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Principles in Quantum-Wiggler Electrodynamics and Analysis of the Smith-Purcell Radiation Based on These Principles

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The principles of quantum-wiggler electrodynamics (QWD) are explained. Based on QWD, we confirm that in the Smith-Purcell configuration, the power of the radiation driven by the wiggling force acting in the beam's direction ('free-electron two-quantum Stark (FETQS) radiation') is independent of the transverse wiggling whereas the power of the radiation driven by the wiggling force acting in the direction perpendicular to the beam's direction ('free-electron two-quantum magnetic-wiggler radiation') is proportional to the square of the transverse wiggling velocity, $\tilde{v}_\perp^2$. Because $\tilde{v}_\perp \propto 1/\gamma$, we find that the ratio of the radiation power of a free-electron radiation device using a magnetic wiggler to that using an electric wiggler scales as $1/\gamma^2$.

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I. INTRODUCTION

In this paper, the radiation power from an electron, P, is the measured radiation power per electron averaged over the measurement time; P is equal to the radiation power from an electron as seen by an observer in the electron's frame [1]. Quantum-wiggler electrodynamics [1–4] is different from conventional quantum electrodynamics in the following aspects:

(1) Any negative-energy state is excluded from any consideration.

(2) The transition-probability amplitude through one route is of random phase with respect to that through any other route. Accordingly, the transition rate from one state to another state is the simple arithmetic sum of the transition rates, each of which represents the transition rate that is appropriate in the absence of any other route.

(3) The transition from a virtual state to another virtual state is strictly forbidden. Here, a virtual state means a state at which the electron's momentum and energy do not satisfy $E = (m^2c^4 + c^2p^2)^{1/2}$. Accordingly, the transition should be completed only through a one-quantum transition or a two-quantum transition.

(4) Unlike the bound-electron energy-level system, in the free-electron energy-level system, for any energy level with energy E_o, there are certainly two energy levels: one with energy $E_h = E_o + h\nu$ and the other with energy $E_l = E_o - h\nu$. The probability for the electron at the level with energy E_o to make a transition to the level with energy E_h through absorption stimulated by incident radiation of frequency ν is exactly equal to that to make a transition to the level with energy E_l through emission stimulated by the same incident radiation. Hence, regardless of whether the population gradient with respect to the electron energy is positive or negative, net stimulated radiation does not take place. Thus, the incident radiation plays the role of a dummy field in emission process. However, the incident radiation can produce an electric wiggler in conjunction with a magnetic wiggler if the latter field is present, and its potential is comparable to or larger than the electron's kinetic energy. The thusly produced electric wiggler drives free-electron two-quantum Stark (FETQS) radiation [5, 6] in cooperation with the electron's intrinsic motivity to change the electron's internal configuration by emitting a photon. Similarly, if both a magnetic wiggler and a uniform magnetic field which is far stronger than the magnetic wiggler are concurrently present, the magnetic wiggler drives free-electron two-quantum magnetic-wiggler radiation [2]. The transition rate through such two-quantum radiation is hundreds of thousands, millions, billions and so on times greater than the transition rate through the one-quantum process in which the foregoing electron's motivity acts only as a first-order perturber. Hence, all radiation phenomena observed in a periodic device in which a wiggler whose potential amplitude is much smaller than the electron's kinetic energy or that of any other present field that can act as a zeroth-order per-

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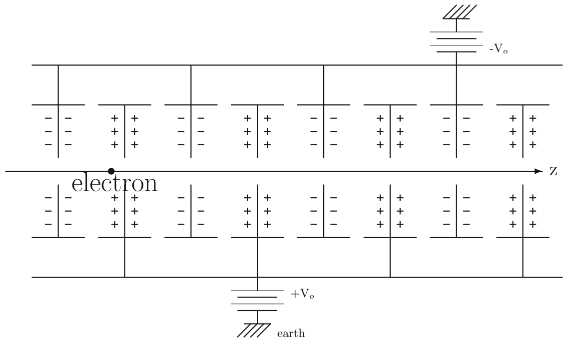

Fig. 1. Schematic showing a free electron traveling in an electric wiggler.

