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Evolutionary Dynamics of the Weighted Voter Model with Opinion Strength on Complex Networks

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The voter model has served to address the emergence of consensus within populations of individuals. However, the dynamics based on the classic voter model has usually been analyzed based on the assumption that the two states in the model are simply equivalent. In this paper, we discuss a mathematical description of the weighted voter model and obtain a series of results for the evolutionary process on complex networks. For homogeneous networks, we study the active link density analytically and find that the opinion strength plays a crucial role in determining whether the system can reach consensus. We also extend our research to heterogeneous networks and discover that the network structure can affect the convergence time but has less influence on the positive proportion. The results can be applied to various pervasive cases in which two conflicting opinions interact with each other.

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I. INTRODUCTION

Collective behavior has recently attracted much interest among the community of physicists working on complex systems. The spreading of rumors, the dynamics of opinions and the diffusion of cultural traits are all collective phenomena that start from a disordered initial configuration and tend to result in an ordered state [1–10]. To analyze this kind of social dynamics from a physical viewpoint, numerous researchers have used the voter model as an insightful starting point due to its simplicity [11–15].

The voter model describes an evolutionary process in which the opinion can be influenced by an individual's direct neighbors. The dynamics in a complex network can be briefly described as follows: Each individual has an opinion characterized by a binary variable $s_i = \pm 1$. Initially, all nodes are assigned to a random opinion with a given probability. At every time step, individual i adopts the state of a randomly chosen neighbor j; *i.e.*, if $s_i \neq s_j$, then s_i is set equal to s_j [11–15]. Studies of the voter model on complex networks have mainly focused on the influence of complex topologies on the dynamical behaviors [15], the relationship between the updating methods (link update or node update) or adopting methods (direct, reverse or random) and the system's dynamic behavior [16].

In the literature, traditional works have mainly assumed that these two conflicting opinions have equal impact and that the two opinions are merely equivalent labels. In reality, however, voters are likely to be influenced by some external factors, such as mass media, election propaganda, racial or religious factors, *etc.*, which can have different influences on both sides, thus causing the changes of opinion from one to another to have different probabilities. Several studies have considered the preference for one of the two states [17, 18]. However, those models are not straightforward and do not gave particular attention to the parameters that are widely concerned in the classic voter model.

In this paper, we aim to delineate this phenomenon directly and discuss some crucial parameters that were not fully studied in the previous literature by adding weights to the two opinions; *i.e.*, we assign different strengths to the two opinions to show their acceptance [19]. If one opinion is more likely to be spread in general, we say it has a stronger strength than the other one. Different from the literature mentioned above, we mainly discuss the influence of the opinion strength on the system's status. Firstly, we give a more precise illustration of our model in Section II. In Section III, the basic equations that control the dynamic behaviors of the system are established. To investigate whether individuals with the same opinions tend to aggregate or separate, we study the active link density, which is the proportion of links

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that connect opposite nodes. Next, we use simulations to extend the results to heterogeneous networks. Finally, conclusions are presented in Section IV.

II. THE MODEL

We consider a connected, undirected network composed of a set of N nodes and M links. Let $\sigma_+(t)$ and $\sigma_-(t)$ be the density of positive nodes (nodes whose state are $+1$) and the density of negative nodes (nodes whose state are -1), respectively, at time t. Initially, nodes are assigned values of 1 or -1 with probabilities given by $\sigma_+(0)$ or $\sigma_-(0) = 1 - \sigma_+(0)$. Denote the nonnegative values f_+, f_- as the strengths of the positive state and the negative state, fulfilling $f_+ + f_- = 1$. We assume that the state strengths remain unchanged during the process.

The weighted voter model is defined as follows:

At every time step, a node i updates its state to a positive node with probability

$$\frac{f_+ m_{i+}}{f_+ m_{i+} + f_- m_{i-}}. \tag{1}$$

Otherwise, it will be negative with probability

$$\frac{f_- m_{i-}}{f_+ m_{i+} + f_- m_{i-}}. \tag{2}$$

Here m_{is} stands for the number of node i's neighbors with state s. This step is repeated until the system necessarily reaches consensus or for long enough time where it never reaches consensus.

