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Analytical Solutions of the Dirac Equation with Effective Tensor Potential

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The analytical solutions of the Dirac equation under spin and pseudospin symmetries with a Hellmann-like tensor potential for a class of Yukawa potential is studied via supersymmetric (SUSY) quantum mechanics (QM). The effect of Hellmann like tensor potential which is a new tensor potential on the energy degeneracy in both the spin and pseudospin symmetries has been investigated in detail. The Hellmann like tensor potential removes the energy degeneracies completely in both the spin and pseudospin symmetries. The popular Coulomb tensor and Yukawa tensor were also deduced from the Hellmann tensor potential.

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I. INTRODUCTION

Yukawa in 1935 published an article describing the interaction of elementary particles [1] with a potential model known today as Yukawa potential. This potential model was proposed for describing the nuclear interaction between protons and neutrons due to pion exchange and has found applications in many branches of physics [2–20]. Due to its simplest model of the interaction between fields, the Yukawa potential is often used as the prototype of more realistic theories in many situations. In fluid, for example, Kadiri *et al.* [2] added the hardcore attractive Yukawa potential to the Lennard-Jones potential model to stabilize the local fluid structure and was shown to have provided a good expression for the free energy and the pair correlation function for the system under study. In Ref. [3], the known regularities in dense fluids were studied using the hardcore one the Yukawa and hardcore double Yukawa potential models. The thermodynamic properties of the freely jointed tangent homonuclear Yukawa chain fluid vapour-liquid equilibria have also been studied [4]. Aside from the use of the hardcore Yukawa potential model for the study of fluids, it has also found application to different systems like chain molecules [6,7], osmotic pressure for proteins [8], and colloidal particles [9]. The potential is also useful in the stabilization of energy levels and the computations of bound-states energies in neutral atoms both in relativistic and non-relativistic equations [10–14]. In relativistic quantum mechanics, the Yukawa potential has been used as a tensor interaction to stabilize the structures of the nucleus [16–20] by removing the degeneracies between the spin and the pseudospin symmetries. For instance, Hamzavi *et al.* [17] applied the inversely quadratic Yukawa potential and a tensor interaction term to the solutions of the approximate spin and pseudospin symmetries of Dirac. Their results show that by the degeneracies between spin and pseudospin state doublets were removed when the tensor interaction was applied. In Ref. [18], the authors applied the Coulomb-like and the Yukawa-like tensor interactions term respectively to the relativistic symmetries of the Deng-Fan and the Eckart potentials. Their findings also reveal the removal of degeneracies when the tensor interaction term is applied. The relativistic symmetries of the multiparameter exponential-type potential within the Coulomb-like and the Yukawa-like tensor interaction terms have also been studied [19], and the removal of degeneracies as a result of the application of the Yukawa tensor interaction term was reported.

The stability of the nuclear structure is very important in the study of atoms; hence, several researchers have used this potential model to obtain the bound-state energies and the wave functions in for the relativistic and the non-relativistic wave equations [20]. Considering the importance/usefulness of the Yukawa potential and Coulomb potential, we intend to study the behavior of Dirac equation in the framework of the spin symmetry and the pseudospin symmetry limit by using combinations of the Yukawa potential, Coulomb potential and inversely quadratic Yukawa potential, called class of Yukawa potentials in the presence of Hellmann-like tensor potential; such as a study yet been reported. The class of Yukawa potential has been studied under non relativistic Schrödinger equation by Onate and Ojonubah

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[21]. The class of Yukawa potentials read:

$$V\left(r\right)=-\frac{b}{r}+\frac{ce^{-\delta r}}{r}-\frac{ae^{-2\delta r}}{r^{2}}. \tag{1}$$

This potential has not received any attention in the relativistic region. Thus, the authors want to study the symmetry limits of the potential with a new tensor interaction which has not yet been reported.

II. DIRAC EQUATION

The Dirac equation for a spin-1/2 particle with mass M moving in the field of a repulsive vector, attractive scalar and tensor potentials ($V(r)$, $S(r)$ and $U(r)$) in relativistic units ($c=\hbar=1$) is given as

$$\begin{aligned}&[\overrightarrow{\alpha}.\overrightarrow{p}+\beta(M+S(r))\\&\quad -i\beta\overrightarrow{\alpha}.\mathbf{r}U(r)+V(r)-E]\,\psi(\overrightarrow{r})=0,\end{aligned} \tag{2a}$$

where $\vec{p}=-i\vec{\nabla}$ is the momentum operator, denotes the relativistic energy of the system, α and β are the 4×4 usual Dirac matrices [22]. Hence, the spinor components can be written as follows:

$$\psi_{nk}(\vec{r})=\begin{pmatrix}f_{nk}(\vec{r})\\ g_{nk}(\vec{r})\end{pmatrix}=\frac{1}{r}\begin{pmatrix}F_{nk}(r)Y^{\ell}_{jm}(\theta,\varphi)\\ iG_{nk}(r)Y^{\bar{\ell}}_{jm}(\theta,\varphi)\end{pmatrix}, \tag{2b}$$

where $f_{nk}(\vec{r})$ is the upper component and $g_{nk}(\vec{r})$ is the lower component of the Dirac spinors. $Y^{\ell}_{jm}(\theta,\varphi)$ and $Y^{\bar{\ell}}_{jm}(\theta,\varphi)$ are spin and pseudospin spherical harmonics, respectively, and m is the projection of the angular momentum on the z-axis. In view of the paper of Ikhdair and Sever [23], we easily obtain

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+T_1\right.\\&-\left[(M+E_{n\kappa}-\Delta(r))\left(M-E_{n\kappa}+\sum(r)\right)\right]\\&\left.+\frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)\right\}F_{n\kappa}(r)=0\end{aligned} \tag{3}$$

$$T_1=\frac{2\kappa U\left(r\right)}{r}-\frac{\mathrm{d}U\left(r\right)}{r}-U^2\left(r\right), \tag{4}$$

for $\kappa\left(\kappa+1\right)=\tilde{\ell}\left(\check{\ell}+1\right)$, $r\in(0,\infty)$, and

