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Stochastic Resonance for a Linear Oscillator with Two Kinds of Fractional Derivatives and Random Frequency

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The stochastic resonance (SR) behavior for a linear oscillator with two kinds of fractional derivatives and random frequency is investigated. Based on linear system theory, and applying with the definition of the Gamma function and fractional derivatives, we derive the expression for the output amplitude gain (OAG). A stochastic multiresonance is found on the OAG curve versus the first kind of fractional derivative exponent. The SR occurs on the OAG as a function of the second kind of fractional exponent, as a function of the viscous damping and the friction coefficients, and as a function of the system's frequency. The bona fide SR also takes place on the OAG curve versus the driving frequency.

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I. INTRODUCTION

A stochastic resonance (SR) is a nonlinear phenomenon induced by the cooperative effect between the signal and the noise in a nonlinear system, and was first put forward by Benzi *et al.* [1], who proposed it as a plausible mechanism for the periodic occurrences of ice ages on Earth. Since its first discovery in the early eighties of the twentieth century, SR has been observed in a great variety of scientific fields such as physics, chemistry, engineering, biology and biomedical sciences [1–5]. The SR behavior has been widely studied in integer-order nonlinear and linear systems with multiplicative and additive noise [1–13]. Researchers investigated the SR in nonlinear systems by using many methods, for instance, experimentation [6], linear-response theory [7], perturbation theory [8], residence time distribution [9], *etc.*. Later, Berdichevsky and Gitterman [10] explored the existence of the SR in integer-order linear systems driven by a colored multiplicative noise. The conventional SR is thought to be a nonlinear effect that accounts for the optimum response of a dynamical system to an external force at a certain noise intensity. The SR in a broad sense means the non-monotonic behavior of the output signal as a function of some characteristics of the noise (noise intensity or noise correlation time) or as a function of a system parameter.

In recent years, investigation of SR behavior in fractional-order system has been a subject of a great attention [14–23]. For one kind of fractional-order derivative system, the stationary response of a Duffing oscillator with hardening stiffness and fractional derivative under Gaussian white noise excitation was studied [14], and the stationary probability densities of the energy envelope and the amplitude envelope were obtained. The SR phenomenon for a fractional linear oscillator with fluctuating frequency [15–18] and with random mass [19,20] was also studied. In addition, the stochastic behavior with two kinds of fractional-order derivatives was investigated. An analytical scheme to determine the statistical behavior of a stochastic system including two terms of fractional derivatives with real, arbitrary, fractional orders was proposed [21]. The constrained optimization harmonic balance method for solving the Duffing oscillator [22], and the primary resonance of the Duffing oscillator [23] with two kinds of fractional order derivative terms were also investigated [23].

However, to our knowledge, little attention has been paid to investigating the SR for a linear oscillator with two kinds of fractional-order derivatives. Thus motivated, we consider a fractional oscillator with two fractional derivative terms subjected to an external periodic

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force and multiplicative and additive noise. The structure of this paper is as follows: Sec. II presents the oscillator investigated. The formula for the output amplitude gain (OAG) of the oscillator is obtained. Sec. III analyzes the behavior of the OAG. The nonlinear phenomenon of the output signal is discussed, and finally some conclusions are presented.

II. FRACTIONAL LINEAR OSCILLATOR AND ITS OUTPUT AMPLITUDE GAIN

As a model for a fractional oscillator coupled with a noisy environment, we consider a fractional oscillator with two kinds of fractional-order derivatives and fluctuating frequency, which can be described using the following stochastic differential equation:

$$\frac{d^2x(t)}{dt^2} + 2r\frac{dx}{dt} + r_1 D^{\alpha_1}[x(t)] + r_2 D^{\alpha_2}[x(t)] + [\omega^2 + \xi(t)]x(t) = A\cos(\Omega t) + \eta(t), \tag{1}$$

