Competitive Learning Behavior in a Stochastic Neural Network

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Stochastic behavior is a natural and inevitable property of biological neurons. The effect of stochastic behavior or thermal fluctuation in neural firings on the learning process in a neural system is investigated. A learning model, which is derived from the stochastic differential equation of the firing-rate model, is presented as an estimate of the gradient flow of free energy. The model reveals that the learning process becomes competitive owing to the effect of entropy even through the synapse modifications only follow the simple Hebbian rule.

I. INTRODUCTION

The basic mechanism for synaptic plasticity proposed by Hebb is that an increase in synapse efficacy arises from the presynaptic cell's repeated and persistent stimulation of the postsynaptic cell [1]. The Hebbian theory attempts to explain associative learning, in which simultaneous activation of cells leads to a pronounced increases in the synaptic strength between those cells and provides a biological basis for errorless learning methods for education and memory rehabilitation. The increase is often simply described as being proportional to the product of the pre- and the postsynaptic neural activities; however, a practical synaptic modulation rule is expected to realize two additional mechanisms because the simple Hebbian model cannot lead to a steady neural learning. One mechanism restricts the connection strength, without which the synaptic weights ould increase exponentially. A variety of learning rules, such as the BCM theory and the Oja's rule, suggest how the synaptic strengths are restricted or normalized [2, 3]. The other is called competitive learning, in which (output) neurons compete for the right to respond to a subset of input patterns so that individual specialization is increased. Without the competitive learning process, neurons would have the same connection pattern as the input neurons in order to response to the most frequent input pattern(s). Traditionally, lateral inhibitory connections between neurons are believed be the essential mechanism for competitive learning in the brain. A winner neuron could inhibit others by an amount proportional to its activation level when neurons may have lateral inhibitory connections. On the other hand, a se-

ries of experiments have revealed that the change in a synapse strength depends on the precise difference between the pre- and the postsynaptic firing timings [4,5]. The so-called spike-timing-dependent plasticity (STDP) mechanism is known to be able to introduce competitive learning to a neural system independently of inhibitory connections and of the restrictions on the number of synaptic connections [6–8].

Meanwhile, the nature of neurons is to make probabilistic responses to external stimuli due to the existence of noisy currents or chaotic behaviors in nonlinear dynamics. Some models, such as the Boltzmann machine, adopt the nature into their representation of neural dynamics [8, 9]. The stochastic (or noise) term is often added to a differential equation for learning dynamics in order to escape from trapping in a local minimum during training.

There are some suggestions that stochastic behavior could be the implemental mechanism of competitive learning in a neural system because stochastic behavior has the character of preventing the elements from staying in the same state. Even though Hebbian competition is weak or lacking, stochastic competition may allow self-organizing cortical map development because the requisite of a maximum entropy distribution forces neurons to avoid connecting to a common target [10]. The sparse function is also defined through a nonlinear function rather than the entropy and is used to prevent the number of few coefficients [11].

Here, the effect of stochastic firing dynamics on learning process is investigated. Stochastic behavior in firing dynamics should affect learning dynamics because neurons make connections with other neurons somewhat randomly when they fire probabilistically. Also, the number of possible states, or the entropy, becomes a mo-

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mentous factor in the determination of the disturbed connection pattern. Interestingly, the derivatives of an energy function can represent both the firing and the learning dynamics. For example, when the energy function is given by $E = -\frac{1}{2}\sum_{\ell,\ell'} W_{\ell\ell'}\phi_\ell\phi_{\ell'}$, the derivative with respect to neural activity, $\Delta\phi_\ell = -\partial E/\partial\phi_\ell = \sum_{\ell'} W_{\ell\ell'}\phi_{\ell'}$, refers to the increase in a neuron's activity depending on the propagation of other neurons' activities via synaptic connections. In addition, the derivative with respect to connection strength, $\Delta W_{\ell\ell'} = -\partial E/\partial W_{\ell\ell'} = \phi_\ell\phi_{\ell'}$, refers to the increase in the synaptic connection in proportional to the product of the pre- and the postsynaptic neural activities, as the simple Hebbian rule says. In this view, examinable problems may be the properties of firing and learning dynamics are represented as the gradient flow of the free energy and the role of entropy in learning dynamics.