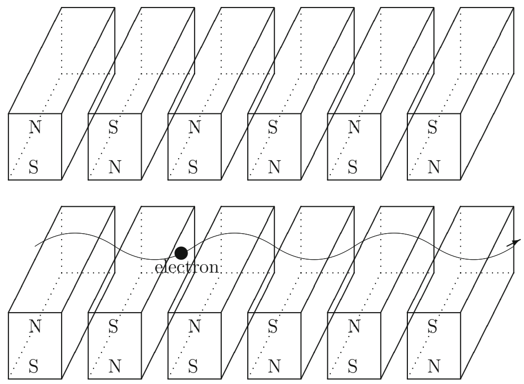

Fig. 2. Schematic showing a free electron traveling in a magnetic.

turber are attributed to two-quantum radiation.

(5) The uncertainty in the dimension of the space in which a photon is produced is independent of the direction from which the space is viewed; *i.e.*, that space is isotropic. Because the z-size of the above space, which is herein called the photon-production space, is on the order of the wiggler period λ_w, the volume of the space is approximately $\pi\lambda_w^3$.

(6) The measured radiation power per electron, P, depends on the radiation frequency ν, the transition rate $\tau = 1/t_{tr}$, and the uncertainty in the electron's energy ΔE [2, 3]. The radiation power in the so-called Smith-Purcell (SM) configuration [7, 8] is given by either $P = h\nu\tau$ if $h\nu \gg h\tau \gg \Delta E$ or $P = h\nu^2$ if $h\tau \gg h\nu \gg \Delta E$.

The above six QWD postulates manifest that as far as free-electron radiation is concerned, any other electrodynamics, *i.e.*, conventional quantum electrodynamics (CQD) or classical electrodynamics (CED) is preposterously wrong, or does not proceed to any sensible conclusion. For example, if the (2) postulates is negated as is implied in CQD, the probability amplitude from one route coherently subtracts that from the other route so that the resulting probability amplitude is extremely small and hence, any two-quantum radiation does not occur to any measurable magnitude [6]. Another example is as follows. The virtual level can significantly contribute to the transition rate only when its energy level is very near to that of a real level [4]. In the bound (atomic) electron system, the energy level is discrete and hence, only a limited numer of the vertual levels must be taken into account. In contrast to the bound electron system, the energy level of the free-electron system is continuous so that there is a real level any near to any virtual level. Accordingly, if the (3) postulate is not adopted, we must consider an infinite number of routes and hence, we can only conclude that free-electron radiation is an incomprehensible phenomenon beyond the human reasoning ability in quantum mechanical approach.

Both the electric wiggler (Cf. Fig. 1) and the magnetic wiggler (Cf. Fig. 2) are spatially periodic fields whose potential amplitudes are much smaller than the electron kinetic energy. The force of magnetic wiggler is acting in the transverse direction while that of electric wiggler in the axial direction. The so-called Smith-Purcell (SM) radiation is the radiation driven by both a magnetic wiggler and an electric wiggler which are concurrently acting on the beam electrons [7,8]. The purpose of this paper is to find whether QWD can adequately explain the SM radiation and if possible, to find the γ-scaling laws of the radiations driven by the electric wiggler and the magnetic wiggler.

II. QUANTUM-WIGGLER ELECTRODYNAMIC EQUATION

The electron spin's state, *i.e.*, either the up ($\uparrow$) or down ($\downarrow$) state, is conserved in a radiative transition and any transition driven by any electric or magnetic wiggler [9]. We find that $\langle\Psi(\boldsymbol{r},t,\boldsymbol{p},\uparrow)|\boldsymbol{\alpha}|\Psi(\boldsymbol{r},t,\boldsymbol{p},\uparrow)\rangle = \boldsymbol{\beta}$, where $\boldsymbol{\beta} = \boldsymbol{v}/c$. From these facts and the foregoing QWD concepts, the QWD equation describing the motion of an electron traveling in a magnetic wiggler (MW) and an electric wiggler (EW) as performing spontaneous emission of photons of wave vector $\boldsymbol{k}$ polarized in the direction $\hat{\boldsymbol{\varepsilon}}_\xi$ ($\xi = 1, 2$) can be written as