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+T_2\right.\\&-\left[(M+E_{n\kappa}-\Delta(r))\left(M-E_{n\kappa}+\sum(r)\right)\right]\\&\left.-\frac{\frac{\mathrm{d}\sum(r)}{\mathrm{d}r}}{M-E_{n\kappa}+\sum(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\right\}G_{n\kappa}(r)=0,\end{aligned} \tag{5}$$

$$T_2=\frac{2\kappa U\left(r\right)}{r}+\frac{\mathrm{d}U\left(r\right)}{r}-U^2\left(r\right), \tag{6}$$

for $\kappa(\kappa-1)=\tilde{\ell}(\check{\ell}+1)$, $r\in(0,\infty)$, where the difference potential is given by $\Delta\left(r\right)=V\left(r\right)-S(r)$, the sum potential is given by $\sum(r)=V(r)+S(r)$, and κ is the orbit-spin quantum number and is related to the orbital quantum numbers ℓ and $\bar{\ell}$ for the spin and the pseudospin symmetric models as

$$\kappa=\begin{cases}-(\ell+1)=-(j+\frac{1}{2})(s_{\frac{1}{2}},p_{\frac{3}{2}},etc.)\\ \qquad j=\ell+\frac{1}{2}\text{ aligned spin }(\kappa<0),\\ +\ell=+(j+\frac{1}{2})(p_{\frac{1}{2}},d_{\frac{3}{2}},etc.)\\ \qquad j=\ell-\frac{1}{2}\text{ unaligned spin }(\kappa>0)\end{cases}$$

respectively. Similarly, κ in the quasi-degenerate doublet structure can be further expressed in terms of $\bar{s}=\frac{1}{2}$ and $\bar{\ell}$ the pseudospin and the pseudo-orbital angular momenta are respectively is given as

$$\kappa=\begin{cases}-\tilde{\ell}=-(j+\frac{1}{2})(s_{\frac{1}{2}},p_{\frac{3}{2}},etc.)\\ \qquad j=\tilde{\ell}-\frac{1}{2}\text{ aligned spin }(\kappa<0),\\ +(\bar{\ell}+1)=+(j+\frac{1}{2})(d_{\frac{3}{2}},f_{\frac{5}{2}},etc.)\\ \qquad j=\bar{\ell}+\frac{1}{2}\text{ unaligned spin }(\kappa>0)\end{cases}$$

where $\kappa=\pm1,\pm2,\pm3,\cdots,j$ is the angular momentum and n is the radial quantum number.

III. BOUND STATE SOLUTIONS

1. Spin symmetry limit

The spin symmetry limit occurs when $\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}=0$, and $\Delta\left(r\right)=C_s=$ constant [24–26]. The sum potential $\sum(r)$ is equal to the class of Yukawa potentials. It is noted that the solution for the ℓ-wave of Eqs. (3) and (5) is not possible due to the centrifugal term with tensor coupling. A higher-order term in a relativistic expansion for tensor coupling increases the spin-orbit coupling. This is an indication that the tensor coupling contributes to the splitting of the pseudospin in the nucleus. This contribution is expected to be close the Fermi surface because the tensor coupling depends on the derivative of a vector potential, which has a peak close to the Fermi surface for typical nuclear mean-field vector potential [27]. To obtain the approximate solution, we apply a new approximation scheme proposed by Qiang *et al.* [28]:

$$\frac{1}{r^2}=\frac{\delta^2}{\left(1-e^{-\delta r}\right)^2}. \tag{7}$$

Due to the inclusion of the tensor potential in Eqs. (3) and (5), we define the Hellmann tensor as

$$U(r)=-\frac{H_{\mathrm{C}}}{r}-\frac{H_Y e^{-\delta r}}{r},r\geq R_{\mathrm{C}},\quad H=\frac{Z_aZ_be^2}{4\pi\epsilon_0} \tag{8}$$

where $R_C = 7.78$ fm is the coulomb radius, Z_a and Z_b denote the charges of the projectile particle a and the target nuclei b respectively, and H is the tensor strength [27]. Substituting Eqs. (1) and (7) into Eq. (3) we obtain

$$\frac{d^2F_{n\kappa}(r)}{dr^2} = \left[V_{\text{effs}} - E_{\text{eff}n\kappa,s}\right] F_{n\kappa}(r) \tag{9}$$

where,

$$V_{\text{eff}} = \frac{V_{T_{S1}}e^{-\delta r}}{1-e^{-\delta r}} + \frac{V_{T_{S2}}e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2} + \frac{V_{T_{S3}}e^{-2\delta r}}{\left(1-e^{-\delta r}\right)^2} \tag{10}$$

$$-E_{\text{eff}n\kappa,s} = \chi_3^S\chi_0^S + \delta^2\chi_1^S, \tag{11}$$

$$V_{T_{S1}} = \delta\chi_V^S\chi_0^S + \chi_1^S\delta^2 + H_Y\delta^2, \tag{12}$$

$$V_{T_{S2}} = \delta^2\left[\chi_1^S + \chi_2^S\right], \tag{13}$$

$$V_{T_{S3}} = \delta^2(H_Y - a\beta) \tag{14a}$$

$$\chi_0^S = M + E_{n\kappa s} - C_s, \tag{14b}$$

$$\chi_V^S = a\delta + b - c, \tag{14c}$$

$$\chi_1^S = (\kappa + H_C)(\kappa + H_C + 1) \tag{14d}$$

$$\chi_2^S = H_Y(1 + 2\kappa + 2H_C), \tag{14e}$$

$$\chi_3^S = M - E_{n\kappa s} - b\delta. \tag{14f}$$

For a bound state, the ground-state function for the upper component $F_{n\kappa}(r)$ can be written in the form of

$$F_{0\kappa}(r) = \exp\left(-\textstyle\int^W (r)\,dr\right), \tag{15}$$

where $W(r)$ is called the superpotential in supersymmetric quantum mechanics [29–32] and satisfies the solution of the Riccati equation given in Eq. (9) as it makes the left side compatible with the right side. Substituting Eq. (15) into Eq. (9), we obtain the following equation for $W(r)$:

$$W^2(r) - W'(r) = \frac{V_{T_{S1}}e^{-\delta r}}{1-e^{-\delta r}} + \frac{V_{T_{S2}}e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2} + \frac{V_{T_{S3}}e^{-2\delta r}}{\left(1-e^{-\delta r}\right)^2} + \chi_0^S\chi_3^S + \delta^2\chi_1^S, \tag{16}$$

where our superpotential $W(r)$ is taken as

$$W(r) = \rho_1 - \frac{\rho_2}{1-e^{-\delta r}}. \tag{17}$$

Substituting Eq. (17) into Eq. (16) leads us to the following relations:

$$\rho_1^2 = \chi_3^S\chi_0^S + \delta^2\chi_1^S, \tag{18}$$

$$\rho_2 = \delta\left[a\chi_0^S - \chi_1^S - \chi_2^S - H_Y^2\right], \tag{19}$$

$$\rho_1 = \frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_1^S}{2\rho_2} - \frac{\rho_2}{2}. \tag{20}$$