In Section II, the correlation-based learning rule, proposed as a feature map formation model in an input-output two layer system [12], is derived from the derivative of the free energy when the system is represented by a Monte Carlo model. The derivative of the internal energy exactly agrees with the correlation-based model derived by means of the firing-rate neuron model and the simple Hebbian rule; however, the entropy does not play any role in the case when only the output neurons are considered to be stochastic and no backward connections are allowed. In Section III, the derivative of the entropy is shown to endow the neural system with competitive learning behavior when not only the output but also the input neurons are considered to fire stochastically. Unfortunately, classical statistical mechanics, ignoring a true 'time' variable, is not proper for predicting the phenomenon (see Appendix). The firing dynamics in a neural system with unidirectional connections cannot be represented by an energy function is also an annoying problem. Instead, the entropy is estimated by means of the connected two-point function in the stochastic differential equation or by the Langevin dynamics version of the firing-rate neuron model. Furthermore, a learning rule, named as the pseudo-stochastic learning (PSL) model, is presented based on the result. The PSL model allows the most probabilistic state in a stochastic neural network to be found through a molecular dynamics simulation instead of a Monte Carlo simulation. In Section IV, the presented learning rule is discussed from other viewpoints and, which reveals how the PSL model relates to (1) the thermodynamic behavior of a stochastic learning model, in which not neural firing, but synaptic plasticity, occurs stochastically, (2) the Gram-Schmidt algorithm, and (3) the informax rule. In Section V, the PSL model is applied to the problem of topographic map formation.

II. PROLOGUE

We suppose that a neural network is composed of two input-output layers, where output neurons have feedforward connections from input neurons and lateral connections with other output neurons. We label input and output neurons with indices (a, b) and (i, j), respectively, and suppose that the continuous function ϕ_i represents the activity of neuron i. We assume that the neural firing dynamics can be expressed as a linear version of the firing-rate model as

$$\frac{d\phi_i}{dt} = -\phi_i + \sum_j J_{ij}\phi_j + \sum_a W_{ia}h_a. \tag{1}$$

Here J, a matrix of dimension $N \times N$, stands for the lateral connections between output neurons, and W, matrices of dimension $N \times M$, stands for the feedforward connections from input to output neurons. N and M are the number of output and input neurons, respectively, and h_a implies the external stimuli to input neuron a.

According to the simple Hebbian rule, the change in the feedforward connection W_{ia} is proportional to $\phi_i\phi_a$. Here, the input neuron's activity ϕ_a is given by h_a. The output neuron's activity at the steady state $d\phi_i/dt = 0$ is

$$\phi_i = \sum_{j,b} K_{ij}W_{jb}h_b, \tag{2}$$

where with $D_{ij} = [I - J]_{ij}$, the effective lateral interaction $K_{ij} = D_{ij}^{-1} = [I + J + J^2 + \ldots]_{ij}$ represents all feasible interactions between output neurons i and j via recursive lateral connections. Finally, if different external inputs are considered during training, the change can be expressed in the form

$$\Delta W_{ia} = \sum_{j,b} K_{ij}W_{jb}Q_{ba}, \tag{3}$$

where Q_{ba} is the average value of $\phi_b\phi_a$ over an ensemble of input patterns in a training set. Q becomes the correlation matrix of input patterns, and Eq. (3) is the so-called correlation-based learning model for feature map formation [12]. The correlation-based learning model is assumed to be able to perform competitive learning through negative variables in K originating from inhibitory lateral connections.