where $x(t)$ is the oscillator displacement, r_1, r_2 the friction coefficients, r the viscous damping of the oscillator, and ω the system frequency. There are several definitions for fractional-order derivative [24,25]. In this paper, the α-order fractional derivative of $x(t)$ with respect to t is adopted in Caputo's sense [26,27] as $D^\alpha[x(t)] = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds$, where $x^{(n)}(s) = \frac{d^n x}{ds^n}$, $n-1 < \alpha < n$, n being a positive integer. $\Gamma(y)$ denotes the Gamma function, and the fractional exponents are α_1, $0 < \alpha_1 < 1$, and α_2, $1 < \alpha_2 < 2$. A and Ω are the amplitude and the frequency of the driving force, respectively. We comment that for the case of $r = r_2 = 0$, Eq. (1) can be simplified as the fractional oscillator with one fractional terms and fluctuating frequency studied in Refs. [15–18], and for $r_1 = r_2 = 0$, Eq. (1) can be simplified as the harmonic oscillator with the ordinary derivative subject to a random frequency investigated in Ref. [28]. The fluctuation $\xi(t)$ is a Markovian dichotomous noise and takes value a and $-a$, $a > 0$, with zero mean and correlation $\langle\xi(t_1)\xi(t_2)\rangle = D\exp(-\lambda|t_1 - t_2|)$, where D and λ are the noise intensity and the correlation rate, respectively. The additive noise $\eta(t)$ is a Gaussian noise with zero mean and coupling strength P with $\xi(t)$, $\langle\xi(t)\eta(t)\rangle = P$.

By virtue of the characteristics both of the Gamma function and the fractional-order derivative [29], Eq. (1) can be rewritten as

$$\frac{d^2x(t)}{dt^2} + 2r\frac{dx}{dt} + \frac{r_1}{\Gamma(1-\alpha_1)}\int_0^t \frac{\dot{x}(u)}{(t-u)^{\alpha_1}}du + \frac{r_2}{\Gamma(1-\beta_2)}\int_0^t \frac{\dot{x}(u)}{(t-u)^{\beta_2}}du + [\omega^2 + \xi(t)]x(t) = A\cos(\Omega t) + \eta(t),\ \beta_2 = \alpha_2 - 1. \tag{2}$$

To obtain the first moment of the system output, one can average both sides of Eq. (2), which yields

$$\frac{d^2\langle x(t)\rangle}{dt^2} + 2r\frac{d\langle x(t)\rangle}{dt} + \frac{r_1}{\Gamma(1-\alpha_1)}\int_0^t \frac{\langle\dot{x}(u)\rangle}{(t-u)^{\alpha_1}}du + \frac{r_2}{\Gamma(1-\beta_2)}\int_0^t \frac{\langle\dot{x}(u)\rangle}{(t-u)^{\beta_2}}du + \omega^2\langle x(t)\rangle + \langle\xi(t)x(t)\rangle = A\cos(\Omega t). \tag{3}$$

Multiplicating $\xi(t)$ on both sides of Eq. (2), using the characteristics of the dichotomous noise $\xi(t)$

$$\langle\xi(t)x(t)\rangle = D\langle x(t)\rangle, \tag{4}$$

and then applying the characteristics of fractional-order derivatives, one gets

$$\left\langle \xi\frac{d^2x}{dt^2}\right\rangle + 2r\frac{\langle\xi x\rangle}{dt} + \frac{r_1}{\Gamma(1-\alpha_1)}e^{-\lambda t}\int_0^t \frac{\langle\xi(u)\dot{x}(u)\rangle e^{\lambda u}}{(t-u)^{\alpha_1}}du + \frac{r_2}{\Gamma(1-\beta_2)}e^{-\lambda t}\int_0^t \frac{\langle\xi(u)\dot{x}(u)\rangle e^{\lambda u}}{(t-u)^{\beta_2}}du + \omega^2\langle x(t)\rangle + D\langle x(t)\rangle = P. \tag{5}$$

By virtue of the Shapiro-Loginov formula [30], for an exponentially-correlated noise $\xi(t)$, one finds

$$\frac{\langle\xi x\rangle}{dt} = \langle\xi\dot{x}\rangle - \lambda\langle\xi x\rangle, \tag{6}$$

$$\frac{\langle\xi\dot{x}\rangle}{dt} = \langle\xi\ddot{x}\rangle - \lambda\langle\xi\dot{x}\rangle. \tag{7}$$

