DOI: 10.3901/CJME.2013.05.851, available online at www.springerlink.com; www.cjmenet.com; www.cjmenet.com.cn

Kinematic Solution of Spherical Stephenson-III Six-bar Mechanism

LIU Yanfang and YANG Suixian*

School of Manufacturing Science and Engineering, Sichuan University, Chengdu 610065, China

Received September 12, 2012; revised February 5, 2013; accepted March 12, 2013

Abstract: A closed-form solution can be obtained for kinematic analysis of spatial mechanisms by using analytical method. However, extra solutions would occur when solving the constraint equations of mechanism kinematics unless the constraint equations are established with a proper method and the solving approach is appropriate. In order to obtain a kinematic solution of the spherical Stephenson-III six-bar mechanism, spherical analytical theory is employed to construct the constraint equations. Firstly, the mechanism is divided into a four-bar loop and a two-bar unit. On the basis of the decomposition, vectors of the mechanism nodes are derived according to spherical analytical theory and the principle of coordinate transformation. Secondly, the structural constraint equations are constructed by applying cosine formula of spherical triangles to the top platform of the mechanism. Thirdly, the constraint equations are solved by using Bezout' s elimination method for forward analysis and Sylvester' s resultant elimination method for inverse kinematics respectively. By the aid of computer symbolic systems, Mathematica and Maple, symbolic closed-form solution of forward and inverse displacement analysis of spherical Stephenson-III six-bar mechanism are obtained. Finally, numerical examples of forward and inverse analysis are presented to illustrate the proposed approach. The results indicate that the constraint equations established with the proposed method are much simpler than those reported by previous literature, and can be readily eliminated and solved.

Key words: spherical Stephenson-III six-bar mechanisms, kinematic analysis, spherical analytical theory, Bezout' s elimination method, Sylvester' s elimination method

1 Introduction

Spherical mechanisms are a type of special mechanism in which the rotation axes of all the links intersect in a signal point located at the center of the mechanism. In recent years, due to their particular characteristics, more and more attention has been paid to parallel spherical mechanisms by researchers. For example, a spherical five-bar mechanism has been applied to an orienting device[1]. CHEN, et al, designed scanning apparatus with a four-degrees-of-freedom hybrid spherical mechanism[2]. Based on a spherical six-bar mechanism, a gearless robotic Pitch-roll wrist has been proposed by HERNANDEZ, et al[3]. VALASEK, et al[4], reported a redundantly actuated parallel spherical mechanism as a new concept of agile telescope by using a three times overactuated structure.

The classic spherical mechanisms, spherical four-bar linkage and the three-degrees-of-freedom spherical parallel mechanisms, have been investigated by researchers[5–10]. Numerous advances have been made in establishing and solving the constraint equations of these mechanisms[7–10]. The configurations and Grashof' s condition of spherical four-bar linkage have been discussed and presented by MURRAY and LAROCHELLE[5]. RUTH and McCARTHY developed a computer-aided design software system for spherical four-bar linkages based on Burmester' s planar theory[6]. It has been studied that the analytical synthesis of function generation of spherical four-bar linkage for five precision points by RASIM, et al[7]. A polynomial approximation method has been presented to determine design parameters. The position equation has been constructed by using triangular relations. LEE, et al[8], explored the motion generation of adjustable spherical four-bar linkage. Based on the input-output (I/O) equation of spherical four-bar linkages, BAI, et al[9], researched the forward-displacement analysis of spherical parallel robots by deposing the closed-loop kinematic chain of a spherical parallel robots into four-bar spherical chains. ENFERADI, et al[10], investigated the forward position problem of a spherical star-triangle parallel manipulator by utilizing a spherical configuration. Constraint equations of the mechanism have been constructed by equivalent angle-axis representation and solved Bezout' s method which leads to a closed-form solution with a polynomial of degree 8. ZHANG, et al[11], presented a three-spherical kinematic chain based parallel mechanism. The constraint of the mechanism has been constructed by means of a virtual symmetric plane based on the three

* Corresponding author. E-mail: yangsx@163.com
This project is supported by National Natural Science Foundation of China(Grant No. 50975186)

virtual centers of the spherical chains. The singularity of the mechanism is identified related to the constraint configuration based on Grassmann line geometry and the dependency of the constraint screw system. The design parameters avoiding the platform singularity have been presented.

Some work have been conducted by researchers on planar six-bar mechanisms, including Stephenson I, II, and III, Watt, etc[6, 12–16]. KIM, et al[12], explored the design problem for six-bar linkages that generate a specific coupler point trajectory. An approach has been proposed by MARIAPPAN and KRISHNAMURTY[13] to design a Stephenson III six-bar linkage that generates a path for a press mechanism by using an optimization procedure. SOH and McCARTHY[14] introduced an approach to synthesize planar six-bar linkages including the Watt I and Stephenson I, II, and III with Denavit–Hartenberg convention. TING and DOU[15] presented a method to identify the effects of both loops on the rotatability of any Stephenson six-bar linkage and developed algorithms to identify its branch condition. The proposed method is based on the rotatability of the common joints between the two loops and no coupler curve is used. By converting a Watt six-bar linkage to an equivalent simple Stephenson linkage using the stretch and rotation of a four-bar loop, TING, et al[16], examined the stretch rotation and complete mobility identification of Watt six-bar chains.