With Eq. (17), we can construct the two partner potentials $V_\pm(r) = W^2(r) \pm \frac{dW(r)}{dr}$ as follows:

$$V_+(r) = \rho_1^2 - \frac{\delta\left(\chi_V^S\chi_0^S - H_Y\delta - \delta\chi_1^S\right)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\delta^2\left(a\chi_0^S + \chi_T^S\right)(a\chi_0^S + \chi_T^S + 1)e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2}, \tag{21}$$

$$V_-(r) = \rho_1^2 - \frac{\delta\left(\chi_V^S\chi_0^S - H_Y\delta - \delta\chi_1^S\right)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\delta^2\left(a\chi_0^S + \chi_T^S\right)(a\chi_0^S + \chi_T^S - 1)e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2}, \tag{22}$$

$$\chi_T^S = -\chi_1^S - \chi_2^S - H_Y^2. \tag{22a}$$

Using the shape invariance technique [33–36], we can readily show that the two partner potentials are shape invariant. The invariant potentials are the same except for a constant. Therefore, the following relationship exists:

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1), \tag{23}$$

where a_0 is an old set of parameters in which the new set of parameters a_1 is obtained from $V_-(r, a_1)$, and $R(a_1)$ is a remainder that is independent of the variable r. Here, $\rho_2 = a_0$ via a mapping of the form $a_1 \to a_0 - \delta$, $a_2 \to a_0 - 2\delta$, $a_3 \to a_0 - 3\delta$. Thus, a generalization is drawn as $a_n \to a_0 - n\delta$. In terms of the parameters of the problem, we obtain the following relations:

$$R(a_1) = \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_0^2}{2a_0}\right] - \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_1^2}{2a_1}\right], \tag{24}$$

$$R(a_2) = \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_1^2}{2a_1}\right] - \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_2^2}{2a_2}\right], \tag{25}$$

$$R(a_3) = \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_2^2}{2a_2}\right] - \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_3^2}{2a_3}\right], \tag{26}$$

$$R(a_n) = \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_{n-1}^2}{2a_{n-1}}\right] - \left[\frac{\delta\chi_V^S\chi_0^S - H_Y\delta^2 - \delta^2\chi_0^S - a_n^2}{2a_n}\right]. \tag{27}$$

Based on the shape invariance approach and formalism [37,38], the ground state energy in relation to the partner potential is given as

$$E_{0,\kappa}^{-}=0, \tag{28}$$

$$E_{n,\kappa}=E_{\rm eff}+E_{0,\kappa}^{-}=\sum_{k=1}^{n}R(a_k)=\left[\frac{\delta\chi_V^S\chi_0^S-H_Y\delta^2-\delta^2\chi_0^S-a_n^2}{2a_n}\right]. \tag{29}$$

This gives energy equation as

$$\chi_3^S\chi_0^S-\chi_1^S=\delta^2\left[\frac{\frac{\chi_0^S\chi_V^S}{\delta}-\left(a\chi_0^S+\chi_1^S+\chi_2^S+H_Y^2+n\right)^2-H_Y-\chi_1^S}{2\left(a\chi_0^S+\chi_1^S+\chi_2^S+H_Y^2+n\right)}\right]^2. \tag{30}$$

We now introduce a new variable of the form $y=e^{-\delta r}$ and writing the wave function which satisfies the boundary conditions as

$$F_{nk}(y)=y^{\chi_4^S}(1-y)^{\chi_5^S}f_{n\kappa}(y), \tag{31}$$

where

$$\chi_4^S=\sqrt{\chi_1^S+\chi_2^S+H_Y^2+\frac{\chi_0^S\chi_3^S}{\delta^2}}, \tag{32}$$

$$\chi_5^S=\frac{1}{2}+\frac{1}{2}\sqrt{1+4\chi_1^S-4a\chi_0^S}. \tag{33}$$

Thus, using formula method for the bound-state problem [39,40], we can easily write the wave spinor for the spin symmetry as

$$F_{n\kappa}(y)=N_{n\kappa}y^{\chi_4^S}(1-y)^{\chi_5^S}{}_2F_1\times\left(-n,n+2\left(\chi_4^S+\chi_5^S\right);2\left(\chi_4^S+\frac{1}{2}\right),y\right), \tag{34}$$

where ${}_2F_1$ is a hypergeometric function.

2. The Pseudospin symmetric limit

For pseudospin symmetry to occur, $\frac{\mathrm{d}\sum(r)}{\mathrm{d}r}=0$ and $\sum(r)=C_{ps}=constant$. Then, we set the sum potential as the class of Yukawa potential, Eq. (1). Substituting Eqs. (1) and (7) into in Eq. (5), we obtain

$$\frac{\mathrm{d}^2G_{n\kappa}(r)}{\mathrm{d}r^2}=\left[V_{\mathrm{eff}n\kappa p}-E_{\mathrm{eff}n\kappa,p}\right]G_{n\kappa}(r), \tag{35}$$

where

$$V_{\mathrm{eff}n\kappa p}=\frac{V_{T_{P1}}e^{-\delta r}}{1-e^{-\delta r}}+\frac{V_{T_{P2}}e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2}+\frac{V_{T_{P3}}e^{-2\delta r}}{\left(1-e^{-\delta r}\right)^2}, \tag{36}$$

$$-E_{\mathrm{eff}n\kappa,p}=\chi_3^P\chi_0^P+\delta^2\chi_1^P, \tag{37}$$

$$V_{T_{P1}}=\delta\chi_V^P\chi_0^P+\delta^2\chi_1^P-H_Y\delta^2, \tag{38}$$

$$V_{T_{P2}}=\left[\chi_1^P+\chi_2^P\right]\delta^2, \tag{39}$$

Table 1. Energy for the spin symmetry with $b=2$, $c=-1$, 1×10^{-5}, $\delta=0.55$, $M=C_s=2b-c$ in the absence of the tensor interaction.