Meanwhile, the same result may be derived by means of the thermodynamics mechanism. We suppose that the activity of neuron i is expressed in the form $\langle\phi_i\rangle = f(v_i)$, where f is a probabilistic function, and that v_i, the scaled membrane potential of neuron i, is defined by the leaky integration of external inputs as follows:

$$v_i = -\phi_i + \sum_j J_{ij}\phi_j + \sum_a W_{ia}h_a. \tag{4}$$

Provided that J is a symmetric matrix, the expectation value of neuron i's activity can be expressed in the form

$$\langle\phi_i\rangle = \frac{1}{Z}\int \mathscr{D}[\phi]\,\phi_i\exp(-E[\phi]/T), \tag{5}$$

with a nonnegative constant T. Moreover, the energy function E and the partition function Z are given by

$$E[\phi] = \frac{1}{2}\sum_{i,j} D_{ij}\phi_i\phi_j - \sum_i h_i\phi_i, \qquad (6)$$

with $h_i = \sum_a W_{ia}h_a$, and

$$Z = \int \mathscr{D}[\phi]\ \exp(-E[\phi]/T). \qquad (7)$$

Provided that ϕ_i can take values in the interval $(-\infty, \infty)$, the notation $\int \mathscr{D}[\phi]$ implies $\prod_k \left[\int_{-\infty}^{\infty}(2\pi T)^{-1/2}d\phi_k \right]$. If ϕ_i takes only 1 or 0, the model system would correspond with the Boltzman machine [9].

The assumption of an infinite interval allows the partition function or the free energy $F = -T\log Z = U - TS$ to be easily calculated [13]. After performing the Gaussian integral, the internal energy and the entropy are obtained as

$$U = -\frac{1}{2}\sum_{i,j} K_{ij}h_ih_j = -\frac{1}{2}\sum_{i,j,a,b} K_{ij}W_{ia}W_{jb}h_ah_b \quad (8)$$

and

$$S = \frac{1}{2}\log\det(K). \qquad (9)$$

Moreover, the expectation value is obtained by taking the derivative of the free energy as

$$\langle\phi_i\rangle = \frac{\partial F}{\partial h_i} = \sum_{j,b} K_{ij}W_{jb}h_b, \qquad (10)$$

which agrees with Eq. (2). Note that $\langle\phi_i\rangle$ becomes identical to ϕ_i when there are no thermal fluctuations.

In addition, the learning rule in Eq. (3) can be derived from the derivative of the free energy as

$$\Delta W_{ia} = -\frac{\partial F}{\partial W_{ia}} = -\frac{\partial U}{\partial W_{ia}} + T\frac{\partial S}{\partial W_{ia}}, \qquad (11)$$

where

$$-\frac{\partial U}{\partial W_{ia}} = \sum_{j,b} K_{ij}W_{jb}h_bh_a \qquad (12)$$

and

$$\frac{\partial S}{\partial W_{ia}} = 0. \qquad (13)$$

Here, the derivative of the free energy corresponds to the correlation-based learning rule in Eq. (3), but the entropy is found to be independent of feedforward connections so that the derivative of the entropy exerts no effect on the learning rule. In the next section, the derivative of the entropy will be demonstrated to be effective when there are thermal fluctuations of input neurons or backward connections from output to input neurons.
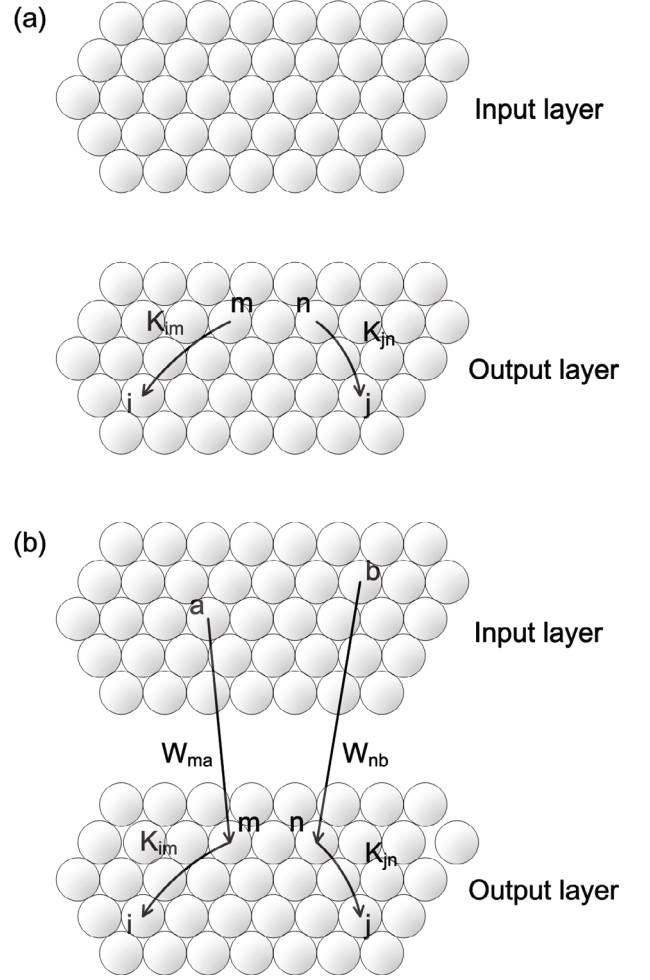