$$i\hbar\frac{\partial\Psi(\boldsymbol{r},t|\boldsymbol{p})}{\partial t} = \Big\{\boldsymbol{\beta}\cdot\big[c\boldsymbol{p}-e\big(\boldsymbol{A}_w(\boldsymbol{r})+\boldsymbol{A}_{spon}(\boldsymbol{r},t|\boldsymbol{k},\xi)\big)\big]+\beta_4 mc^2+e\phi(\boldsymbol{r})\Big\} \times \Psi(\boldsymbol{r},t|\boldsymbol{p}), \tag{1}$$

where $\beta_z = 1$,

$$\boldsymbol{\beta}_\perp = \frac{\boldsymbol{v}_\perp}{c} \approx \frac{\boldsymbol{\Gamma}_w}{\gamma}, \quad \boldsymbol{\Gamma}_w = \frac{e\boldsymbol{A}_w(\boldsymbol{r})}{mc^2} \tag{2}$$

[Cf. Eq. (43) of Ref. 10],
$\beta_4 = \langle \Psi(\boldsymbol{r}, t, \boldsymbol{p}, \uparrow) | \alpha_4 | \Psi(\boldsymbol{r}, t, \boldsymbol{p}, \uparrow) \rangle = mc^2/E$, with

$$\alpha_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (3)$$

Here,

$$\boldsymbol{A}_{spon}(\boldsymbol{r}, t|\boldsymbol{k}, \xi) = \hat{\boldsymbol{\varepsilon}}(\boldsymbol{k}, \xi) \left(\frac{2\pi\hbar c}{kV_{ph}} \right)^{1/2} \exp[-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)], \quad (4)$$

where $\hat{\boldsymbol{\varepsilon}}(\boldsymbol{k}, \xi)$ is the polarization vector, $\omega = ck$, and $V_{ph} \simeq \pi\lambda_w^3$ in accordance with postulate (5), is the potential that simulates spontaneous radiation, which is driven by the electron's intrinsic motivity to change its internal configuration by emitting a photon.

In two-quantum radiation processes, the electron's intrinsic motivity is a first-order perturber. The other first-order perturber is either a magnetic wiggler or an electric wiggler. As is seen in Eq. (1), the salient difference between the electric wiggler (EW) and the magnetic wiggler (MW) is that the Hamiltonian representing the interaction of the electron with a magnetic wiggler of potential $\boldsymbol{A}_w$ is proportional to $\boldsymbol{\beta} \cdot \boldsymbol{A}_w$ while that representing the interaction of the electron with the electric wiggler of potential ϕ is independent of $\boldsymbol{\beta}$ so that the ratio of the radiation power from an electron using a magnetic wiggler as the first-order perturber to that from an electron using an electric wiggler as the first-order perturber can be written as

$$\frac{P^{\mathrm{MW}}}{P^{\mathrm{EW}}} = \frac{(\boldsymbol{\beta} \cdot \boldsymbol{A}_o^{\mathrm{MW}})^2}{(\phi_o^{\mathrm{EW}})^2}, \quad (5)$$

where the subscript 'o' denotes the amplitude.

We find from Eqs. (2) and (5) that in a radiation device using a magnetic wiggler, the radiation power satisfies

$$P^{\mathrm{MW}} \propto (\frac{\Gamma_w}{\gamma})^2, \quad \Gamma_w \approx \frac{F_o^w \lambda_w}{2\pi mc^2}, \quad (6)$$

where F_o^w is the amplitude of the wiggling force acting in the perpendicular direction and λ_w is the distance between rulings. The wavelength scales as $1/\gamma^2$ as the radiation power does so. Hence, using a magnetic wiggler in order to built a free-electron device emitting short-wavelength radiation while its power is as large as possible is unreasonable. In this paper, we confirm the validity of Eq. (5) by investigating the radiation of the first SM configuration [7], which can be conceived as a device in which both a magnetic wiggler and an electric wiggler simultaneously act on a beam electron.

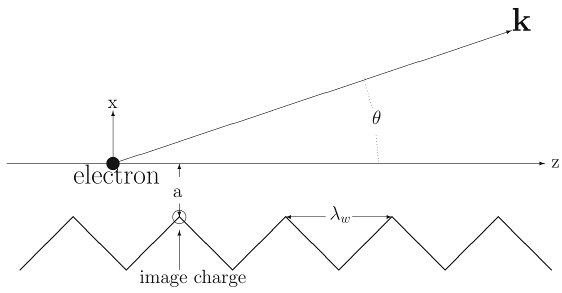

Fig. 3. Schematic describing Smith-Purcell radiation.