In the classic voter model, a node adopts its neighbor's opinion randomly. If most of its neighbors are positive nodes, it will have a higher probability to become (or remain) positive. Consequently, the positive updating probability can be understood as $m_{i+}/(m_{i+} + m_{i-})$. Therefore, we name our model the weighted voter model. To note, this model is more suitably categorized as a majority rule model, but we still name it the weighted voter model because this model is compared with the classic voter model in this paper.

Naturally, the system may reach a consensus situation in which all nodes hold the same state or may never reach this situation. Hence, we are concerned with the conditions that the system can reach a fully-ordered state. If these conditions are met, we will focus on the mean time to converge; otherwise, we will take ordering parameters such as the density of positive nodes, the density of active links and the magnetization into account. The detailed definitions of these parameters will be provided in the following sections.

III. RESULTS

1. The Master Equation

In order to have a basic understanding of this model, we firstly develop the differential equation to describe the evolution of the positive density. For simplicity, we consider the uncorrelated *Erdös − Rényi* (ER) network in this subsection.

According to the updating rule, the probability $P(- \to +)$ that a randomly-chosen node changes state from -1 to +1 can be approximated as $(1 - \sigma_+) \cdot \frac{f_+\sigma_+}{f_+\sigma_+ + f_-\sigma_-}$. Conversely, we obtain $P(+ \to -) = \sigma_+ \cdot \frac{f_-\sigma_-}{f_+\sigma_+ + f_-\sigma_-}$. The number of positive nodes is $N_+ = N\sigma_+$.

Therefore, the evolution equation for the number of positive nodes in the mean-field approximation reads

$$\frac{\mathrm{d}N^+}{\mathrm{d}t} = N[P(- \to +) - P(+ \to -)], \tag{3}$$

which can be rewritten as

$$\frac{\mathrm{d}\sigma_+}{\mathrm{d}t} = \frac{A\sigma_+ - A(\sigma_+)^2}{A\sigma_+ + B}, \tag{4}$$

where $A = 2f_+ - 1, B = 1 - f_+$. The stationary solutions of this equation can be easily obtained and are $\sigma_+ = 0$ or $\sigma_+ = 1$ under the condition of $A \neq 0$. Particulary, $\sigma_+ = 1$ is stable whenever $f_+ > 1/2$, and $\sigma_+ = 0$ is stable if $f_+ < 1/2$, which is in accord with common sense.

However, if $A = 0$ ($f_+ = 1/2$), the right-hand side of Eq.(4) equals zero, which means that both the positive nodes and the negative nodes keep their initial densities. Actually, our model turns out to be the classic voter model if $A = 0$. The conclusion of the classic voter model on this network topology is $\sigma_+(t) \to \sigma_+(0)$ when $N \to \infty$, and the system never reaches the fully-ordered state [14]. However, if the fluctuation is considered, the system may be driven to the consensus state.

To confirm this, we plot the positive node density for every 10 steps of one realization. The results are shown in Fig. 1. Without loss of generality, we only need to consider $f_+ \geq f_-$ in the following analysis. As shown in Fig. 1, the further f_+ deviates from $1/2$, the easier the system converges. When $f_+ = 1/2$, the positive node density fluctuates around $1/2$, in agreement with the classic conclusion.