The spherical six-bar mechanism is a special type of multi-bar mechanism. Comparative research in the analysis and synthesis of six-bar mechanisms has been conducted by MAKHSUDYAN, et al regarding spherical and planar mechanisms[17]. The comparison was carried out by studying three generalized non-dimensional indexes: velocity, acceleration and dynamic power. The results show that spherical linkages have better properties than planar linkages. A synthesis approach has been proposed by YANG and XU for spherical six-bar path generation mechanisms, in which the mechanism is divided into several link groups[18]. The synthesis equations of two link groups for the spherical mechanisms have been established and the constraint conditions and objective functions presented as well. ZHANG, et al[19], proposed a method for optimal trajectory synthesis of an adjustable spherical Stephenson-III six-bar mechanism. The multi-task synthesis equations were derived and the optimization model of mechanism synthesis was established based on the virus evolutionary genetic algorithm to obtain comprehensive results. SANCISI, et al[20], presented a validation approach to a one degree-of-freedom spherical model for kinematic analysis of the human ankle joint. GREGORIO[21] researched the analytical method for the singularity analysis, and exhaustive enumeration of the singularity conditions in single-DOF spherical mechanisms by exploiting the properties of instantaneous pole axes. The exhaustive enumeration of the geometric conditions which occur for all the singularity types is given, and a general analytical method based on this enumeration is carried out for implementing the singularity analysis.

Kinematics analysis is one of the fundamental problems of spatial mechanism analysis. Establishing constraint equations is the first step to analyze, synthesize and evaluate a spatial mechanism. Analytical and numerical methods can be used to solve the displacement analysis problem of a mechanism. By using an analytical method, a closed-form solution of displacement analysis can be obtained for some mechanisms with simple structures, which is helpful for carrying out performance analysis and configuration design intuitively. However, it is difficult to obtain closed-form symbolic solutions for both forward and inverse displacement analysis of parallel spatial mechanisms and the inverse solution of serial mechanisms unless the constraint equations are established with an appropriate method and solved in a proper way. ZHAO, et al[22], examined the generation of closed-form inverse kinematics for reconfigurable robots by means of the screw and product-of-exponentials formula.

A number of studies have been carried out relating to kinematics of spherical mechanisms. For example, WAMPLER[23] addressed the displacement analysis of spherical mechanism having three or fewer loops by using rotation matrices or quaternion, including the classical pentad mechanism which has eight solutions. The solutions were obtained by using modified Sylvester' s elimination and numerical calculation via standard eigenvalue routines. BAKER[24] presented the displacement-closure equations of the unspecialized double-Hooke' s-joint linkage with focus on the general relationship between input and output shaft angles. A set of geometric constraint equations of the 6R double-centered overconstrained mechanisms has been constructed by CUI and DAI[25]. The axis constraint equation of the 6R double-centered overconstrained mechanisms was obtained after applying the Sylvester' s dialytic elimination method. The input-output equation of a spherical Stephenson-III six-bar mechanism has been reported by HERNANDEZ[3] and the dimensional synthesis of the mechanism was conducted. However, the power of the input-output equation is on the high side which means that there are some extraneous roots. BOMBIN, et al[26], presented an approach to deal with the computation of the direct kinematics of parallel spherical mechanisms using Bernstein polynomials. The direct kinematics of parallel spherical mechanisms with l legs was converted to solving systems of l–1 second-order multinomials. KONG and GOSSELIN[27] investigated the forward displacement analysis(FDA) of a quadratic spherical parallel manipulator: the Agile Eye. An alternative formulation of the kinematic equations of the Agile Eye was presented and the singularity analysis of the Agile Eye was examined. RODRIGUEZ and RUGGIU[28] explored the forward displacement problem(FDP) of several common spherical parallel manipulators(SPMs). Quaternion algebra was employed to express the FDP as a system of equations and the Dixon determinant procedure to construct univariate polynomials whose roots can be found either numerically or analytically. The solutions of the system were obtained analytically by a symbolic method exploiting symmetries. Ref. [29] reports the forward kinematics of a 3-DOF spherical parallel manipulator. An algebraic solution was presented by using a transformational matrix to construct the geometric constraint equations. The mechanism is with a simple geometric structure, so its forward kinematic solution can be reached easily in a univariate quartic polynomial equation. HUANG and YAO[30] studied the kinematics of a generalized 3-DOF spherical parallel manipulator, of which each leg consists of two rotating bars respectively. The inverse kinematic solution of the mechanism was obtained by using spherical analytical theory in concise form. The forward kinematics of the mechanism was also reached in a closed-form solution.

In this paper, analytical methods are used to obtain a closed-form input-output equation of the spherical Stephenson-III six-bar mechanism. The displacement analysis constraint equations have been derived by utilizing spherical analytical theory[31]. Bezout’ s elimination method and Sylvester’ s resultant elimination method[32–33] are applied to solve the constraint equations, and implemented using the computer symbolic systems, Mathematica and Maple respectively.

This paper is organized as follows: The constraint equations of the spherical Stephenson-III six-bar mechanism are derived in section 2. Based on the established constraint equations, forward and inverse displacement analyses are conducted in section 3. Discussions and conclusion are presented at the end of the paper.