III. SMITH-PURCELL RADIATION

In the first SP configuration [7], the electron's kinetic energy is 300 keV, and the largest potential energy is $e^2/a = 1.4 \times 10^{-6}$ keV, where a is the closest distance between the electron and its image charge as shown in Fig. 3. Smith and Purcell assumed $a = \lambda_w/10$ [7], which we adopt. Hence, the electron just travels on a straight path when we average over the transverse wiggling. The potential of the Coulomb force acting on an electron from its image charge has the largest magnitude e/a at $z = N\lambda_w$. Hence, the field acting on the electron can be expressed by using the following Fourier series:

$$\boldsymbol{A}(z, t) = [A_o^w \cos(k_w z) + \sum_l A_l^w \cos(lk_w z)]\hat{\mathbf{x}}, \quad (7)$$

where $A_o^w = e/2a$, $k_w = 2\pi/\lambda_w$, and $l = 1, 2, 3\cdots$ is the order of harmonics. We do not have any knowledge about the detailed geometric shape of the grating and the image charge distribution, which would allow us to estimate with sufficiently accurately A_l^w, $l = 2, 3, 4\cdots$. Hence, in this paper, we only calculate the fundamental harmonics approximately. The Hamiltonian representing the radiative interaction exerted by the force perpendicular to the electron's path ('transverse force') from the image charge on the electron is written as

$$H^w(z) = H_o^w \exp(-ik_w z) + S(-k_w), \quad (8)$$

where $H_o^w = -e\beta_x A_o^w/2$ with $\beta_x = v_x/c$. $S(-k_w)$ is of the same form as the preceding term except that k_w is replaced by $-k_w$.

Because $v_y = 0$, the wavevector and the polarization of the radiation can be written as

$$\begin{aligned} &\hat{\mathbf{k}}(\theta) = \boldsymbol{k}(\theta)/k = \sin\theta\, \hat{\mathbf{x}} + \cos\theta\, \hat{\mathbf{y}}, \\ &\hat{\boldsymbol{\varepsilon}}(\theta, 1) = \hat{\mathbf{y}}, \\ &\hat{\boldsymbol{\varepsilon}}(\theta, 2) = -\cos\theta\, \hat{\mathbf{x}} + \sin\theta\, \hat{\mathbf{z}}. \end{aligned} \quad (9)$$

From Eq. (1), we find that the interaction Hamiltonian simulating the work done by the electron's intrinsic motivity to change its internal configuration by spontaneously emitting a photon with wavevector $\boldsymbol{k}$ and with polarization in the ξ direction, as denoted in Eq. (9), can be written as

$$H^{spon}(\boldsymbol{r},t|\boldsymbol{k},\xi) = -e[\boldsymbol{\beta}\cdot\hat{\boldsymbol{\varepsilon}}(\theta,\xi)]\left(\frac{2\pi\hbar c}{kV_{ph}}\right)^{1/2}\exp[-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]. \quad (10)$$

Because $\tilde{\beta}_y = \tilde{\beta}_z = 0$, $\boldsymbol{\beta}\cdot\hat{\boldsymbol{\varepsilon}}(\theta,1) = 0$ and $\boldsymbol{\beta}\cdot\hat{\boldsymbol{\varepsilon}}(\theta,2) = -\tilde{\beta}_x\cos\theta+\sin\theta$, radiation polarized in the $\xi = 1$ direction (*i.e.*, y-direction) does not take place, and the radiation power from an electron is proportional to $(\tilde{\beta}_x\cos\theta+\sin\theta)^2$. Accordingly, we approximate $\boldsymbol{\beta}\cdot\hat{\boldsymbol{\varepsilon}}(\theta,2)$ to be equal to $\sin\theta$ at θ, which is practically distinguishable from zero. This fact that the power of the radiation emitted into per unit solid radian increases with $\sin^2\theta$ is perspicuously observed in the SP experiment [7].