Fig. 1. Illustration of the output neurons' activation induced by other autonomously firing neurons. (a) $\sum_n K_{in}\eta_n$ is the firing probability of neuron i due to the firing propagation via lateral connections from other autonomously firing output neurons, and $\sum_{m,n} K_{im}K_{jn}\langle\eta_m\eta_n\rangle = T[KK^{\dagger}]_{ij}$ is its correlation. (b) $\sum_{a,m} K_{im}W_{ma}\eta_a$ is the firing probability of neuron i due to the firing propagation via feedforward and lateral connections from autonomously firing input neurons, and $\sum_{a,b,m,n} K_{im}W_{ma}K_{jn}W_{nb}\langle\eta_a\eta_b\rangle = T[KWW^{\dagger}K^{\dagger}]_{ij}$ is its correlation.

III. THE MODEL

Suppose the neural firing dynamics can be expressed in the form

$$\tau\frac{d\phi_i}{dt} = -\phi_i + \sum_j J_{ij}\phi_j + \sum_a W_{ia}(h_a + \eta_a) + \eta_i. \quad (14)$$

Here η_ℓ, referring to endogenous firings of output or input neuron ℓ, has the properties $\langle\eta_\ell\rangle = 0$, $\langle h_a\eta_{\ell'}\rangle = 0$, and $\langle\eta_\ell\eta_{\ell'}\rangle = T\delta_{\ell\ell'}$. At the steady state $\partial\phi_i/\partial t = 0$ or

$$\phi_i = \sum_j K_{ij}\Big[\sum_a W_{ia}(h_a+\eta_a) + \eta_j\Big], \qquad (15)$$

the expectation value of output neuron i's activity becomes $\langle \phi_i \rangle = \sum_{j,b} K_{ij} W_{jb} h_b$, as in Eq. (10). Moreover, if the synaptic plasticity rule is assumed to follow the simple Hebbian rule, the change in feedforward connections is obtained as $\Delta W_{ia} = \langle \phi_i \rangle \langle \phi_a \rangle = \sum_{j,b} K_{ij} W_{jb} h_b h_a$, which agrres with Eq. (3) or Eq. (12).

Meanwhile, for Eq. (11), suppose that the synaptic plasticity rule has an additional term such as

$$\Delta W_{ia} = \langle \phi_i \rangle \langle \phi_a \rangle + T \frac{\partial S}{\partial W_{ia}}. \tag{16}$$

Here, the entropy(-like) function S is defined by

$$S = \frac{1}{2} \log \det(T^{-1} G), \tag{17}$$

and the *connected two-point function* G_{ij} is defined by

$$G_{ij} = \langle \phi_i \phi_j \rangle - \langle \phi_i \rangle \langle \phi_j \rangle. \tag{18}$$

The connected two-point function relates to the correlation between two neurons that are activated by other autonomously firing neurons in the absence of external stimuli.