$n,k<0$	(ℓ,j)	$H_C=H_Y=0$	$n,k>0$	(ℓ,j)	$H_C=H_Y=0$
0, −1	$0S1/2$	0.912500756	0, 1	$0P1/2$	3.879763952
1, −1	$1S1/2$	1.779269037	1, 1	$1P1/2$	4.631047143
2, −1	$2S1/2$	3.675367809	2, 1	$2P1/2$	4.924637269
3, −1	$3S1/2$	4.540090195	3, 1	$3P1/2$	4.999955229
0, −2	$0P3/2$	3.879763952	0, 2	$0d3/2$	4.899994026
1, −2	$1P3/2$	4.631047143	1, 2	$1d3/2$	4.720178525
2, −2	$2P3/2$	4.924637269	2, 2	$2d3/2$	4.442935528
3, −2	$3P3/2$	4.999955229	3, 2	$3d3/2$	4.014130752
0, −3	$0d5/2$	4.899994026	0, 3	$0f5/2$	2.901530174
1, −3	$1d5/2$	4.720178525	1, 3	$1f5/2$	2.903129629
2, −3	$2d5/2$	4.442935528	2, 3	$2f5/2$	2.904403796
3, −3	$3d5/2$	4.014130752	3, 3	$3f5/2$	2.905434959
0, −4	$0f7/2$	2.901530174			
1, −4	$1f7/2$	2.903129629			
2, −4	$2f7/2$	2.904403796			
3, −4	$3f7/2$	2.905434959			

$$V_{T_{P3}}=\left(a\chi_0^P+H_Y^2\right)\delta^2, \tag{40a}$$

$$\chi_0^P=M-E_{n\kappa P}+C_p. \tag{40b}$$

$$\chi_1^P=(k+H_{\rm C})(\kappa+H_{\rm C}-1), \tag{40c}$$

$$\chi_V^P=b-c-a\delta, \tag{40d}$$

$$\chi_2^P=H_Y(2\kappa+2H_{\rm C}-1) \tag{40e}$$

$$\chi_3^P=E_{n\kappa,P}+M+b\delta. \tag{40f}$$

To avoid repetition of work, we can easily the energy equation for the spin symmetric limit by following the previous procedure [32,41,42]:

$$\chi_3^P\chi_0^P+\delta^2\chi_1^P=\delta^2\left[\frac{\frac{\chi_0^P\chi_V^P}{\delta}-\left(n+\chi_1^P+\chi_2^P+H_Y^2+a\chi_0^P\right)^2-H_Y-\chi_1^P}{2\left(n+\chi_1^P+\chi_2^P+H_Y^2+a\chi_0^P\right)}\right]^2. \tag{41}$$

Using the formula method for bound-state problem [39, 40], the corresponding wave functions is written in the form

$$G_{nk}(y)=N_{nk}y^{\chi_4^P}(1-y)^{\chi_5^P}{}_2F_1\times\left(-n,n+2\left(\chi_4^P+\chi_5^P\right);2\left(\chi_4^P+\frac{1}{2}\right),y\right). \tag{42}$$

where

$$\chi_4^P=\sqrt{\chi_1^P+H_Y^2+\frac{\chi_3^P\chi_0^P}{\delta^2}}, \tag{43}$$

$$\chi_5^P=\frac{1}{2}+\frac{1}{2}\sqrt{1+4\chi_1^P+4\chi_2^P+4a\chi_0^P}. \tag{44}$$

Table 2. Energy for the spin symmetry with $b = 2$, $c = -1$, 1×10^{-5}, $\delta = 0.55$, $M = C_s = 2b - c$ in the presence of the Hellmann-like tensor interaction.

$n, k < 0$	(ℓ, j)	$H_C = H_Y = 0.5$	$H_C = H_Y = 1$	$n, k > 0$	(ℓ, j)	$H_C = H_Y = 0.5$	$H_C = H_Y = 1$
0, −1	$0S1/2$	0.458333949	3.780014709	0, 1	$0P1/2$	4.923440481	2.887427687
1, −1	$1S1/2$	1.836006199	4.586797645	1, 1	$1P1/2$	4.758338466	2.891085893
2, −1	$2S1/2$	3.702005064	4.908062172	2, 1	$2P1/2$	4.497938250	2.894000312
3, −1	$3S1/2$	4.551999181	4.999028663	3, 1	$3P1/2$	4.099094335	2.896359088
0, −2	$0P3/2$	0.229167844	0.183334146	0, 2	$0d3/2$	2.893773804	2.909525707
1, −2	$1P3/2$	2.052256303	1.999648265	1, 2	$1d3/2$	2.896505572	2.909798738
2, −2	$2P3/2$	3.805400093	3.780014709	2, 2	$2d3/2$	2.898681878	2.910035521
3, −2	$3P3/2$	4.598089330	4.586797645	3, 2	$3d3/2$	2.900443229	2.910242175
0, −3	$0d5/2$	4.087555134	0.550004728	0, 3	$0f5/2$	2.912858516	2.914815964
1, −3	$1d5/2$	4.721738072	2.398674876	1, 3	$1f5/2$	2.912823268	2.914667155
2, −3	$2d5/2$	4.956122444	3.974874806	2, 3	$2f5/2$	2.912792587	2.914531967
3, −3	$3d5/2$	4.997601092	4.672863737	3, 3	$3f5/2$	2.912765701	2.914408781
0, −4	$0f7/2$	4.846567773	4.312528891				
1, −4	$1f7/2$	4.638831301	4.816018940				
2, −4	$2f7/2$	4.326784589	4.983413442				
3, −4	$3f7/2$	3.824660336	4.985998070				

IV. THEORETIC QUANTITIES AND THE CLASS OF YUKAWA POTENTIAL

To study any theoretic quantity, first we must obtain the non-relativistic limit of the spin symmetry, which is equal to the solution of the Schrödinger equation. Use of transformations, $\kappa \to \ell$, $H = C_S = 0$, $M - E_{n\kappa s} \to E_{n\ell}$, $F_{n\kappa} \to R_{n\ell}$ and $M + E_{n\kappa s} \to \frac{2\mu}{\hbar^2}$, turn energy equation, Eq. (30) into

$$E_{n\ell} = \delta\left(\frac{\delta\ell(\ell+1)\hbar^2}{2\mu} - b\right) - \frac{\delta^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu\delta(b-c-a\delta)}{\delta\hbar^2} - (n+\ell+1)^2 - \ell(\ell+1)}{2(n+\ell+1)}\right]^2, \quad (45)$$

with the corresponding wave function being

$$R_{n\ell}(y) = N_{n\ell}^2 y^{2\eta}(1-y)^{2\lambda-1} \times \left[P_n^{(\lambda,\eta)}(x)\right]^2, \quad (46)$$

where

$$\eta = \sqrt{\ell(\ell+1) - \frac{2\mu(\delta b + E_{n\ell})}{\delta^2\hbar^2}}, \quad (47)$$

$$\lambda = \sqrt{1 + \ell(\ell+1) - \frac{2\mu a}{\delta\hbar^2}}, \quad (48)$$

and $P_n^{(\lambda,\eta)}$ is a Jacobi polynomial.