Eqs. (3) and (5) - (7) form a system of linear equations for variables $x_1 = \langle x\rangle$, $x_2 = \langle\dot{x}\rangle$, $x_3 = \langle\xi x\rangle$, $x_4 = \langle\xi\dot{x}\rangle$, and $x_5 = \langle\xi\ddot{x}\rangle$. Taking the Laplace transformation (LT) of these equations, $x_i LT X_i$, $i = 1, 2, \ldots, 5$, one can obtain, in the long-time limit ($t \to \infty$), five linear equations:

$$s^2X_1 + 2rX_1 + r_1 s^{\alpha_1-1}X_2 + r_2 s^{\beta_2-1}X_2 + \omega^2 X_1 + X_3 = LT[A\cos(\Omega t)], \tag{8}$$

$$X_2 = sX_1, \tag{9}$$

$$(s+\lambda)^2X_3 + 2r(s+\lambda)X_3 + r_1(s+\lambda)^{\alpha_1-1}X_4 + r_2(s+\lambda)^{\beta_2-1}X_4 + \omega^2X_3 + DX_1 = LT[P], \tag{10}$$

$$sX_3 = X_4 - \lambda X_3, \tag{11}$$

$$sX_4 = X_5 - \lambda X_4. \tag{12}$$

For the sake of the stability of the first-order moment, $\langle x(t)\rangle$, one must ensure that the characteristic equation of the corresponding equations obtained from Eqs. (8) - (12) cannot have positive real roots, which means that the inequality

$$(\omega^2 + bD)(\lambda^2 + \omega^2 + bD) > a^2D \tag{13}$$

must hold. Henceforth, in this work, we shall assume that this condition is fulfilled. The system OAG, defined by the ratio between the output amplitude of the system

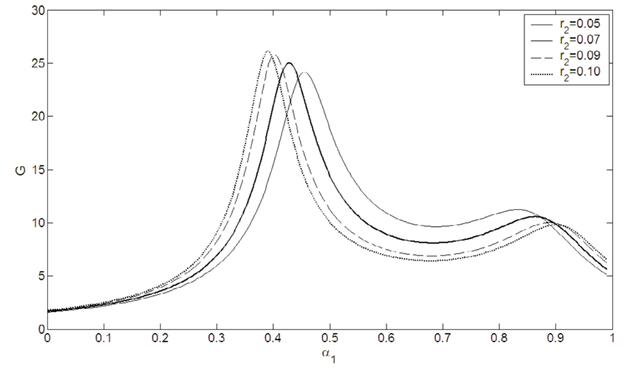

Fig. 1. OAG versus the fractional exponent α_1 for $\omega = 0.6$, $r = 0.1$, $D = 0.05$, $\Omega = 1.5$, $r_1 = 0.9$, and $\alpha_2 = 1.1$ for different values of the friction coefficient r_2.

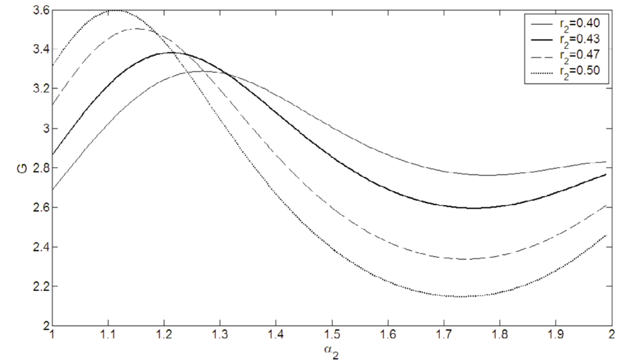

Fig. 2. OAG versus the fractional exponent α_2 for $\omega = 0.7$, $r = 0.5$, $\lambda = 3$, $D = 0.1$, $\Omega = 1.7$, $r_1 = 0.05$, and $\alpha_1 = 0.9$ for different values of the friction coefficient r_2.