2. The Density of Active Links

Now, we discuss the density of active links. Naturally, two types of links are included in the voter system: active links that connect nodes of different states and inert links that connect nodes of the same state. The density of active links $\rho = L/M$ measures the level of activity in

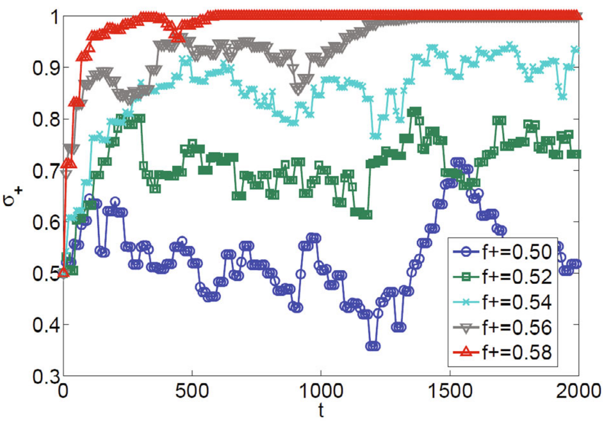

Fig. 1. (Color online) The relationship between the positive node density and time in one realization. The initial condition is $\sigma_+(0) = 1/2$. We adopt the ER network with $N = 1000$ and mean degree $\mu = 2.75$.

the system, where L is the number of active links [14]. In this subsection, we consider networks with any given degree distribution.

Let $P(k)$ stand for the degree distribution of nodes. A positive state node i with degree k is chosen. We denote by n the number of active links connected to it before the update. After the update, the probability of node i remaining positive is

$$P_+ = \frac{f_+\frac{k-n}{k}}{f_+\frac{k-n}{k} + f_-\frac{n}{k}}, \tag{5}$$

and the probability of it flipping its status is

$$P_- = \frac{f_-\frac{n}{k}}{f_+\frac{k-n}{k} + f_-\frac{n}{k}}. \tag{6}$$

The expectation for the number of active links after updating is $E(L_{after}) = nP_+ + (k-n)P_-$. Therefore, the expectation value for the increase in the number of active links after updating for a positive node is

$$\begin{aligned} E(\Delta L|+) &= E(L_{after}) - n \\ &= \frac{f_- n - 2f_-\frac{n^2}{k}}{f_+(1-\frac{n}{k}) + f_-\frac{n}{k}}. \end{aligned} \tag{7}$$

Likewise, for a negative node,

$$E(\Delta L|-) = \frac{f_+ n - 2f_+\frac{n^2}{k}}{f_-(1-\frac{n}{k}) + f_+\frac{n}{k}}. \tag{8}$$

Assembling these factors, we can describe the change in the number of active links in a single time interval $\mathrm{d}t = 1/N$ as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_k P_k \frac{\mathrm{d}L}{\mathrm{d}t}\Big|_k, \tag{9}$$

where $\frac{\mathrm{d}L}{\mathrm{d}t}\big|_k$ is the average change in L when a node of degree k is chosen.

Here, we assume that the distribution of the state of a node's neighbors is independent of the node's state itself. We define $B(n,k)$ as the probability that n of the k links connected to a node are active ($\Sigma_k B(n,k) = 1$) and $B(n,k|s)$ as the corresponding conditional probability given that the node has state s. Introducing these expressions, we can rewrite Eq. (9) as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_k \frac{P_k}{1/N} \sum_{n=0}^{k} \sum_{s=\pm} E(\Delta L|s) B(n,k|s)\sigma_s. \tag{10}$$

The conditional probability $B(n,k|s)$ is a binomial distribution with $P(-s|s)$ as the single event probability, where $P(-s|s)$ is the probability that a node changes state from s to $-s$. Let $\mu = 2M/N$ stand for the mean degree of the network. The total number of active links in the network is $L = \rho\mu N/2$, among which $\sigma_s \mu N$ connect nodes with state s. Therefore, $P(-s|s) = \frac{\rho\mu N/2}{\sigma_s \mu N} = \frac{\rho}{2\sigma_s}$, leading to

$$B(n,k|s) = \binom{n}{k} \left(\frac{\rho}{2\sigma_s}\right)^n \left(1 - \frac{\rho}{2\sigma_s}\right)^{k-n}. \tag{11}$$