2 Constraint Equations

As shown in Fig. 1, a spherical Stephenson-III six-bar mechanism can be decomposed into a spherical four-bar linkage and a spherical two-bar unit. Assuming that A_0 is the input reference point and C_0 is the output reference point, therefore, θ_1 and θ_5 are the input and output angles, respectively. The reference coordinate systems are set according to the mechanism’ s structural characteristics as shown in Fig. 1. Point O is the global center. The X-axis of the original coordinate system O-XYZ is perpendicular to the plane (O,α_{12}) ; Z-axis coincides with the axis OB_0 and Y-axis is in the plane (O,α_{12}) in Cartesian coordinates. The coordinate system of a four-bar mechanism is assumed to coincide with the original coordinate system. The X_1 -axis of the coordinate system O-$X_1Y_1Z_1$ of the spherical two-bar unit is perpendicular to the plane (O,α_{12}) ; Z_1-axis coincides with the axis OC_0 ; and Y_1 is following Cartesian coordinates. Denoting $S_i=\sin\theta_i$; $C_i=\cos\theta_i$; $S_{ij}=\sin\alpha_{ij}$; $C_{ij}=\cos\alpha_{ij}$, where α_{ij} is the spherical central angle between points i and j. The transformation matrix rotating around the X-axis and Z-axis can be expressed as follows:

$$\boldsymbol{R}\left(X,\alpha_{ij}\right)=\begin{pmatrix}1 & 0 & 0\\ 0 & C_{ij} & -S_{ij}\\ 0 & S_{ij} & C_{ij}\end{pmatrix}, \tag{1}$$

$$\boldsymbol{R}\left(Z,\theta_i\right)=\begin{pmatrix}C_i & -S_i & 0\\ S_i & C_i & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2}$$

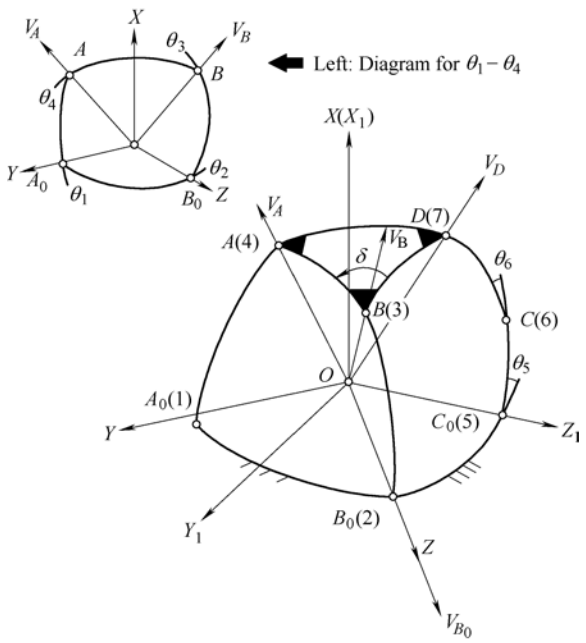


Fig. 1. Coordinate systems of the mechanism

The coordinate system O-$X_1Y_1Z_1$ can be transferred to the original coordinate system O-XYZ by rotating α_{25} clockwise with respect to the X-axis. Thus, the transformation matrix is written as follows:

$$\boldsymbol{R}\left(X,\alpha_{25}\right)=\begin{pmatrix}1 & 0 & 0\\ 0 & C_{25} & -S_{25}\\ 0 & S_{25} & C_{25}\end{pmatrix}. \tag{3}$$

According to Ref. [31], vector $\boldsymbol{V}_B$ in coordinate system O-XYZ can be obtained by the loop equation of the spherical polygon of the spherical four-bar linkage:

$$\boldsymbol{V}_B=\left(x_B,\ y_B,\ z_B\right)^{\mathrm{T}}, \tag{4}$$

where

$$x_B=\overline{X}_4C_1-\overline{Y}_4S_1,$$
$$y_B=C_{12}\ (\overline{X}_4S_1+\overline{Y}_4C_1)-S_{12}\overline{Z}_4,$$
$$z_B=S_{12}\ (\overline{X}_4S_1+\overline{Y}_4C_1)\ +C_{12}\overline{Z}_4;$$

$\overline{X}_4$, $\overline{Y}_4$, and $\overline{Z}_4$ are defined as in Ref. [31], $\overline{X}_4 = S_{34}S_4$, $\overline{Y}_4 = -(S_{41}C_{34} + C_{41}S_{34}C_4)$, $\overline{Z}_4 = C_{41}C_{34} - S_{41}S_{34}C_4$.

Similarly, vector $\boldsymbol{V}_A$ in O-*XYZ* can be expressed as follows:

$$\boldsymbol{V}_A = \left(x_A, y_A, z_A\right)^{\mathrm{T}}, \tag{5}$$

where $x_A = S_{41}S_1$, $y_A = S_{41}C_1C_{12} + C_{41}S_{12}$, $z_A = -S_{41}\ C_1S_{12} + C_{41}C_{12}$.