At the steady state in Eq. (15), the connected two-point function becomes

$$G_{ij} = T \left[KK^\dagger + K\Sigma K^\dagger \right]_{ij}, \tag{19}$$

with the matrix $\Sigma = WW^\dagger$. Here, $T[KK^\dagger]$ refers to the correlation originating from the firing propagation via lateral connections from autonomously firing output neurons, and $T[K\Sigma K^\dagger]$ refers to the correlation originating from the firing propagation via feedforward and lateral connections from autonomously firing input neurons (see Fig. 1).

The substitution of Eq. (19) into Eq. (16) produces the learning rule

$$\Delta W_{ia} = \left[KWQ + T(I+\Sigma)^{-1} W \right]_{ia}. \tag{20}$$

Here, a series expansion is possible and yields $[I+\Sigma]^{-1} = q[I - (I-qI-q\Sigma)]^{-1} = q[I + (I-qI-q\Sigma) + \ldots]$ when the absolute of all the eigenvalues of $I - qI - q\Sigma$ are less than 1 for a constant q. Furthermore, $-q^2 T\Sigma W$, the most effective term in the expansion, introduces the competitive relationship between feedforward connections, and the degree of competition increases with the degree of endogenous neural firings (or the parameter T). Alternatively, if the output neurons' autonomous firings are ignored, being not effective in the learning dynamics, the connected two-point function would be $G_{ij} = T[K\Sigma K^\dagger]_{ij}$, and the learning rule would become

$$\Delta W_{ia} = \left[KWQ + T\Sigma^{-1} W \right]_{ia}. \tag{21}$$

Note that the output neurons' autonomous firing would affect the learning dynamics when there are backward connections from output to input layer so that the firing propagation from output to output neurons via input neurons is possible. The presented learning rule is named the pseudo-stochastic learning (PSL) model and reflects the effect of a stochastic neural network on the learning dynamics not through a Monte Carlo simulation but a molecular dynamics simulation.

IV. UNDERSTANDING THE MODEL IN OTHER VIEWS

1. Stochastic Learning Model

The PSL model is based on the learning behavior in a stochastic dynamics model system in which (input) neurons fire stochastically, but connection strengths change determinately. Not only firing dynamics but also learning dynamics may possibly occur stochastically in a neural model system [8]. Consider a stochastic learning model system in which neurons fire determinately, but synaptic connections change stochastically. Provided that Eq. (3) is the derivative of an energy function with respect to connection strength, the energy function would read

$$E[W] = -\frac{1}{2} \sum_{i,j,a,b} K_{ij} W_{ia} Q_{ab} W_{jb}. \tag{22}$$

Let us suppose that the connection strengths change stochastically and their probabilities of them follow the Boltzmann distribution

$$P(W) = \frac{e^{-E[W]/T}}{\int \mathscr{D}[W'] e^{-E[W']/T}}. \tag{23}$$

Calculating the correlation function $\langle W_{ia} W_{jb} \rangle = \int \mathscr{D}[W] W_{ia} W_{jb} P(W)$ is helpful for understanding the statistical behavior of the model system, where the calculation becomes convenient by adding an external source term $E_{\text{souce}} = -\sum_{i,a} B_{ia} W_{ia}$ to the energy function. If W_{ia} is assumed to be able to take a value in the interval $(-\infty, \infty)$ and the Gaussian integral is performed, the calculation result can be expressed in the form

$$\langle W_{ia} W_{jb} \rangle = \langle W_{ia} \rangle \langle W_{jb} \rangle - T K_{ij}^{-1} Q_{ab}^{-1}, \tag{24}$$

where $\langle W_{ia} \rangle = \int \mathscr{D}[W] W_{ia} P(W)$ vanishes when $B = 0$. Meanwhile, the learning rule in Eq. (21) arrives at the steady state for the condition

$$K_{ij} Q_{ab} + T\Sigma_{ij}^{-1} \delta_{ab} = 0. \tag{25}$$

Here, $\Sigma_{ij} = \sum_a W_{ia} W_{ja}$ would be identical to $\sum_a \langle W_{ia} W_{ja} \rangle$ if the thermal fluctuation in W is ignored. Namely, the steady state of the PSL model corresponds to one of the most probable states in the stochastic learning model.