Substituting these expressions into Eq.(10), we obtain

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_k \frac{P_k}{1/N} \sum_{s=\pm} \sigma_s \left(\alpha_s \langle n \rangle_{k,s} - \frac{\beta_s}{k} \langle n^2 \rangle_{k,s}\right). \tag{12}$$

We have used a Taylor expansion in the above equation. Here, $\alpha_+ = \frac{f_-}{f_+}$, $\beta_+ = \frac{f_-}{(f_+)^2}$, $\alpha_- = \frac{f_+}{f_-}$, $\beta_- = \frac{f_+}{(f_-)^2}$ and

$$\langle n \rangle_{k,s} = \sum_{n=0}^{k} B(n,k|s) n = \frac{kL}{\mu N \sigma_s}, \tag{13}$$

$$\langle n^2 \rangle_{k,s} = \sum_{n=0}^{k} B(n,k|s) n^2 = \frac{kL}{\mu N \sigma_s} + \frac{k(k-1)L^2}{\mu^2 N^2 (\sigma_s)^2}. \tag{14}$$

The property of a binomial distribution is used when calculating Eq. (13) and Eq. (14). Combining these equations with Eq. (12), we find that the stationary solutions of this equation are $L_1 = 0$ or

$$L_2 = \frac{\mu N \sigma_+ \sigma_- (f_+^3(\mu f_- - 1) + f_-^3(\mu f_+ - 1)}{(\mu - 1)(\sigma_+ f_+^3 + \sigma_- f_-^3)}. \tag{15}$$

Using $\rho = 2L/\mu N$, we finally obtain the density of stationary active links $\rho_1 = 0$ or

$$\rho_2 = \frac{2\sigma_+\sigma_-(f_+^3(\mu f_- - 1) + f_-^3(\mu f_+ - 1))}{(\mu - 1)(\sigma_+ f_+^3 + \sigma_- f_-^3)}. \tag{16}$$

Therefore, we can easily show that ρ_1 is stable if

$$\mu < \mu_c = \frac{f_+^3 + f_-^3}{f_+ f_- (f_+^2 + f_-^2)}, \tag{17}$$

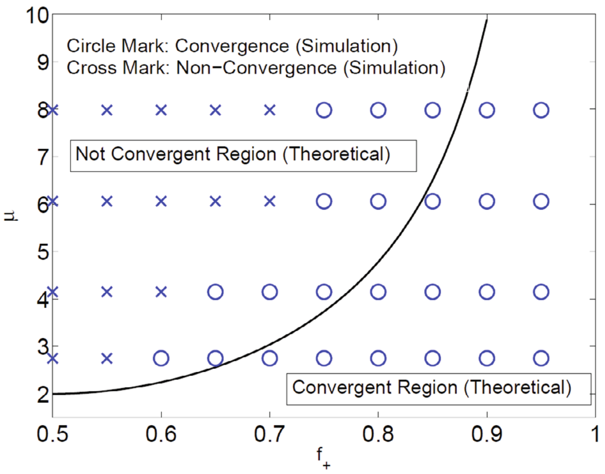

Fig. 2. (Color online) The relationship between μ_c and f_+ in our model. The curve is an analytical result derived from Eq. 17. The circles and crosses stand for the convergent state and the non-convergent state, respectively, during the simulation. We adopt the ER network with $N = 1000$. The average degrees are 7.98, 6.06, 4.15, and 2.75 (from top to bottom).