Table 3. Energy for the pseudospin symmetry with $b = 2$, $c = -1$, 1×10^{-5}, $\delta = 0.55$, $M = -C_s = 2b - c$ in the absence of the tensor interaction.

$n, k < 0$	(ℓ, j)	$H_C = H_Y = 0$	$n, k > 0$	(ℓ, j)	$H_C = H_Y = 0$
1, −1	$1S1/2$	−8.057055900	1, 2	$0d3/2$	−8.057055900
1, −2	$1P3/2$	−8.812992127	1, 3	$0f5/2$	−8.812992127
1, −3	$1d5/2$	−8.276254031	1, 4	$0g7/2$	−8.276254031
1, −4	$1f7/2$	−6.268199121	1, 5	$0h9/2$	−6.268199121
2, −1	$1S1/2$	−8.543579392	2, 2	$1d3/2$	−8.543579392
2, −2	$2P3/2$	−8.807557605	2, 3	$1f5/2$	−8.807557605
2, −3	$2d5/2$	−8.128636585	2, 4	$1g7/2$	−8.128636585
2, −4	$2f7/2$	−5.869980610	3, 5	$1h9/2$	−5.869980610

1. Onicescu information energy and the Class of Yukawa Potential

The Onicescu information energy is defined as [43]

$$E(\rho) = 4\pi \int_0^\infty \rho^2(r). \quad (49)$$

where $\rho(r)$ is the probability density and is equal to the squared of the radial wave function:

$$\rho(y) = N_{nk} y^{2\lambda}(1-y)^{2\eta} \times \left[P_n^{(\lambda,\eta)}(1-2y)\right]^2, \quad (50)$$

Table 4. Energy for the pseudospin symmetry with $b = 2$, $c = -1$, 1×10^{-5}, $\delta = 0.55$, $M = -C_s = 2b - c$ in the presence of the Hellmann-like tensor interaction.

$n, k < 0$	(ℓ, j)	$H_C = H_Y = 0.5$	$H_C = H_Y = 1$	$n, k > 0$	(ℓ, j)	$H_C = H_Y = 0.5$	$H_C = H_Y = 1$
1, −1	$1S1/2$	−4.420912179	−4.459802726	1, 2	$0d3/2$	−8.851402330	−8.354981192
1, −2	$1P3/2$	−7.926127556	−4.145083478	1, 3	$0f5/2$	−8.319856191	−6.405654197
1, −3	$1d5/2$	−8.739747755	−7.759779526	1, 4	$0g7/2$	−6.342575564	−2.073606788
1, −4	$1f7/2$	−8.207386447	−8.652089433	1, 5	$0h9/2$	−2.071786783	−2.078022112
2, −1	$1S1/2$	−7.018827033	−7.041623306	2, 2	$1d3/2$	−8.841803377	−8.206520154
2, −2	$2P3/2$	−8.452985591	−6.856689634	2, 3	$1f5/2$	−8.171754942	−6.019479342
2, −3	$2d5/2$	−8.742424436	−8.338192691	2, 4	$1g7/2$	−5.950971423	−2.074457428
2, −4	$2f7/2$	−8.060612931	−8.664748160	3, 5	$1h9/2$	−2.072749151	−2.078445284

Table 5. Energy of the spin symmetry for the Hellman potential with Hellmann-like tensor interaction with $b = 2$, $c = -1$, 1×10^{-5}, $\delta = 0.55$, $M = C_s = 2b - c$.

$n, k < 0$	(ℓ, j)	$H_C = H_Y = 0$	$H_C = H_Y = 0.25$	$n, k > 0$	(ℓ, j)	$H_C = H_Y = 0$	$H_C = H_Y = 0.25$
0, −1	$0S1/2$	2.500000000	1.106582950	0, 1	$0P1/2$	4.965099438	4.927210840
1, −1	$1S1/2$	4.205614468	3.663151543	1, 1	$1P1/2$	4.993794690	4.800671193
2, −1	$2S1/2$	4.906681764	4.835972843	2, 1	$2P1/2$	4.921300501	4.619346840
3, −1	$3S1/2$	4.999986119	4.995743854	3, 1	$3P1/2$	4.788598250	4.371832674
0, −2	$0P3/2$	4.965099438	4.077696207	0, 2	$0d3/2$	4.458785634	2.764731860
1, −2	$1P3/2$	4.993794690	4.907083976	1, 2	$1d3/2$	4.167024724	2.641491803
2, −2	$2P3/2$	4.921300501	4.999997675	2, 2	$2d3/2$	3.736663966	2.640785213
3, −2	$3P3/2$	4.788598250	4.952402313	3, 2	$3d3/2$	2.639538462	2.640250791
0, −3	$0d5/2$	4.458785634	4.884458024	0, 3	$0f5/2$	2.644367418	2.642631513
1, −3	$1d5/2$	4.167024724	4.745062101	1, 3	$1f5/2$	2.643353028	2.642037614
2, −3	$2d5/2$	3.736663966	4.552367111	2, 3	$2f5/2$	2.642547771	2.641541130
3, −3	$3d5/2$	2.639538462	4.290555184	3, 3	$3f5/2$	2.641897891	2.641121869
0, −4	$0f7/2$	2.644367418	2.647823434				
1, −4	$1f7/2$	2.643353028	2.645819672				
2, −4	$2f7/2$	2.642547771	2.644347005				
3, −4	$3f7/2$	2.641897891	2.643233168				

$$E(\rho) = -\frac{4\pi}{\delta}\int_1^0 \rho^2(y)\mathrm{d}y, \quad y = e^{-\delta r}, \tag{51}$$

$$E(\rho) = \frac{2\pi}{\delta}\int_{-1}^{1} \rho^2(s)\mathrm{d}s, \quad s = 1 - 2y, \tag{52}$$

$$\rho(s) = N_{n\ell}^2\left(\frac{1-s}{2}\right)^{2\eta}\left(\frac{1+s}{2}\right)^{2\lambda-1} \times \left[P_n^{(\lambda,\eta)}(s)\right]^2, \tag{53}$$