2. Gram-Schmidt Algorithm

The Gram-Schmidt algorithm is a method for orthogonalizing a set of vectors. If the vectors are normalized to unity, the Gram-Schmidt algorithm can be written as

$$\Delta W_{ia} = -\sum_{j=1}^{i-1}(\sum_b W_{ib}W_{jb})W_{ja} \qquad (26)$$

or

$$\Delta W_{ia} = -[lower(\Sigma)W]_{ia}. \qquad (27)$$

Here, *lower* is the function setting all matrix elements on or above the diagonal to 0. If the *lower* function is ignored, this corresponds with the most effective competition term in the PSL model, $-q^2T\Sigma W$. Namely, the competition process in the PSL model relates to that in the Gram-Schmidt algorithm. A difference is that the Gram-Schmidt algorithm obtains the most principal component first and finds the next principal components successively by subtracting the vector projection onto the more principal components. On the other hand, the PSL model does the subtraction between the components simultaneously without priority.

3. Informatix Rule

In the view of informatics, neural network learning is expected to maximize the information transfer from input to output neurons. Provided that the output of neuron i is expressed in the form $\phi_i = g(v_i)$, with $v_i = \sum_a W_{ia}h_a$ and g being the (nonlinear) activation function, the information transfer can be measured by using the joint entropy

$$H(\phi) = H(\phi_1,\ldots,\phi_N) = -\int \mathscr{D}[\phi]P(\phi)\log P(\phi) \quad (28)$$

The joint entropy can be rewritten as

$$H(\phi) = \sum_{i=1}^{N} H(\phi_i) - I(\phi), \qquad (29)$$

where $H(\phi_i)$, the marginal entropy of the output ϕ_i, is

$$H(\phi_i) = -P(\phi)\log P(\phi) \qquad (30)$$

and $I(\phi)$, the mutual information of the outputs, is

$$I(\phi) = \int \mathscr{D}[\phi]P(\phi)\log \frac{P(\phi)}{\prod_{k=1}^{N}P(\phi_i)}. \qquad (31)$$

Provided that the numbers of input and output neurons are the same, the derivative of the joint entropy is given by

$$\frac{\partial H(\phi)}{\partial W_{ia}} = \frac{\partial I(\phi)}{\partial W_{ia}} = \frac{\partial}{\partial W_{ia}}\log\det(\mathcal{J}), \qquad (32)$$

where the other terms are assumed to be independent of W. Here, $\mathcal{J}_{ia} = P(h_a)/P(\phi_i)$ is the Jacobian of the transformation from h_a to ϕ_i. If $\partial\phi_i/\partial h_a = W_{ia}(\partial\phi_i/\partial v_i)$ when no lateral connections exist between output neurons, the Jacobian can be rewritten as

$$\mathcal{J}_{ia} = W_{ia}\prod_{i=1}^{N}\frac{\partial\phi_i}{\partial v_i}. \qquad (33)$$

Finally, the learning rule, maximizing the joint entropy, is obtained by

$$\Delta W_{ia} = \frac{\partial}{\partial W_{ia}}\log\det(W) + \frac{\partial}{\partial W_{ia}}\sum_{i=1}^{N}\log\frac{\partial\phi_i}{\partial v_i}. \quad (34)$$

A further manipulation of this equation gives the learning informax rule [14] and is related to the algorithm of independent component analysis (ICA), which is the most popular model for blind source separation [15]. Note that the first term in Eq. (34) corresponds with the second term in Eq. (21) because $\log\det(W) = \frac{1}{2}\log\det(WW^{\dagger})$ for a square matrix W. Namely, the competition process in the PSL model is related to the maximization of the joint entropy or the information transfer.