Vector $\boldsymbol{V}_{B_0}$ in O-*XYZ* is denoted as follows:

$$\boldsymbol{V}_{B_0} = [0,\ 0,\ 1]^{\mathrm{T}}. \tag{6}$$

According coordinate transformation theory, vector $\boldsymbol{V}_D$ in $O\text{-}X_1Y_1Z_1$, denoting as $\boldsymbol{V}_{D_1}$, is written as follows:

$$\boldsymbol{V}_{D_1} = \boldsymbol{R}(z,\theta_5)\boldsymbol{R}(x,\alpha_{56})\boldsymbol{R}(z,\theta_6)\boldsymbol{R}(x,\alpha_{67})\left(0,\ 0,\ 1\right)^{\mathrm{T}}.$$

Therefore, $\boldsymbol{V}_D$ in O-*XYZ* can be expressed as follows:

$$\boldsymbol{V}_D = \boldsymbol{R}(X,\alpha_{25})\boldsymbol{V}_{D_1}. \tag{7}$$

Applying the cosine formula of spherical triangles to the platform ABD of the spherical Stephenson-III six-bar mechanism, the structural constraint equations can be obtained as follows:

$$\begin{cases} \boldsymbol{V}_B \bullet \boldsymbol{V}_D = \cos\alpha_{73}, \\ \boldsymbol{V}_A \bullet \boldsymbol{V}_D = \cos\alpha_{34}\cos\alpha_{73} - \sin\alpha_{34}\sin\alpha_{73}\cos\delta, \\ \boldsymbol{V}_B \bullet \boldsymbol{V}_{B_0} = \cos\alpha_{23}. \end{cases} \tag{8}$$

Eq. (8) is the constraint equation set of the mechanism, which involve variables θ_1, θ_4, θ_5 and θ_6. While the input angle θ_1 is given, Eq. (8) can be solved.

3 Displacement Analysis

3.1 Forward displacement analysis

θ_5 is taken as the output angle of the mechanism. Forward displacement analysis of the spherical Stephenson-III six-bar mechanism is defined as that to solve Eq. (8) to obtain the value of output angle θ_5 when the input angle θ_1 is given.

Denoting $\tan(\theta_i/2) = t_i$, the following equation can be obtained:

$$C_i = (1-t_i^2)/(1+t_i^2);\ S_i = 2t_i/(1+t_i^2)\quad (i = 1, 4, 5, 6). \tag{9}$$

Substituting Eq. (9) into Eq. (7), according to trigonometric transformation and by the aid of symbolic software Mathematica[34], Eq. (7) can be expressed as follows:

$$\begin{cases} a_{11}t_6^2 + b_{11}t_6 + c_{11} = 0, \\ a_{22}t_6^2 + b_{22}t_6 + c_{22} = 0, \\ a_{33}t_4^2 + b_{33}t_4 + c_{33} = 0, \end{cases} \tag{10}$$

where

$$a_{11} = a_1t_5^2 + a_2t_5 + a_3,\ b_{11} = a_4t_5^2 + a_5t_5 + a_6,$$
$$c_{11} = a_7t_5^2 + a_8t_5 + a_9,\ a_{22} = b_1t_5^2 + b_2t_5 + b_3,$$
$$b_{22} = b_4t_5^2 + b_5t_5 + b_6,\ c_{22} = b_7t_5^2 + b_8t_5 + b_9,$$

a_i and $b_i (i = 1,\ 2,\ \cdots,\ 9)$ are expressions of t_4 and the structural parameters of the mechanism, a_{33}, b_{33}, c_{33} are the expressions of the structural parameters of the mechanism.

t_4 is obtained by solving the third equation in Eq. (10):

$$t_4 = \frac{-b_{33} \pm \sqrt{(b_{33})^2 - 4a_{33}c_{33}}}{2a_{33}}. \tag{11}$$

The Bezout' s elimination method is traditionally used for reducing a set of polynomials of multiple variables into a polynomial of only one variable. Appling Bezout' s elimination method to the first two equations of Eq. (9) and eliminating t_6, the following equation is obtained:

$$(a_{11}c_{22} - a_{22}c_{11})^2 - (a_{11}b_{22} - a_{22}b_{11})(b_{11}c_{22} - b_{22}c_{11}) = 0 \tag{12}$$

Eq. (12) can be expressed in a univariate quartic polynomial equation as follows:

$$\begin{aligned} &d_1t_5^8 + d_2t_5^7 + d_3t_5^6 + d_4t_5^5 + d_5t_5^4 + \\ &\quad d_6t_5^3 + d_7t_5^2 + d_8t_5 + d_9 = 0, \end{aligned} \tag{13}$$

where $d_1, \cdots, d_9$ are expressions about $t_4\ (C_4, S_4)$.

Substituting the two solutions of t_4 obtained from Eq. (11) into Eq. (13), sixteen solutions of t_5 are derived, which are the forward displacement solutions of the spherical Stephenson-III six-bar mechanism.

Substituting solutions of t_4 and t_5 into the first and second equations of Eq. (10), t_6 is obtained:

$$t_6 = \frac{a_{22}c_{11} - a_{11}c_{22}}{a_{11}b_{22} - a_{22}b_{11}}. \tag{14}$$

After that, with Eq. (9), θ_4, θ_5 and θ_6 can be calculated.

3.2 Numerical example of forward displacement analysis

The structural parameters of the spherical Stephenson-III six-bar mechanism are given as α_{12}=135°; α_{23}=85°; α_{34}=70°; α_{41}=85°; α_{25}=135°; α_{56}=85°; α_{67}=75°; α_{73}=90°, and the input angle is set as $\theta_1 = 60°$. After computing coefficients in Eq. (10) and Eq. (13) with the given structural parameters and the value of input angle, t_4 and t_5 are obtained, as listed in Table 1.