and the amplitude of the driving force, can be derived from Eqs. (8) - (12), and has the form

$$G = \sqrt{\frac{f_2^2 + f_4^2}{(f_1 f_2 + f_3 f_4 - D)^2 + (f_1 f_4 + f_2 f_3)^2}}, \tag{14}$$

where

$$\begin{aligned} f_1 &= -\Omega^2 + r_1 \Omega^{\alpha_1} \cos(\pi\alpha_1/2) \\ &\quad + r_2 \Omega^{\alpha_2 - 1} \cos(\pi(\alpha_2 - 1)/2) + \omega^2, \end{aligned} \tag{15}$$

$$\begin{aligned} f_2 &= -\Omega^2 + \lambda^2 + 2r\lambda \\ &\quad + r_1 (\Omega^2 + \lambda^2)^{\alpha_1/2} \cos(\alpha_1 \arctan(\Omega/\lambda)) \\ &\quad + r_2 (\Omega^2 + \lambda^2)^{(\alpha_2 - 1)/2} \cos((\alpha_1 - 1) \arctan(\Omega/\lambda)) \\ &\quad + \omega^2, \end{aligned} \tag{16}$$

$$\begin{aligned} f_3 &= 2r\Omega + r_1 \Omega^{\alpha_1} \sin(\pi\alpha_1/2) \\ &\quad + r_2 \Omega^{\alpha_2 - 1} \sin(\pi(\alpha_2 - 1)/2), \end{aligned} \tag{17}$$

$$\begin{aligned} f_4 &= 2\lambda\Omega + 2r\Omega \\ &\quad + r_1 (\Omega^2 + \lambda^2)^{\alpha_1/2} \cos(\alpha_1 \arctan(\Omega/\lambda)) \\ &\quad + r_2 (\Omega^2 + \lambda^2)^{(\alpha_2 - 1)/2} \cos((\alpha_2 - 1) \arctan(\Omega/\lambda)). \end{aligned} \tag{18}$$

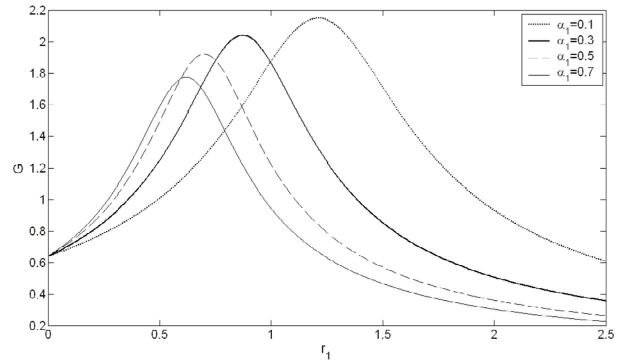

Fig. 3. OAG versus the friction coefficient r_1 for $\omega = 0.1$, $r = 0.7$, $\lambda = 5$, $D = 0.05$, $\Omega = 2.2$, $r_2 = 0.1$, and $\alpha_2 = 1.6$ for different values of the fractional exponent α_1.

III. DISCUSSION

By virtue of Eqs. (14) - (18), we obtain the explicit expression of the system output gain G for any combination of the system parameters α_1, α_2, r, r_1, r_2, and ω, the driving frequency Ω, and the noise parameters D and λ. Figures 1 and 2 depict the OAG versus the fractional exponents α_1 and α_2 for different values of the friction coefficient r_2, respectively. In these two figures, one can find that the OAG varies nonmonotonously with various fractional exponents; *i.e.*, the SR phenomenon occurs. At the same time, a comparison between these figures shows that the effects of the two fractional exponents on the OAG have some differences. Three extreme values exist on each of the OAG curves versus α_1, *i.e.*, two maximum values and one minimum value, a similar result to that observed in Ref. [18]. With increasing α_1, the OAG increases to reach its first maximum value; then, it decreases to a minimum value, after which it increases again to reach its second maximum value and finally decreases monotonously. However, for the curves of OAG versus α_2, two extreme values, *i.e.*, one maximum and one minimum value, exist. The OAG first achieves a maximum value and then reaches its minimum value as α_2 rises. In other words, a stochastic multiresonance takes place on the curve of the OAG versus α_1 while the conventional SR occurs on the curves of the OAG versus α_2.