V. APPLICATION TO TOPOGRAPHIC MAP FORMATION

The direct application of Eq. (21) requires very much computation time in acquiring the matrix inverse per update of weights. If the in-coming connection vectors are constrained to be unity, i.e., $|\mathbf{W}_i|^2 = \sum_a |W_{ia}|^2 = 1$ for all i, the approximation $\Sigma^{-1} \simeq 2I - \Sigma$ (or $[I+\Sigma]^{-1} \simeq I - \Sigma$) can be adopted, and an approximated version of the PSL model can be expressed in the form

$$\Delta W_{ia} = [KWQ - T(\Sigma-\alpha)W]_{ia}. \qquad (35)$$

Here, α is set to be 2 or 1, or it could be ignored because $\alpha W_{ia} = (\alpha/2)(\partial/\partial W_{ia})|\mathbf{W}_i|^2$ becomes negligible owing to the normalization constraint.

The formation of topographic maps between input and output square lattices is examined in Fig. 2. The topographic map formation problem has been studied in the other learning model, and the conditions for a proper map formation are presented [16]. First, both the lateral interactions between output neurons and the input correlations between input neurons should decrease with increasing distance between neurons. The property yields the metric between neurons through which close output neurons have connections with close input neurons. Second, both the out-going connections from an input neuron and the in-coming connections to an output neurons should have a competitive relationship. In the simulations, the lateral interactions and the input correlations are modeled by using Gaussian functions,
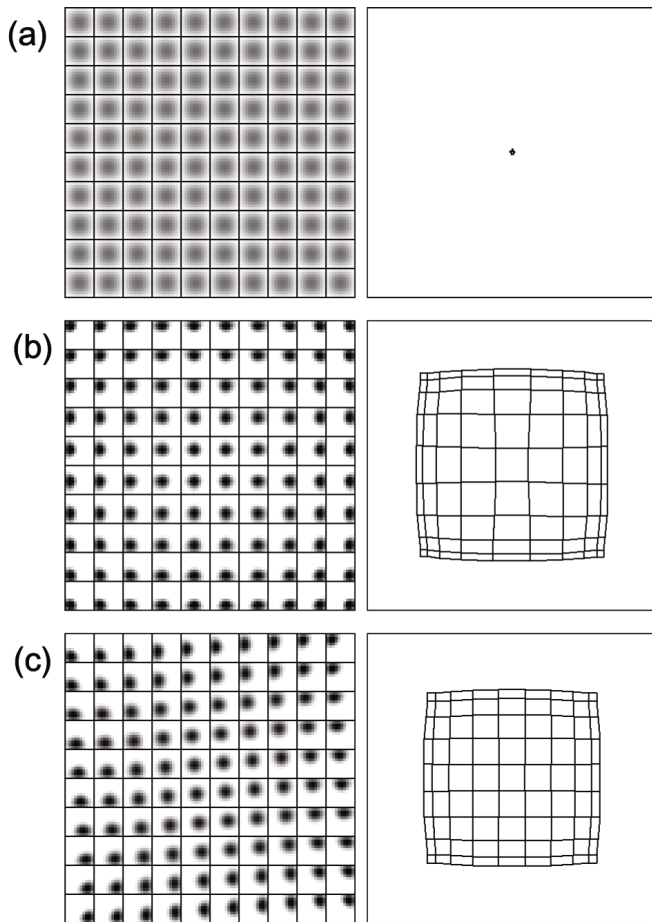
Fig. 2. Emergent topographic structures between input and output square lattices of size M=10×10 and N=10×10. The left figures show the distribution of the output neurons' receptive field onto the input layer, and the right figures show the connections between neighboring receptive field centers. Equation (35) was applied in the simulations, where K_{ij}=$\exp(-|\mathbf{r}_i-\mathbf{r}_j|^2)-\beta$, Q_{ab}=$\exp(-|\mathbf{r}_a-\mathbf{r}_b|^2)$, and $\alpha=0$. The interaction parameters are (a) $(\beta,T)=(0,0)$, (b) $(0.02,0)$, and (c) $(0,0.1)$.

where the competition between output-connections is achieved by using negative lateral interactions controlled by β, or by using stochastic competition controlled by T. Anso, the competition between in-coming connections is achieved by using the normalization condition $|\mathbf{W}_i| = 1$ for all i. When there is no (or very small) competition, the output neurons have the same receptive fields covering the entire input layer (Fig. 2(a)). The receptive fields become localized and form a proper topographic map when the degree of negative lateral interactions is large enough (Fig. 2(b)). Finally, the network forms a proper topographic structure even though the competition between out-going connections is achieved not by lateral interactions but by stochastic competition (Fig. 2(c)).