Table 1. Solutions of forward displacement analysis

Variable t_4	Variable t_5	
	Real root	Imaginary root
−0.288 823	−0.600 069	i
	0.296 822	i
	0.897 892	−i
	23.948	−i
1.624 46	−3.033 83	i
	−0.598 56	i
	0.642 809	−i
	0.924 906	−i

Substituting t_4 and the real roots of t_5 into Eq. (9) respectively, the solutions of joint angle θ_4 and the output angle θ_5 of the spherical Stephenson-III six-bar are obtained and presented in Table 2.

Table 2. Forward solutions

Joint angle θ_4/(°)	Output angle θ_5/(°)
−32.22	−61.93
	33.06
	83.84
	175.22
116.77	−143.51
	−61.81
	65.47
	85.53

With the results of the forward displacement analysis, configurations of the mechanism were calculated. Of the eight real solutions, two solutions meet the mechanism' s configuration requirement, which is shown in Fig. 2. The other solutions will cause interference between the links.

3.3 Inverse displacement analysis

Inverse displacement analysis is defined as a process whereby the input angle θ_1 of the mechanism is determined by solving the displacement constraint equations when the output angle θ_5 of the mechanism is given.

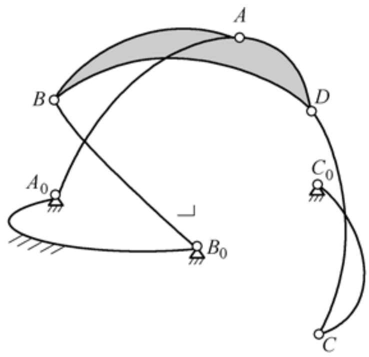
Denoting $\tan(\theta_i/2) = t_i$ $(i = 1,\ 4,\ 6)$, Eq. (8) can be transformed into Eq. (15) with trigonometric transformation:

$$\begin{cases} M_1 t_6^2 + M_2 t_6 + M_3 = 0, \\ L_1 t_6^2 + L_2 t_6 + L_3 = 0, \\ \left(n_1 t_1^2 + n_2\right) t_4^2 + n_3 t_1 t_4 + n_4 t_1^2 + n_5 = 0, \end{cases} \tag{15}$$

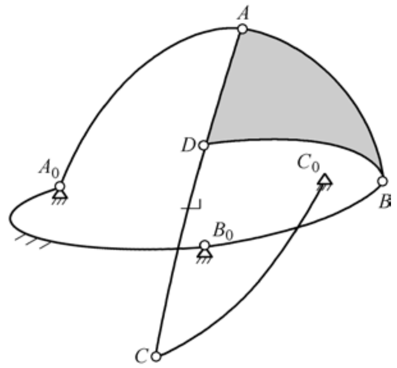
where

$$M_1 = m_0 t_1 t_4 + m_3 t_1^2 t_4 + m_6 t_1^2 t_4^2 + m_7 t_1 t_4^2 + m_8 t_4^2 + m_9 t_1^2 t_4 + m_{10} t_4 + m_{11} t_1^2 + m_{12} t_1 + m_{13},$$
$$M_2 = m_1 t_1 t_4^2 + m_3 t_1^2 t_4^2 + m_4 t_1^2 t_4 + m_{14} t_1 t_4 + m_{15} + m_{16} t_4^2 + m_{17} t_4 + m_{18} t_1^2 + m_{19} t_1,$$
$$M_3 = m_2 t_1 + m_5 t_1^2 + m_{20} t_1^2 t_4^2 + m_{21} t_1 t_4^2 + m_{22} t_4^2 + m_{23} t_1^2 t_4 + m_{24} t_1 t_4 + m_{25} t_4 + m_{26},$$
$$L_1 = l_1 t_1^2 + l_2 t_1 + l_3,$$
$$L_2 = l_4 t_1^2 + l_5 t_1 + l_6,$$
$$L_3 = l_7 t_1^2 + l_8 t_1 + l_9,$$

$l_1, \cdots, l_9$, $m_0, \cdots, m_{26}$, and $n_1, \cdots, n_5$ are all of expressions about the structural parameters of the mechanism.



(a) Output angle θ_5=−143.51°



(b) Output angle θ_5=−61.93°

Fig. 2. Configurations of forward solutions

Eq. (15) can be solved with Sylvester' s resultant elimination method by the aid of computer symbolic systems Mathematica and Maple. The solving process is as follows.

Step 1. Eliminating t_6 with the first two equations of Eq. (15), Eq. (16) is obtained:

$$p_0 t_4^4 + p_1 t_4^3 + p_2 t_4^2 + p_3 t_4 + p_4 = 0, \tag{16}$$

where $p_0, \cdots, p_4$ are all polynomials with the highest degree of t_1^8.

Step 2. Eliminating t_4 with Eq. (16) and the third equation of Eq. (15) by resultant elimination, an equation with the 24th-degree of t_1 is obtained as follows:

$$k_0 t_6^{24} + k_1 t_1^{23} + k_2 t_1^{22} + \cdots + k_{22} t_1^2 + k_{23} t_1 + k_{24} = 0, \quad (17)$$

where $k_0, \cdots, k_{24}$ are all of expressions about the structural parameters of the mechanism.