In Figs. 3 - 6, the behaviors of the OAG for various values of the friction coefficients, r_1 and r_2, are depicted. One can easily conclude that the OAG exhibits a nonmonotonic dependence on both of the friction coefficients, which has also been observed in many other works [15,18,20,31]. The positions of the maxima move monotonically in the direction of smaller r_1 as α_1 and α_2 rises, as shown in Figs. 3 and 4. However, a comparison between Fig. 5 and Fig. 6 shows that, with increasing friction coefficient r_2, the effect of α_2 and that of α_1 on the OAG are some what different. With increasing fractional exponent α_1, the direction of the peaks of

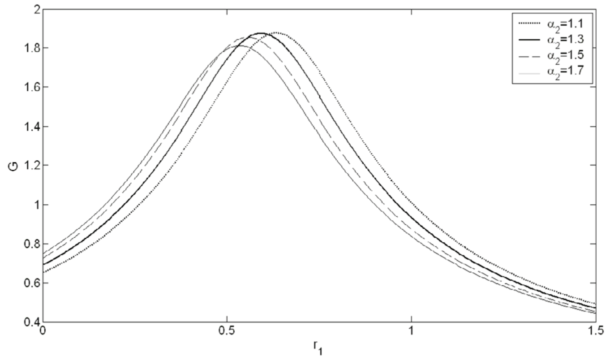

Fig. 4. OAG versus the friction coefficient r_1 for $\omega = 0.1$, $r = 0.7$, $\lambda = 5$, $D = 0.05$, $\Omega = 2.2$, $r_2 = 0.2$, and $\alpha_1 = 0.62$ for different values of the fractional exponent α_2.

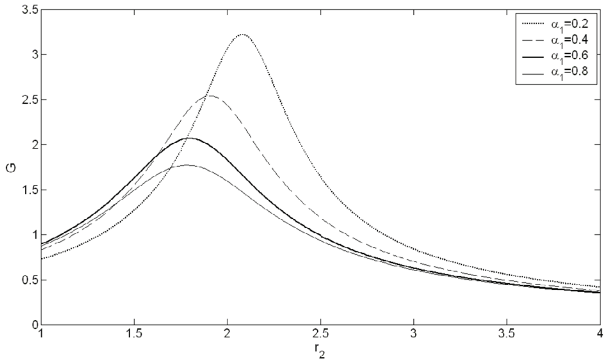

Fig. 5. OAG versus the friction coefficient r_2 for $\omega = 0.5$, $r = 0.1$, $\lambda = 3$, $D = 0.1$, $\Omega = 2$, $r_1 = 0.6$, and $\alpha_2 = 1.1$ for different values of the fractional exponent α_1.

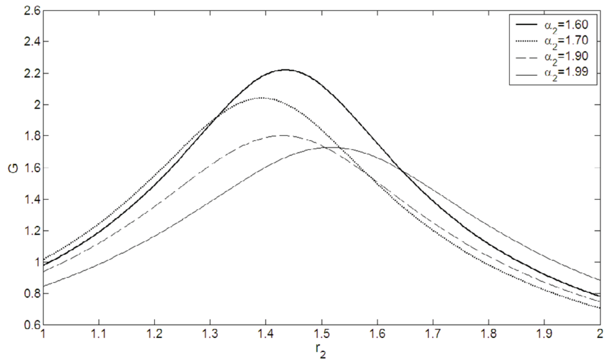

Fig. 6. OAG versus the friction coefficient r_2 for $\omega = 0.5$, $r = 0.1$, $\lambda = 3.5$, $D = 0.1$, $\Omega = 2$, $r_1 = 0.2$, and $\alpha_1 = 0.4$ for different values of the fractional exponent α_2.

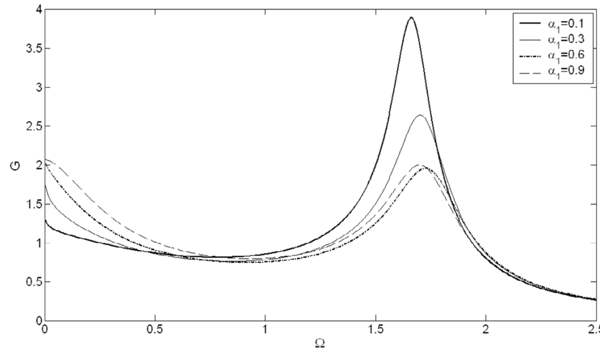

Fig. 7. OAG versus the driving frequency Ω for $\omega = 0.7$, $r = 0.5$, $\lambda = 3$, $D = 0.1$, $r_1 = 0.5$, $r_2 = 0.1$, and $\alpha_2 = 1.9$ for different values of the fractional exponent α_1.