Using a standard integral of the form

$$\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{a-1}\left(\frac{1+x}{2}\right)^{b} \times \left[P_n^{(a,b)}(x)\right]^2\mathrm{d}x = \frac{2\Gamma(a+n+1)\Gamma(b+n+1)}{n!a\Gamma(a+b+n+1)}, \tag{54}$$

the information energy in Eq. (48) finally becomes

$$E(\rho) = 8\pi\left[\frac{\lambda\Gamma(2\lambda+n+1)\Gamma(2\lambda+2\eta+n+1)}{(n!)^2\Gamma(2\lambda+1)^2\Gamma(2\eta+n+1)} \times \frac{\Gamma(2\lambda+n+1)\Gamma(2\eta+2\lambda+n+1)}{(2\lambda-1)\Gamma(2\eta+2\lambda+n)}\right]^2 \tag{55}$$

For the ground state,

$$E(\rho) = 8\pi\left[\frac{\lambda\Gamma(2\lambda+2\eta+1)}{\Gamma(2\eta+1)} \times \frac{\Gamma(2\eta+2\lambda+1)}{(2\lambda-1)\Gamma(2\eta+2\lambda)}\right]^2. \tag{56}$$

Table 6. Energy of the spin symmetry for the Yukawa potential with the Yukawa-like tensor interaction with $b = 0$, $c = 1$, $a = 0 \times 10^{-5}$, $\delta = 0.55$, $M = C_s = 5c$.

$n, k < 0$	(ℓ, j)	$H_Y = 0$	$H_Y = 0.5$	$n, k > 0$	(ℓ, j)	$H_Y = 0$	$H_Y = 0.5$
$0, -1$	$0S1/2$	2.500000000	0.252500000	$0, 1$	$0P1/2$	4.163800037	4.208004496
$1, -1$	$1S1/2$	3.763926359	3.952507983	$1, 1$	$1P1/2$	4.308216912	4.051227891
$2, -1$	$2S1/2$	4.382088460	4.376113753	$2, 1$	$2P1/2$	4.264914944	3.803863139
$3, -1$	$3S1/2$	4.448429075	4.408894175	$3, 1$	$3P1/2$	4.127441859	3.420268543
$0, -2$	$0P3/2$	4.163800037	3.497053681	$0, 2$	$0d3/2$	3.634304075	2.345202112
$1, -2$	$1P3/2$	4.308216912	4.178390788	$1, 2$	$1d3/2$	3.228983337	2.348405045
$2, -2$	$2P3/2$	4.264914944	4.272866598	$2, 2$	$2d3/2$	2.340466926	2.350794726
$3, -2$	$3P3/2$	4.127441859	4.208004496	$3, 2$	$3d3/2$	2.345076923	2.352624740
$0, -3$	$0d5/2$	3.634304075	3.928702150	$0, 3$	$0f5/2$	2.346967071	2.353762710
$1, -3$	$1d5/2$	3.228983337	3.784673189	$1, 3$	$1f5/2$	2.349261448	2.354745540
$2, -3$	$2d5/2$	2.340466926	3.512579759	$2, 3$	$2f5/2$	2.351082803	2.355571442
$3, -3$	$3d5/2$	2.345076923	3.008336744	$3, 3$	$3f5/2$	2.352552719	2.356272122
$0, -4$	$0f7/2$	2.346967071	2.335588128				
$1, -4$	$1f7/2$	2.349261448	2.340571217				
$2, -4$	$2f7/2$	2.351082803	2.344289059				
$3, -4$	$3f7/2$	2.352552719	2.347136175				

Table 7. Energy of the spin symmetry for the Coulomb potential with a Coulomb-like tensor interaction with $b = 2$, $c = 0$, $a = 0 \times 10^{-5}$, $\delta = 0.55$, $M = C_s = 2b + 1$.

$n, k < 0$	(ℓ, j)	$H_C = 0$	$H_C = 0.5$	$n, k > 0$	(ℓ, j)	$H_C = 0$	$H_C = 0.5$
$0, -1$	$0S1/2$	2.500000000	0.183416084	$0, 1$	$0P1/2$	4.540067872	4.999853717
$1, -1$	$1S1/2$	2.761306300	1.907522146	$1, 1$	$1P1/2$	4.918972850	4.953381998
$2, -1$	$2S1/2$	4.384809460	4.114209258	$2, 1$	$2P1/2$	4.999968789	4.823907866
$3, -1$	$3S1/2$	4.869196387	4.787110513	$3, 1$	$3P1/2$	4.950526310	4.620280417
$0, -2$	$0P3/2$	4.540067872	2.224519290	$0, 2$	$0d3/2$	4.726342943	3.530408853
$1, -2$	$1P3/2$	4.918972850	4.221286256	$1, 2$	$1d3/2$	4.484823223	2.771161353
$2, -2$	$2P3/2$	4.999968789	4.823477471	$2, 2$	$2d3/2$	4.137513066	2.771836461
$3, -2$	$3P3/2$	4.950526310	4.988073817	$3, 2$	$3d3/2$	3.563847326	2.772348315
$0, -3$	$0d5/2$	4.726342943	4.999853717	$0, 3$	$0f5/2$	2.778620690	2.781248432
$1, -3$	$1d5/2$	4.484823223	4.953381998	$1, 3$	$1f5/2$	2.778088235	2.780527192
$2, -3$	$2d5/2$	4.137513066	4.823907866	$2, 3$	$2f5/2$	2.777664975	2.779923868
$3, -3$	$3d5/2$	3.563847326	4.620280417	$3, 3$	$3f5/2$	2.777323009	2.779414111
$0, -4$	$0f7/2$	2.778620690	3.530408853				
$1, -4$	$1f7/2$	2.778088235	2.771161353				
$2, -4$	$2f7/2$	2.777664975	2.771836461				
$3, -4$	$3f7/2$	2.777323009	2.772348315				

