9). The results from numerical simulations match quite well with the boundaries of fully locked states. The partial locked state and the fully drifting state are divided by green curves(**—**) in Fig. 3, which are obtained from numerical simulations.

Next, we investigate an effect of different coupling strength distributions on the phase synchrony. The five different set of coupling strengths $\{K_i\}$ are chosen randomly from three Gaussian distributions whose the standard deviations are 2.2×10^{-3} (Fig. 3(b)), 4.4×10^{-3} (Fig. 3(a)) and 6.6×10^{-3} (Fig. 3(c)).

We observe the qualitative difference in bifurcation behaviours of the fully locked state for two different values of the standard deviation; a smaller one (Fig. 3(b)) and a larger one (Fig. 3(c)). With the increase of σ_K, we observe the squeeze in the region for fully locked states, while the mean of coupling strengths is fixed (Fig. 3(b) $\to$ (a) $\to$ (c)). On the other hand, the green curves(**—**), dividing partial synchrony and fully drifting states, move to higher σ_ω as σ_K increases.

When the ratio of σ_K to σ_ω is fixed, the pattern of the phase transition between phase locked states is relatively simple as shown in Fig. 4. The order parameter decreases as the standard deviation increases. Even if σ_K is large, the system can become fully asynchronous for large σ_ω.

We find that the appearance of full synchrony for large σ_K is due to the inhomogeneity in the frequency distribution. If ω is constant, then the effective frequency of strongly coupled oscillators is lower than one of weakly coupled oscillators. Because the natural frequencies and

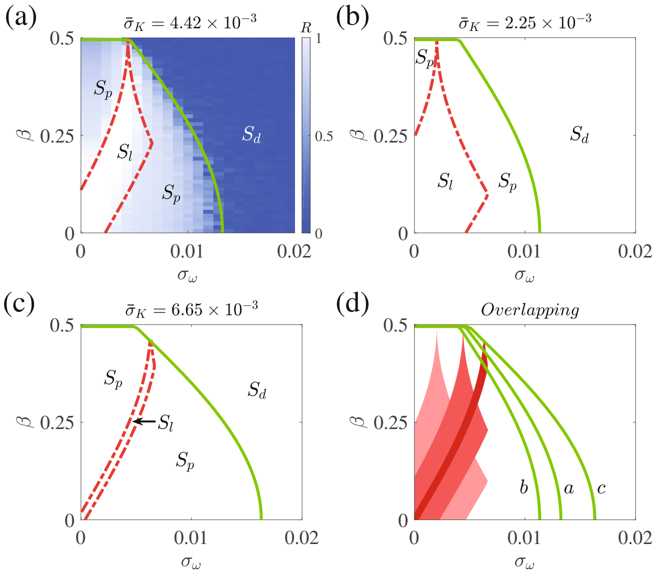

Fig. 3. (Color online) The phase diagram of the coupled oscillators with the Gaussian coupling strength distribution as a function of σ_ω and β. The boundaries **- -** and **—** divide the fully locked region and the fully drifting region, respectively. Note that only the red dotted curve, **- -** are computed analytically. (a) The phase diagram with the color-coded order parameter R. (a) - (c) A set of phase diagrams for different standard deviations of the coupling strength distribution: 4.42×10^{-3}, 2.25×10^{-3}, 6.65×10^{-3}, respectively. (d) All boundaries in (a) - (c) are overlapped to show the overall changes. The larger σ_K gets, the narrower the region of the fully locked state. In addition, the green boundaries for the fully drifting region moves to the right for larger σ_K.

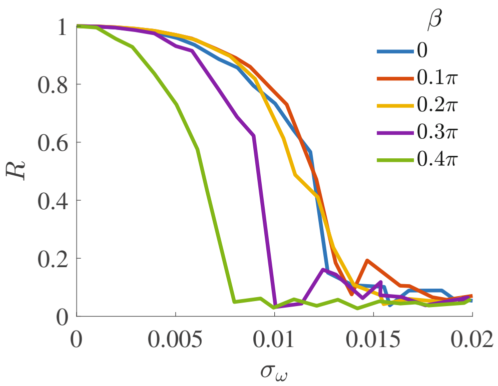

Fig. 4. (Color online) The distribution of the order parameters, R depending on σ_ω when σ_K/σ_ω is fixed to 2. The order parameter decreases as σ_ω increases. The system becomes a fully asynchronous state for small σ_ω when β is large. For these simulations, the mean of the coupling strength is set to 0.05.

the coupling strengths are both correlated with each other, strongly coupled oscillators can take advantage of an additional frequency to catch up with the fastest oscillators, leading to full synchrony. We hope that further study of the uncorrelated or anticorrelated cases can help to better understand the synchrony behaviors of the complex systems of coupled oscillators.

IV. CONCLUSION

We have analyzed the phase dynamics of the coupled oscillator system with both the inhomogeneous coupling strengths and the natural frequencies. Two inhomogeneous sets are taken from the Gaussian distributions and they are correlated with each other. The bifurcation conditions for the fully locked state are obtained analytically, which are in a good agreement with the numerical simulations. Moreover, we observe that a synchrony can break up and the system becomes drifting when the inhomogeneity in the natural frequencies becomes large. The features of the synchrony break-up are found to be different depending on the phase delay. Our studies on the effects of the multiple inhomogeneities on the phase dynamics will help to understand how synchronous behaviors can break down due to inhomogeneities in diverse coupled oscillator-like systems in a real world.

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