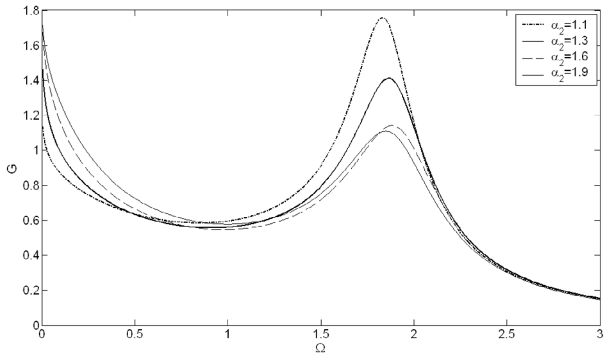

Fig. 8. OAG versus the driving frequency Ω for $\omega = 0.7$, $r = 0.5$, $\lambda = 3$, $D = 0.1$, $r_1 = 0.5$, $r_2 = 0.5$, and $\alpha_1 = 0.3$ for different values of the fractional exponent α_2.

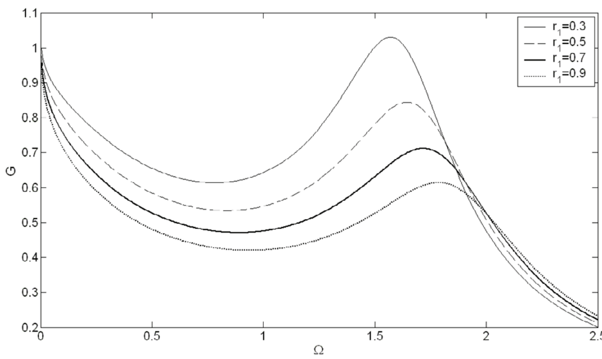

Fig. 9. OAG versus the driving frequency Ω for $\omega = 0.9$, $r = 0.5$, $\lambda = 2$, $D = 0.1$, $r_2 = 0.3$, $\alpha_1 = 0.4$, and $\alpha_2 = 1.1$ for different values of the friction coefficient r_1.

the curves for the OAG versus α_1 shift monotonously to smaller r_2, as shown in Fig. 5, while the maxima of the OAG versus α_2 shift to small values of r_2 for relatively small values of α_2 ($\alpha_2 \in (1.6, 1.7)$) and then shift to large r_2 for relatively large values of α_2 ($\alpha_2 \in (1.9, 1.99)$), as shown in Fig. 6.

Using Figs. 7 - 10, we analyze the dependence of the OAG on the driving frequency. One can see that one maximum value and one minimum value exist on each of the OAG curves; *i.e.*, bona fide SR occurs. The OAG first obtains its minimum value and then increases untill it reaches its maximum value as Ω rises. For small fractional exponent α_1 ($\alpha_1 \in (0.1, 0.3)$), with increasing α_1, the curve of the OAG becomes flatter, *i.e.*, the system bandwidth becomes wider as α_1 rises; on the other hand, for large α_1 ($\alpha_1 \in (0.6, 0.9)$), the OAG curve becomes

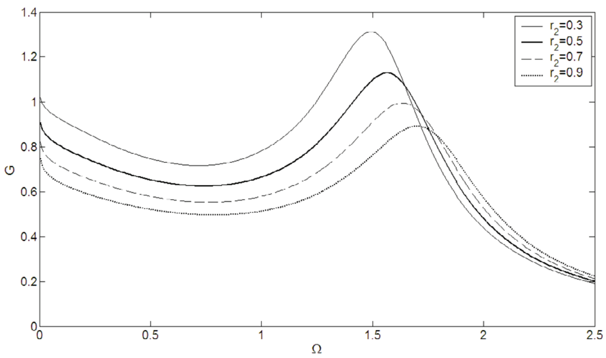

Fig. 10. OAG versus the driving frequency Ω for $\omega = 0.9$, $r = 0.5$, $\lambda = 2$, $D = 0.1$, $r_1 = 0.1$, $\alpha_1 = 0.4$, and $\alpha_2 = 1.1$ for different values of the friction coefficient r_2.