For the first excited state,

$$E(\rho) = 8\pi \left[\frac{\lambda\Gamma(2\lambda+2)\Gamma(2\lambda+2\eta+2)}{\Gamma(2\lambda+1)^2\Gamma(2\eta+2)} \times \frac{\Gamma(2\lambda+2)\Gamma(2\eta+2\lambda+2)}{(2\lambda-1)\Gamma(2\eta+2\lambda+1)} \right]^2, \quad (57)$$

and for the second excited state,

$$E(\rho) = 1.571 \left[\frac{\lambda\Gamma(2\lambda+3)\Gamma(2\lambda+2\eta+3)}{\Gamma(2\lambda+1)^2\Gamma(2\eta+3)} \times \frac{\Gamma(2\lambda+3)\Gamma(2\eta+2\lambda+3)}{(2\lambda-1)\Gamma(2\eta+2\lambda+2)} \right]^2. \quad (58)$$

Table 8. Energy of the pseudospin symmetry for the Hellmann potential with a Hellmann like tensor interaction with $b = 2$, $c = -1$, $a = 0 \times 10^{-5}$, $\delta = 0.55$, $M = -C_s = 2b - c$.

$n, k < 0$	(ℓ, j)	$H_C = H_Y = 0$	$H_C = H_Y = 0.25$	$n, k > 0$	(ℓ, j)	$H_C = H_Y = 0$	$H_C = H_Y = 0.25$
1, −1	$1S1/2$	−8.057003775	−6.530900800	1, 2	$0d3/2$	−8.057003775	−8.693980106
1, −2	$1P3/2$	−8.812992666	−8.628057889	1, 3	$0f5/2$	−8.812992666	−8.692084514
1, −3	$1d5/2$	−8.276266742	−8.639126154	1, 4	$0g7/2$	−8.276266742	−7.581622032
1, −4	$1f7/2$	−6.268223546	−7.515944046	1, 5	$0h9/2$	−6.268223546	−2.883583613
2, −1	$2S1/2$	−8.543555197	−7.889381976	2, 2	$1d3/2$	−8.543555197	−8.825294931
2, −2	$2P3/2$	−8.807561507	−8.772394570	2, 3	$1f5/2$	−8.807561507	−8.608942046
2, −3	$2d5/2$	−8.128650687	−8.558673640	2, 4	$1g7/2$	−8.128650687	−7.350588730
2, −4	$2f7/2$	−5.870007125	−7.283408611	3, 5	$1h9/2$	−5.870007125	−2.067990858

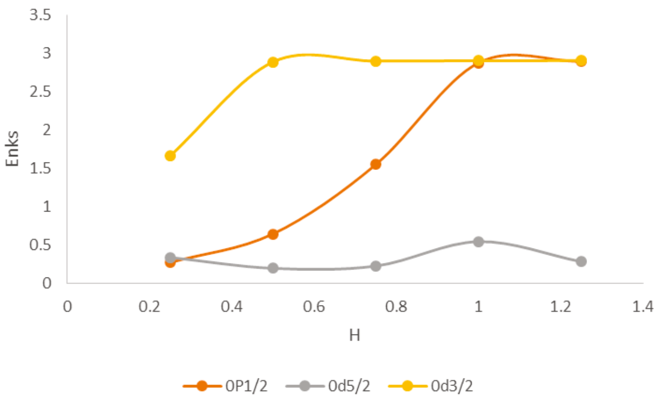

Fig. 1. (Color online) Energy of the spin symmetry E_{nks} for $0p_{1/2}$, $0d_{3/2}$ and $0d_{5/2}$with the Hellmann potential.

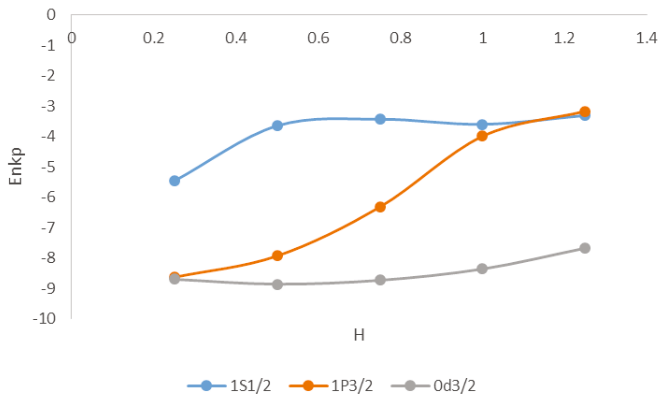

Fig. 2. (Color online) Energy of the pseudospin symmetry E_{nkp} for $1s_{1/2}$, $1p_{3/2}$ and $0d_{3/2}$with the Hellmann potential.

V. DISCUSSION

In Tables 1 and 2, we presented the energy eigenvalues of the spin symmetry for the class of Yukawa potentials in the absence and presence of a Hellmann-like tensor potential, respectively. In the absence of the Hellmann like tensor potential, the following energy degeneracies occur $0p_{3/2} = 0p_{1/2}$, $1p_{3/2} = 1p_{1/2}$, $2p_{3/2} = 2p_{1/2}$, $3p_{3/2} = 3p_{1/2}$, $0d_{5/2} = 0d_{3/2}$, $1d_{5/2} = 1d_{3/2}$, $2d_{5/2} = 2d_{3/2}$, $3d_{5/2} = 3d_{3/2}$, $0f_{5/2} = 0f_{7/2}$, $1f_{5/2} = 1f_{7/2}$, $2f_{5/2} = 2f_{7/2}$, and $3f_{5/2} = 3f_{7/2}$. In the presence of the Hellmann-like tensor potential, these degeneracies are completely removed.

In Figs. 1 and 2, we plot energy eigenvalue of the spin symmetry and pseudospin symmetry respectively of some states against the Hellmann-like tensor potential.