Step 3. Solving Eq. (17) to discover reverse displacement solutions of the mechanism, Eq. (17) has twenty-four solutions in total. After t_1 is obtained, substituting t_1 into the second and the third equations of Eq. (15), the solutions of t_4 and t_6 will be obtained as follows:

$$t_4 = \frac{-(n_3 t_1 + n_4) \pm \sqrt{(n_3 t_1 + n_4)^2 - 4\left(n_1 t_1^2 + n_2\right) n_5}}{2\left(n_1 t_1^2 + n_2\right)}, \quad (18)$$

$$t_6 = \frac{L_1 M_3 - L_3 M_1}{L_2 M_1 - L_1 M_2}. \quad (19)$$

Furthermore, configurations of the inverse displacement analysis of mechanism can be calculated.

Coefficients a, b, d in Eqs. (10)−(13) are shown in the appendix. Coefficients l, m, n, p, k in Eqs. (15)−(17) are too complex to present in this paper, which can be requested through email: yuanzhang198621@163.com.

3.4 Numerical example of inverse displacement analysis

The structural parameters of the mechanism are set in the same way as those in the numerical example of forward displacement analysis. The output parameter of the mechanism is set as θ_5=−143.51° (t_5=−3.033 83). Substituting t_5 and all the structural parameters into Eq. (17), twenty-four solutions of t_1 which are the solutions of the inverse displacement analysis have been obtained, and shown in Table 3.

Table 3. Solutions of inverse displacement analysis (t_1)

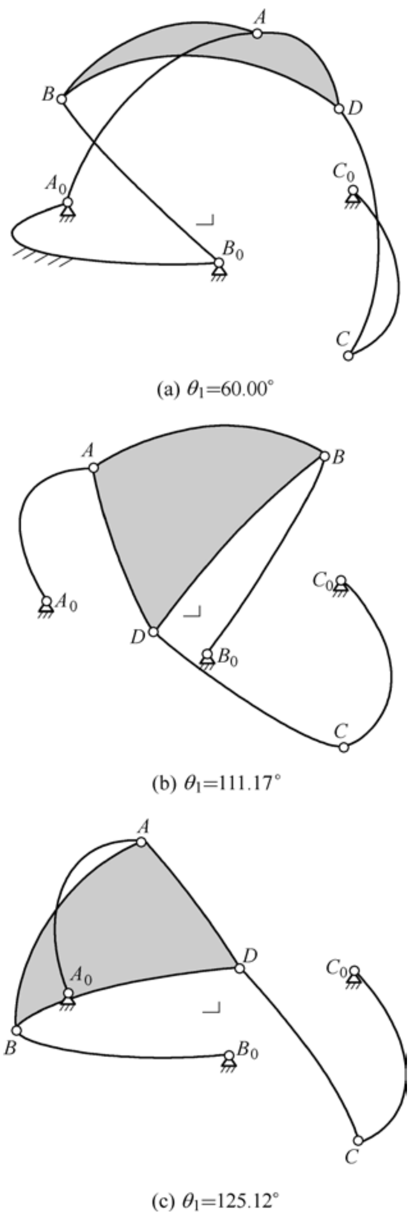
	Parameter t_1	
Real root	−0.006 395	0.577 353
	1.459 67	1.925 87
Imaginary root	0.263 218+0.531 167i	0.263 218−0.531 167i
	0.076 878+0.356 91i	0.076 878+0.356 91i
	0.002 612+1.003 03i	0.002 612−1.003 03i
	−1.006 17+2.114 76i	−1.006 17−2.114 76i
	−1.343 723+1.091 6i	−1.343 723−1.091 6i
	−1.509 42+1.211 36i	−1.509 42−1.211 36i
	−0.735 87+0.929 349i	−0.735 87+0.929 349i
	0.003 017+0.997 41i	0.003 017+0.997 41i
	−0.003 059+1.002 59i	−0.003 059+1.002 59i
	−0.002 57+0.996 961i	−0.002 57−0.996 961i

With the four real roots in Table 3, the values of the input angle θ_1 of the mechanism have been calculated by the aid of trigonometric transformation. The solutions are listed in Table 4.

Table 4. Inverse solutions of input angle (θ_1)

θ_1	−139.88°	60.00°	111.17°	125.12°

The corresponding mechanism configurations of the inverse solutions are presented in Fig. 3 except the first solution in Table 3. The mechanism configuration of the first solution in Table 3 will result in interference between bars B_0B and A_0A. The configuration shown in Fig. 3(a) is the same as the forward result shown in Fig. 2(a).



(a) θ_1=60.00°

(b) θ_1=111.17°

(c) θ_1=125.12°

Fig. 3. Configurations of inverse solutions

4 Discussion

Compared with the results presented in Ref. [3], the constraint equations constructed in this paper are polynomials with the highest order about $t_1^2 t_4^2 t_6^2$ which is lower than the highest order of the constraint equation polynomials, $t_1^4 t_4^4 t_6^4$, presented in Ref. [3]. Therefore, the constraint equations established in this paper are much simpler, and can be readily eliminated and solved. They also reduce the number of extraneous roots, which is the main contribution of this paper.

5 Conclusions

(1) A symbolic kinematic solution of the spherical Stephenson-III six-bar mechanism is obtained by dividing the mechanism into a spherical four-bar mechanism and a spherical two-bar unit according to the structural characteristics of the mechanism.