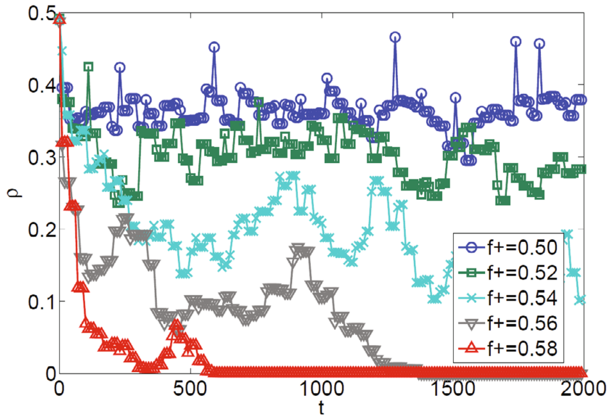

Fig. 3. (Color online) The relationship between the density of active links and time in one realization. The initial condition is $\sigma_+(0) = 1/2$. We adopt the ER network with $N = 1000$ and $\mu = 2.75$.

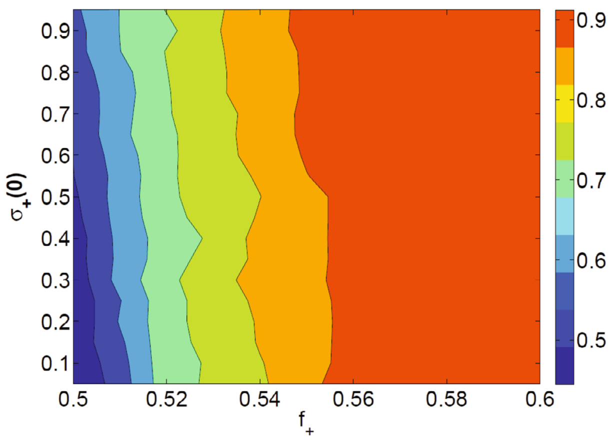

Fig. 4. (Color online) The contour map of $\sigma_+(1000)$ as a function of f_+ and $\sigma_+(0)$ for the ER network. We adopt the ER network with $N = 1000$ and mean degree $\mu = 2.75$. The results have been averaged for 10^2 experiments.

ρ_2 is stable otherwise.

Let us review the relative conclusions of the classic voter model [14], which also has two solutions. The solution $\rho_1 = 0$ is stable if $\mu < \mu_c = 2$, and when $\mu > 2$, the stable solution is $\rho_2 = 4\xi\sigma_+\sigma_-$, where $\xi = \frac{\mu-2}{2(\mu-1)}$ is a constant value only related to the mean degree. That is to say, if the mean degree of the network is no more than 2, no links will be active after a long time. Otherwise, the density of active links tends to be a constant nonzero value $4\xi\sigma_+(0)\sigma_-(0)$ in the thermodynamic limit. Naturally, substituting $f_+ = f_- = 1/2$ into Eq. (16), we obtain this result.

Therefore, compared with the classic voter model, the weighted voter model can be understood as an expectation of the convergence area. From Eq. (17), we obtain the curve in Fig. 2. This curve distinguishes the not-convergent region (upper curve) from the convergent region. During the simulation, if the system can indicate a convergent state (ρ is small enough after $5N$ steps), we draw a circle. Otherwise, we plot a cross. As shown in the figure, a larger mean degree indicates an increased difficulty in obtaining a consensus. Due to fluctuation and finite size effect, errors are unavoidable. Therefore, Eq. (17) is a rough estimate of whether the system can reach convergence. In Fig. 3, we consider the evolutionary of the active link density with time. The results are similar to the results in Fig. 1 because ρ and σ_+ can be related by Eq. (16).

In the analysis above, we find that $\sigma_+(0)$ and f_+ both affect σ_+ after a long time. In Fig. 4, we draw a contour map to test their significance. The figure is divided by a series of vertical lines instead of horizontal lines, which means that f_+ is the key parameter that determines the system's evolution. In contrast, $\sigma_+(0)$ has little effect on the system. Intuitively, high values of $\sigma_+(0)$ and f_+ can both increase σ_+, but a high value of $\sigma_+(0)$ leads to an unstable high value of σ_+ in the beginning, while a high value of f_+ helps to maintain this result.