The wavefunction representing the unperturbed momentum state can be written as

$$\Psi(\boldsymbol{r},t;\boldsymbol{p}) = \frac{1}{\sqrt{V_i}}\exp\left[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Et)\right], \quad (11)$$

where V_i is the volume of the space in which the electron can be found during the interaction. The probability amplitude for a transition from state 1 with momentum $\boldsymbol{p}_1$ to state 2 with momentum $\boldsymbol{p}_2$ through the emission of a photon of wavevector $\boldsymbol{k}$ during a time T can be written as

$$\begin{aligned}\mathcal{A}(\boldsymbol{p}_1\to\boldsymbol{p}_2|\boldsymbol{k}) &= \frac{1}{(i\hbar)^2}\sum_i\Bigg(\exp(i\Lambda_i^I)\int_0^T dt\langle\Psi(\boldsymbol{r},t|\boldsymbol{p}_2)|H^{spon}(\boldsymbol{r},t|\boldsymbol{k},2)|\Psi(\boldsymbol{r},t|\boldsymbol{p}_i)\rangle\int_0^t dt'\langle\Psi(\boldsymbol{r},t'|\boldsymbol{p}_i)|H^w(z)|\Psi(\boldsymbol{r},t'|\boldsymbol{p}_1\rangle \\ &\quad+\exp(i\Lambda_i^{II})\int_0^T dt\langle\Psi(\boldsymbol{r},t|\boldsymbol{p}_2)|H^{spon}(\boldsymbol{r},t|\boldsymbol{k},2)|\Psi(\boldsymbol{r},\boldsymbol{t}|\boldsymbol{p}_i)\rangle\int_0^t dt'\langle\Psi(\boldsymbol{r},t'|\boldsymbol{p}_i)|H^w(z)|\Psi(\boldsymbol{r},t'|\boldsymbol{p}_1)\rangle\Bigg) \\ &= \frac{\alpha e\beta_x\sin\theta}{4i\hbar a\beta_z k_w}\left(\frac{2\pi\hbar c}{kV_{ph}}\right)^{1/2}[\exp(i\Lambda_1)-\exp(i\Lambda_2)]\delta_{\boldsymbol{p}_2,\boldsymbol{p}_1-\hbar(\boldsymbol{k}+k_w\hat{\boldsymbol{z}})}\int_0^T\exp\Big(ic[k(1-\beta_z\cos\theta-\beta_x\sin\theta)-\beta_z k_w]t\Big)dt,\end{aligned} \quad (12)$$

where $\alpha = e^2/c\hbar = 1/137$ is the fine-structure constant. Here, we have used, in advance, $k(1-\beta_z\cos\theta-\beta_x\sin\theta)-\beta_z k_w = 0$, which expresses momentum and energy conservation up to the first-order in the $\hbar$ series expansion. The rate of transition by spontaneous emission of the wavevector $\boldsymbol{k}$ can be written as

$$\tau(\boldsymbol{p}_1\to\boldsymbol{p}_2|\boldsymbol{k}) = \frac{c\alpha^3\beta_x^2\sin^2\theta}{16\pi^2a^2}\delta_{\boldsymbol{p}_2,\boldsymbol{p}_1-\hbar(\boldsymbol{k}+k_w\hat{\boldsymbol{z}})}\delta[k-k_w/(1-\beta_z\cos\theta)]. \quad (13)$$

The argument of the Dirac delta function in the above equation shows $\lambda = \lambda_w(\beta^{-1}-\cos\theta)$, which is called the Smith-Purcell formula. Because the wavevector is given by $k_x = 2\pi n_x/\lambda_w$, $k_y = 2\pi n_y/\lambda_w$, $k_z = 2\pi n_z/\lambda_w$, where n_x, n_y, and n_z are all integers, the number of wavevector states between k and $k+dk$ and θ and $\theta+d\theta$ is $\lambda_w^3k^2dk\sin\theta\,d\theta/(2\pi)^2$. With this fact, the rate of the transition accompanied by the emission of a photon from an electron with momentum $\boldsymbol{p}$ is given by