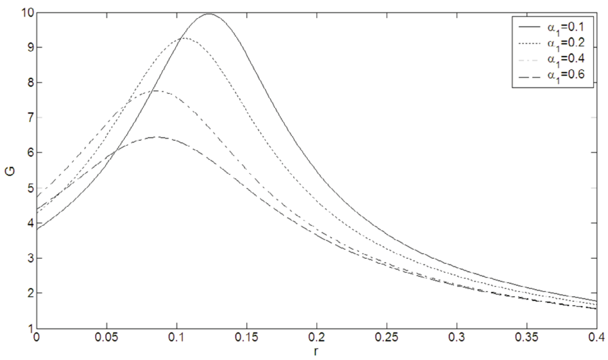

Fig. 11. OAG versus the viscous damping r for $\omega = 0.4$, $\lambda = 1.5$, $D = 0.2$, $\Omega = 1$, $r_1 = 0.3$, $r_2 = 0.2$, and $\alpha_2 = 1.4$ for different values of the fractional exponent α_1.

relatively sharper as α_1 increases, which means that the system bandwidth becomes narrower. Moreover, from Fig. 7 and Fig. 8, one can see that the peaks of the OAG neither monotonically move to lower nor monotonically shift to higher Ω as α_1 and α_2 rise, which is some what similar to that shown in Fig. 6. In addition, from Figs. 9 and 10, one can find that for low frequency ($\Omega < 1.5$), the system OAG decreases as the friction coefficients r_1 and r_2 rise while for high frequency ($\Omega > 2$ for Fig. 9, $\Omega > 1.8$ for Fig. 10) , the system output increases as r_1 and r_2 rise. This suggests that in order to improve the system output signal, one should select small r_1 and r_2 when Ω is slow or choose large r_1 and r_2 when Ω is relatively high. With increasing r_1 and r_2, the OAG curves become flatter, and the peak values of OAG shift monotonously to higher frequency. This means that large values of r_1 and r_2 can increase the system bandwidth, and at the same time, the driving frequency Ω should be tuned to a higher value to achieve the system resonance and, thus, to improve the system's output signal. Furthermore, Figs. 11 and 12 show that the OAG also varies nonmonotonically with the viscous damping r and the system frequency ω.

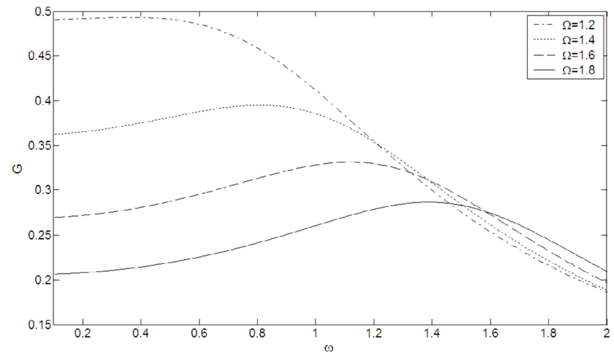

Fig. 12. OAG versus the system frequency ω for $r = 1$, $\lambda = 1$, $D = 0.01$, $r_1 = 0.2$, $r_2 = 0.1$, $\alpha_1 = 0.4$, and $\alpha_2 = 1.9$ for different values of the driving frequency Ω.

IV. CONCLUSION

In this work, we have studied the SR phenomenon for a fractional oscillator with two kinds of fractional terms. By virtue of the characteristics of the Gamma function and the fractional derivative, as well as those of the dichotomous noise, the closed term of the OAG was obtained. Under some conditions, a stochastic multiresonance was manifested in the dependence of the OAG upon the first fractional exponent α_1. In addition, a SR was observed on the OAG curves versus the second fractional exponent α_2, versus the viscous damping, versus the two friction coefficients, versus the driving frequency, and versus the system frequency. We point out that the results obtained in this paper can be easily expanded to wider linear oscillators with more kinds of fractional exponents. We believe that the investigation of linear oscillator with two kinds of fractional derivatives is of general significance.

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