In Tables 3 and 4, we present the energy eigenvalues of the pseudospin symmetry for the class of Yukawa potential in the presence and absence of a Hellmann like tensor potential respectively. In the absence of the Hellmann-like tensor potential, the usual energy degeneracies $1s_{1/2} = 0d_{3/2}$, $1p_{3/2} = 0f_{5/2}$, $1d_{5/2} = 0g_{7/2}$, $1f_{7/2} = 0h_{9/2}$, $2s_{1/2} = 1d_{3/2}$, $2p_{1/2} = 1f_{5/2}$, $2d_{5/2} = 1g_{7/2}$, $2f_{7/2} = 1h_{9/2}$,occur. In the presence of the Hellmann-like tensor potential, the all degeneracies disappeared. In Table 5, we present the energy eigenvalues of the spin symmetry for the Hellmann potentials in the absence and presence of a Hellmann-like tensor potential. This is achieved by putting $a = 0$. The effects observed in Tables 1 and 2 are also observed. In Table 6, the energy eigenvalues of the spin symmetry for the Yukawa potential in the absence and presence of a Yukawa-like tensor potential are presented. This is done by putting $a = b = H_C = 0$. In the absence of the Yukawa-like tensor potential, the usual degeneracies for the spin symmetry occur. In the presence of the Yukawa-like tensor potential, the only degeneracy that occurs is $3p_{3/2} = 0p_{1/2}$. In Table 7, we presented the energy of the spin symmetry for the Coulomb potential in the absence and presence of Coulomb like tensor potential. This is done when $a = c = H_Y = 0$. In the absence of the Coulomb like tensor potential, the usual degeneracies occurred. In the presence of the Coulomb like tensor potential, the following degeneracies are observed: $1p_{3/2} = 0d_{3/2}$, $1d_{5/2} = 0f_{5/2}$, $1f_{7/2} = 0g_{7/2}$, $2p_{3/2} = 1d_{3/2}$, $2d_{5/2} = 1f_{5/2}$, $2f_{7/2} = 1g_{7/2}$. In Table 8, we presented energy eigenvalue of the pseudospin symmetry for Hellmann potential in the absence and the presence of Hellmann-like tensor potential. In the absence of tensor term, the usual degeneracies occurred

Table 9. Energy of the pseudospin symmetry for the Yukawa potential with a Yukawa-like tensor interaction with $b = 0$, $c = 1$, $a = 0 \times 10^{-5}$, $\delta = 0.55$, $M = -C_s = 5c$.

$n, k < 0$	(ℓ, j)	$H_Y = 0$	$H_Y = 0.5$	$n, k > 0$	(ℓ, j)	$H_Y = 0$	$H_Y = 0.5$
1, −1	$1S1/2$	−9.971468114	−9.891053522	1, 2	$0d3/2$	−9.971468114	−9.973929323
1, −2	$1P3/2$	−9.857189064	−9.953227672	1, 3	$0f5/2$	−9.857189064	−9.561563088
1, −3	$1d5/2$	−9.145279786	−9.500737136	1, 4	$0g7/2$	−9.145279786	−8.343707574
1, −4	$1f7/2$	−7.173189514	−8.236147929	1, 5	$0h9/2$	−7.173189514	−3.951261585
2, −1	$1S1/2$	−9.999989585	−9.987590496	2, 2	$1d3/2$	−9.999989585	−9.928313466
2, −2	$2P3/2$	−9.782062744	−9.901316976	2, 3	$1f5/2$	−9.782062744	−9.444404068
2, −3	$2d5/2$	−8.986644891	−9.381540467	2, 4	$1g7/2$	−8.986644891	−8.103467153
2, −4	$2f7/2$	−6.799205331	−7.990988074	3, 5	$1h9/2$	−6.799205331	−2.640407026

Table 10. Energy of the pseudospin symmetry for the Coulomb potential with a Coulomb-like tensor interaction with $b = 2$, $c = 0$, $a = 0 \times 10^{-5}$, $\delta = 0.55$, $M = -C_s = 2b + 1$.

$n, k < 0$	(ℓ, j)	$H_C = 0$	$H_C = 0.5$	$n, k > 0$	(ℓ, j)	$H_C = 0$	$H_C = 0.5$
1, −1	1S1/2	−8.866870907	−7.956416623	1, 2	$0d3/2$	−8.866870907	−9.158621729
1, −2	1P3/2	−9.150102641	−9.158621729	1, 3	$0f5/2$	−9.150102641	−8.939460167
1, −3	1d5/2	−8.512239820	−8.939460167	1, 4	$0g7/2$	−8.512239820	−7.777325100
1, −4	1f7/2	−6.496842829	−7.777325100	1, 5	$0h9/2$	−6.496842829	−3.180340169
2, −1	1S1/2	−9.123519000	−8.819867380	2, 2	$1d3/2$	−9.123519000	−9.204554967
2, −2	2P3/2	−9.109353093	−9.204554967	2, 3	$1f5/2$	−9.109353093	−8.842613476
2, −3	2d5/2	−8.358486290	−8.842613476	2, 4	$1g7/2$	−8.358486290	−7.544231131
2, −4	2f7/2	−6.103779103	−7.544231131	3, 5	$1h9/2$	−6.103779103	−2.212396424

while in the absence of tensor term, there are no degeneracies occur. In Table 9, we presented the energy eigenvalu of the pseudospin symmetry for the Yukawa potential in the absence and presence of Yukawa-like tensor term. The usual degeneracies occur in the absence of Yukawa-like tensor term. The degeneracies disappeared in the presence of Yukawa-like tensor term. In Table 10, we present energy eigenvalue of the pseudospin symmetry for the Coulomb potential in the absence and presence of Coulomb-like tensor potential. In the absence of the Coulomb-like tensor potential, the usual degeneracies for the pseudospin symmetry occur, but in the presence of the Coulomb-like tensor potential, the following degeneracies occur: $2p_{3/2} = 1d_{3/2}$, $2d_{5/2} = 1f_{5/2}$, $2f_{7/2} = 1g_{7/2}$.

VI. CONCLUSION

In this study, we have shown the analytical expression for the bound states energy of the Dirac equation under spin and pseudospin symmetries with a new tensor interaction for a class of the Yukawa potential in the framework of supersymmetry quantum mechanics. The effect of tensor potential which removes the energy degeneracy doublets in both the spin and pseudospin symmetries are studied in detail. We note that the Hellmann-like tensor potential removes the whole degeneracies from both the spin and pseudospin symmetries while the Yukawa-like tensor potential removes all degeneracies in the pseudospin symmetry but a degeneracy doublet in the spin symmetry. However, the popularly used Coulomb-like tensor potential still have some energy degeneracies in both the spin and the pseudospin symmetries. Thus, the inclusion of a Hellmann-like tensor interaction is better for atomic stability because the removal of the degeneracy results in atomic stability. Finally, the analytical expression for the Onicescu information energies for the ground state and the excited states has also been provided. The result obtained so far can be applied in the study of atoms and molecules because of the stabilizing force of the Yukawa potential model.

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