$$\begin{aligned}\tau(\boldsymbol{p}) &= \sum_{n_x,n_y,n_z}\tau(\boldsymbol{p}\to\boldsymbol{p}_2|\boldsymbol{k}) \\ &= \int_0^{\pi/2}\sin\theta d\theta\int k^2dk\frac{\lambda_w^3\tau(\boldsymbol{p}\to\boldsymbol{p}_2|\boldsymbol{k})}{(2\pi)^2} \\ &= \frac{c\alpha^3\beta_x^2(\lambda_w/d)^2}{16\pi^2\lambda_w}\int_0^{\pi/2}\frac{\sin^3\theta\,d\theta}{(1-\beta_z\cos\theta)^2} \\ &\sim \frac{180c\alpha^3\beta_x^2}{16\pi^2\lambda_w},\end{aligned} \quad (14)$$

which is 2.3×10^{-9}/sec in the SP configuration. Here, $\int_0^{\pi/2}\sin^3\theta\,d\theta/(1-\beta_z\cos\theta)^2 = -[2\beta_z+\beta_z^2+2\ln(1-\beta_z)]/\beta_z^3 = 1.8$ has been used.

The frequency of the axial radiation, which can be written as

$$\omega = ck_w/(1-\beta_z), \quad (15)$$

is 5.1×10^{15} radian/sec. Because $\omega \gg \tau$, the power from an electron is given by [2]

$$P^{\mathrm{MW}} = \hbar\omega\tau. \quad (16)$$

Unlike in the quantum-mechanical concept, in the classical concept, the radiation power from an electron, which is formulated as the Larmor formula, varies with the local position inside of one wiggling period. If the image charge is assumed to be constantly located at the summit of the hill on a time scale of λ_w/v_z, as shown in Fig. 1, the radiation power from an electron at position z can be written as

$$P^{\rm Larmor}(z) = \frac{2}{3}\frac{e^2\dot{v}(z)^2}{c^3}, \tag{17}$$

where $\dot{v} = v^2/R$, with R being the radius of curvature. If R is assumed to be equal to the distance between the electron and its image charge, the SP radiation power from an electron in the classical concept is given by

$$P^{cl} \approx \frac{1}{5a}\int_0^{5a} P^{\rm Larmor}(z)dz \approx \frac{0.1ce^2r_e^2}{a^4}, \tag{18}$$

where a is the distance of the grating summit from the electron trajectory. Accordingly, we find

$$\frac{P^{\rm MW}}{P^{cl}} = 3.4 \times 10^{-5}. \tag{19}$$

A recognized fact is that Smith and Purcell found their measured radiation power to be 10^{14} times larger than their classically-calculated radiation power. Accepting this fact leads us to hold a strong conviction that the SP radiation is not free-electron magnetic-wiggler radiation and to conjecture that the SP radiation is FETQS radiation [5,6].

From $-\frac{\partial\phi(z)}{\partial z} = \frac{F_z}{e} = \frac{ez}{(a^2+z^2)^{3/2}}$, we find that $\phi(z) = \frac{e}{(a^2+z^2)^{1/2}}$. Hence, we can approximate the potential of the electric wiggler as

$$\phi(z) = \frac{e\cos(k_m z)}{a}. \tag{20}$$

Then, we find that the ratio R of the power of FETQS radiation due to the electric wiggler to the radiation power calculated with the Larmor formula in the first SP configuration is given by

$$\begin{aligned} R &= \frac{P^{\rm EW}}{P^{cl}} = \frac{P^{\rm EW}}{P^{\rm MW}}\frac{P^{\rm MW}}{P^{cl}} = (\frac{\phi_o}{A_o^w}\frac{1}{\beta_\perp^w})^2\frac{P^{\rm MW}}{P^{cl}} \\ &\approx 2.4 \times 10^{12}, \end{aligned} \tag{21}$$

which can be considered to be in good agreement with the foregoing Smith-Purcell estimate.

IV. CONCLUSION

We find that a feature conspicuously implied by QWD based on the Dirac equation: the power of the radiation driven by the axial force is proportional to ϕ_o^2 while that of the radiation driven by the transverse force is proportional to $(\tilde{\beta}_\perp A_o^w)^2$, is undoubtedly manifested in the original SP radiation. Accordingly, the ratio of the radiation power driven by the electric wiggler to that driven by the magnetic wiggler scales as γ^2.

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