From Eq. (16), we can obtain more of the parameters, such as the magnetization [14], that were studied in the voter model. Magnetization is another parameter that measures the level of order in the network. Although the density of active link ρ is indicative of whether the system will reach a fully-ordered state, it cannot denote which one of the two states will be reached. For this reason, the node magnetization $m = \sigma_+ - \sigma_-$ is introduced where $m = 1$ ($m = -1$) represents the + (−) fully-ordered state. The link magnetization has been proven to be simply the node magnetization [14], *i.e.*,

$$m = \sigma_+ - \sigma_- = \rho_{++} - \rho_{--} = 2\sigma_+ - 1, \qquad (18)$$

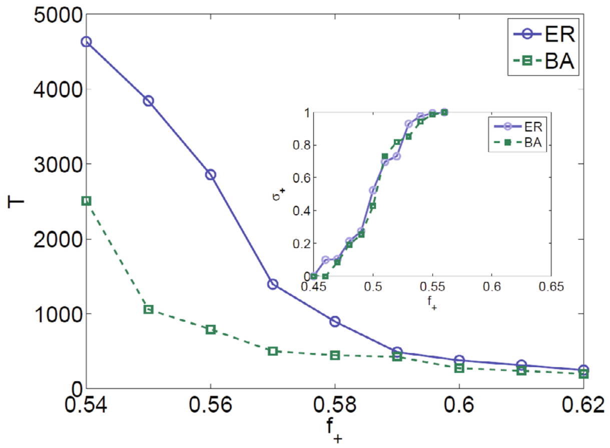

Fig. 5. (Color online) The positive density $\sigma_+(5000)$ (inset) and the convergence time T (main) as functions of f_+ in ER and BA networks. For both networks, $N = 1000$ and mean degree $\mu = 4.15$. The results have been averaged for 10^2 experiments.

where ρ_{++} (ρ_{--}) is the density of links connecting two nodes with state +1 (−1).

If Eq. (18) and Eq. (16) are combined, ρ and m are related through

$$\rho = \frac{(f_+^3(\mu f_- - 1) + f_-^3(\mu f_+ - 1))(1 - m^2)}{(\mu - 1)[(1 + m)f_+^3 + (1 - m)f_-^3]}. \quad (19)$$

Compared with the classic result $\rho = \xi(1 - m^2)$ [14], the result here is more complicated and shows the influence of the state strengths.

3. Simulation on Heterogeneous Networks

Next, we extend our model to a heterogeneous substrate such as a BA network [21]. The degree distribution of the BA network is skewed and highly variable. In Fig. 5, we compare the positive density σ_+ and the convergence time T as functions of f_+ in ER and BA networks. As shown in the figure, the convergence time in the BA network is shorter than that in the ER network. The reason lies in the role of hubs, which have more neighbors than normal nodes , in the BA network. The presence of a hub helps to accelerate the consensus process because its many neighbors tend to become equal to it. However, the difference in σ_+ when the network has not reached consensus is slight.

IV. CONCLUSION

To summarize, we have investigated a weighted voter model that introduces an opinion weight into the classic voter model. It expands the application limits of the classic voter model. The relationship among the state density, the active link density, the mean degree and the strength of opinion has been presented in Eq. (16). A slight difference in the positive state strength can make a huge difference in the system. If f_+ is sufficiently close to 0.5, achieving consensus with the system is difficult. The results have been verified by many numerical simulations. Then, we extend our research to heterogeneous networks. The presence of hub nodes has been found to help the consensus process accelerate, but has less influence on the positive proportion.

To note, the actual meaning of opinion strengths is not involved in the discussion above. Therefore, this model can be applied to various cases in which two conflicting opinions have unequal status.

Except for all the work we have done, various topics still attract the interest of scientists, such as the survival probability and the conservation laws. Besides, the dynamical scaling laws (*e.g.*, node update or link update) and the degree exponent γ in the scale-free network may influence the final results. However, to concentrate on opinion strength, we have not discussed all these factors. These will be investigated in the future to improve our model.

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