(2) The constraint equations of the mechanism are constructed by utilizing spherical analytic geometry and coordinate transformation theory and solved by Bezout' s elimination method and Sylvester' s resultant elimination method. The solving process is implemented by using the computer symbolic systems Mathematica and Maple respectively.

(3) Numerical examples are presented as an illustration to validate the proposed approach. The mechanism configurations of the forward and inverse solutions are also presented.

References

[1] GOSSELIN C M, CARON F. *Two degree-of-freedom spherical orienting device*[P]. U.S. Patent US596691, 1999.

[2] CHEN C, LI P Y, ERDMAN A G. Design and analysis of a four-degree-of-freedom hybrid Spherical scanning apparatus[C]// *ASME 2000 International Design Engineering Technical Conferences and Computers and information in Engineering Conference*, Baltimore, Maryland. September 10–13, 2000, DETC2000/MECH-14130: l–6.

[3] HEMANDEZ S, BAI S P, ANGELES J. The design of a chain of spherical Stephenson mechanisms for a gearless robotic pitch-roll wrist[J]. *Journal of Mechanical Design*, 2006, 128(2): 422–429.

[4] VALASEK M, ZICHA J, KARASEK M, et al. Hexasphere—redundantly actuated parallel spherical mechanism as a new concept of agile telescope[J]. *Advances in Astronomy*, Vol. 2010, Article ID 348286, 6 pages. doi:10.1155/2010/348286.

[5] MURRAY A P, LAROCHELLE P M. A classification scheme for planar 4R, spherical 4R, and spatial RCCC linkages to facilitate computer animation[C]//*Proceedings of DETC' 98, 1998 ASME Design Engineering Technical Conferences*, Atlanta, Georgia, USA. September 13–16, 1998, DETC98/MECH-5887: 1–7.

[6] RUTH D A, McCARTHY J M. The design of spherical 4R linkages for four specified orientations[J]. *Mechanism and Machine Theory*, 1999, 34(5): 677–692.

[7] RASIM I, ALIZADE, OZGUR K. Analytical synthesis of function generating spherical four-bar mechanism for the five precision points[J]. *Mechanism and Machine Theory*, 2005, 40(7): 863–878.

[8] LEE W T, RUSSELL K, SHEN Q, et al. On adjustable spherical four-bar motion generation for expanded prescribed positions[J]. *Mechanism and Machine Theory*, 2009, 44(1): 247–254.

[9] BAI S, HANSEN M, ANGELES J. A robust forward-displacement analysis of spherical parallel robots[J]. *Mechanism and Machine Theory*, 2009, 44(12): 2 204–2 216.

[10] ENFERADI J, TOOTOONCHI A A. A novel spherical parallel manipulator: forward position problem, singularity analysis, and isotropy design[J]. *Robotica*, 2009, 27(5): 663–676.

[11] ZHANG K, FANG Y, FANG H, et al. Geometry and constraint analysis of the three-spherical kinematic chain based parallel mechanism[J]. *J. Mechanisms Robotics,* August 2010, Vol. 2, Issue 3, 031014(7 pages).

[12] KIM H S, HAMID S, SONI A H. Synthesis of six-link mechanisms for point path generation[J]. *Journal of Mechanisms*, 1971, 6(4): 447–461.

[13] MARIAPPAN J, KRISHNAMURTY S. A generalized exact gradient method for mechanism synthesis[J]. *Mechanism and Machine Theory*, 1996, 31(4): 413–421.

[14] SOH G S, McCARTHY J M. The synthesis of six-bar linkages as constrained planar 3R chains[J]. *Mechanism and Machine Theory*, 2008, 43(2): 160–170.

[15] TING K L, DOU X. Classification and branch identification of Stephenson six-bar chains[J]. *Mechanism and Machine Theory*, 1996, 31(3): 283–295.

[16] TING K L, XUE C, WANG J, et al. Stretch rotation and complete mobility identification of Watt six-bar chains[J]. *Mechanism and Machine Theory*, 2009, 44(10): 1 877–1 886.

[17] MAKHSUDYAN N, DJAVAKHYAN R, ARAKELIAN V. Comparative analysis and synthesis of six-bar mechanisms formed by two serially connected spherical and planar four-bar linkages[J]. *Mechanics Research Communications*, 2009, 36(2): 162–168.

[18] YANG Zhongfu, XU Liju. Synthesis of spherical six-bar path generation mechanisms[J]. *Journal of Sichuan University (Engineering Science Edition)*, 2005, 32(3): 48–50. (in Chinese)

[19] ZHANG Junfu, XU Liju, WANG Jie. Synthesis of adjustable spherical six-bar mechanisms for path generation[J]. *Chinese Journal of Mechanical Engineering*, 2007, 43(11): 50–55. (in Chinese)

[20] SANCISI N, PARENTI-CASTELLI V, BALDISSERRI B, et al. Validation of a one degree-of-freedom spherical model for kinematics analysis of the human ankle joint[J]. *Journal of Foot and Ankle Research*, 2012, 5(Supp. 1): 13.

[21] GREGORIO R. Analytical method for the singularity analysis, and exhaustive enumeration of the singularity conditions in single-DOF spherical mechanisms[J]. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, December 27, 2012, 0(0): 1–11. doi: 10.1177/0954406212469328

[22] ZHAO J, WANG W, GAO Y, et al. Generation of closed-form inverse kinematics for reconfigurable robots[J]. *Chinese Journal of Mechanical Engineering*, 2006, 42(8): 210–214.