VI. DISCUSSION

Here, the foundation of competitive learning has been considered in the view of statistical mechanics. The suggestion based on thermal fluctuations in or probabilistic responses of neural activities could be the origin of competitive learning in a a neural system. Presented is a learning rule, the named PSL model, which shows that the endogenous firing of input neurons becomes the key to stochastic competition in a two-layer neural system. The endogenous firing of output neurons may exert a similar effect on network learning when the firings fluctuate the input neurons' activity via backward connections from the output to the input layer. Such random firing of input neurons prevents output neurons from having the same connection strengths with input neurons, and the entropy, depending on the connection strengths, becomes an important factor in the network structure modulation.

The model has been demonstrated to lead to the formation of a proper topographic map through an effective competition process. How the competition process of the model is related to that of the other learning rules has also been presented. Nevertheless, the model needs to be improved in future by means of more advanced theoretical methods. The PSL model is derived from the stochastic differential equation or the Langevin dynamics version of the firing-rate neuron model. The method provides the advantage of estimating the effect of the entropy even though the neural interactions cannot be represented by an energy function; however, the model is built on the basis of the several postulates for neural network modulation. An annoying problem is that the derivative of the entropy bears different results depending on how the neural firing dynamics is expressed by the energy function or the Langevan equation, where the difference originates because the energy function does not adopt a true 'time' variable (see Appendix). We also did not answer how the stochastic competition mechanism is related to the learning behavior of the biological synaptic plasticity rule, which produces different change in a synapse depending on the exact difference between the pre- and the postsynaptic spike timings. Finally, the effect of stochastic neural firings on network learning after the effect of exact firing timings on neural firings is considered needs to be studied.

In addition, we expect to develop a nonlinear version of the PSL model in future. Some phenomena (or learning problems) cannot be explained (or solved) without nonlinear terms. It is often difficult for a learning model to extract proper features from an input correlation matrix, so it should include terms with higher-order input correlations. The ability of blind source separation for the informax rule in Eq. (34) also depends on the form of the nonlinearity in the activation function [15]. The development of a nonlinear PSL model may require more sophisticated methods in statistical mechanics, such as perturbation theory or renormalization group theory.

APPENDIX

Equation (9) becomes $\frac{1}{2}\log\det(KK^\dagger)$ when there is no η_a in Eq. (14) so that no input but only output neurons fire stochastically. That this is not in accord with Eq. (17) may be a suspect property. The leak of agreement originates from the energy function in Eq. (6) not adopting a true 'time' variable. Essentially, K_{ij} relates to the propagator $K(r_i, t_i; r_j, t_j)$ referring to the probability of that neuron i at position r_i fires at time t_i when neuron j at position r_j fires at time t_j. Nevertheless, the matrix K has somewhat different properties with the (time ordered) propagator. The propagation from $(\mathbf{r}_j, t_j)$ to $(\mathbf{r}_i, t_i)$ may be regarded as the result of propagation from $(\mathbf{r}_j, t_j)$ to all available intermediate points $\mathbf{r}$ followed by propagation from $(\mathbf{r}, t)$ to $(\mathbf{r}_i, t_i)$ so that the propagator usually satisfies the property $K(r_i, t_i; r_j, t_j) = \sum_r \int dt K(r_i, t_i; r, t) K(r, t; r_j, t_j)$. However, the matrix K in Eq. (9) does not satisfy the property $[KK]_{ij} = K_{ij}$. In addition, the matrix K in Eq. (9) is a symmetric and real matrix so that $K = K^\dagger$. However, $K_{ij}^\dagger$ relates to the time-reversed propagation via the reciprocal connection from neuron j to i. The notation of the complex conjugation will be more meaningful if the propagator is represented in the Fourier space.

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