[23] WAMPLER C. Displacement analysis of spherical mechanisms having three or fewer loops[J]. *Journal of Mechanical Design*, 2004, 126(1): 93–100.

[24] BAKER J E. Displacement-closure equations of the unspecialised double-Hooke' s-joint linkage[J]. *Mechanism and Machine Theory*, 2002, 37 (10): 1 127–1 144.

[25] CUI L, DAI J S. Axis constraint analysis and its resultant 6R double-centered overconstrained mechanisms[J]. *Journal of Mechanisms and Robotics*, 2011, 3(3): 031004 (9 pages).

[26] BOMBIN C, ROS L, THOMAS F. On the computation of the direct kinematics of parallel spherical mechanisms using Bernstein polynomials[C]//*Proceedings of the 2001 IEEE International Conference on Robotics & Automation*, Seoul, Korea, May 21–26, 2001: 3 332–3 337.

[27] KONG X, GOSSELIN C M. A formula that produces a unique solution to the forward displacement analysis of a quadratic spherical Parallel manipulator: The agile eye[J]. *J. Mechanisms Robotics*, November 2010, 2(4): 044501 (4 pages).

[28] RODRIGUEZ J, RUGGIU M. A novel method for the solution of the forward displacement problem of spherical parallel manipulators[J]. *ZAMM—Journal of Applied Mathematics and Mechanics*, 2013, 93(1): 73–82. doi: 10.1002/zamm.201200059.

[29] JI P, WU H. Algebraic solution to forward kinematics of a 3-DOF spherical parallel manipulator[J]. *Journal of Robotic Systems*, 2011, 18(5): 251–257.
[30] HUANG Z, YAO Y L. A new closed-form kinematics of the generalized 3-DOF spherical parallel manipulator[J]. *Robotica*, 1999, 17(5): 475–485.
[31] DUFFY J. *Analysis of mechanisms and robot manipulators*[M]. New York: John Wiley, 1980.
[32] TSAI M S, SHIAU T N, TSAI Y J, et al. Direct kinematic analysis of a 3-PRS parallel mechanism[J]. *Mechanism and Machine Theory*, 2003, 38(1): 71–83.
[33] EMIRIS Z I, MOURRAIN B. Matrices in elimination theory[J]. *Journal of Symbolic Computation—JSC* , 1999, 28(1–2): 3–44.
[34] WOLFRAM S. *The Mathematica book*[M]. 4th Ed. Cambridge, UK: Cambridge University Press, 1999.

Biographical notes

LIU Yanfang, born in 1986, is currently an engineer at *Dongfang Electric Corporation Limited, China.* She received her master’ s degree on industrial design in *Sichuan University, China*, in 2011. Her research interests are kinematics of spherical mechanisms and mechanical design.
E-mail: yuanzhang198621@163.com

YANG Suixian, born in 1965, is currently a professor at *School of Manufacturing Science and Engineering, Sichuan University, China*. He received his PhD degree from *Sichuan University, China*, in 1997. His research interests include spatial mechanisms, computer aided design, product design and nondestructive testing.
Tel: +86-28-85405309; E-mail: yangsx@163.com

Appendix

Coefficients a, b, d in Eq. (10)–(13) are as follows:

$$\begin{aligned} a_1 = {} & C_{12}C_{25}C_{34}C_{41}C_{56}C_{67} - C_{73} - C_{34}C_{41}C_{56}C_{67}S_{12}S_{25} - \\ & C_1C_{25}C_4C_{41}C_{56}C_{67}S_{12}S_{34} - C_1C_{12}C_4C_{41}C_{56}C_{67}S_{25}S_{34} + \\ & C_{25}C_{56}C_{67}S_1S_{12}S_{34}S_4 + C_{12}C_{56}C_{67}S_1S_{25}S_{34}S_4 - \\ & C_1C_{25}C_{34}C_{56}C_{67}S_{12}S_{41} - C_1C_{12}C_{34}C_{56}C_{67}S_{25}S_{41} - \\ & C_{12}C_{25}C_4C_{56}C_{67}S_{34}S_{41} + C_4C_{56}C_{67}S_{12}S_{25}S_{34}S_{41} + \\ & C_{25}C_{34}C_{41}C_{67}S_{12}S_{56} + C_{12}C_{34}C_{41}C_{67}S_{25}S_{56} + \\ & C_1C_{12}C_{25}C_4C_{41}C_{67}S_{34}S_{56} - C_1C_4C_{41}C_{67}S_{12}S_{25}S_{34}S_{56} - \\ & C_{12}C_{25}C_{67}S_1S_{34}S_4S_{56} + C_{67}S_1S_{12}S_{25}S_{34}S_4S_{56} + \\ & C_1C_{12}C_{25}C_{34}C_{67}S_{41}S_{56} - C_1C_{34}C_{67}S_{12}S_{25}S_{41}S_{56} - \\ & C_{25}C_4C_{67}S_{12}S_{34}S_{41}S_{56} - C_{12}C_4C_{67}S_{25}S_{34}S_{41}S_{56} - \\ & C_{25}C_{34}C_{41}C_{56}S_{12}S_{67} - C_{12}C_{34}C_{41}C_{56}S_{25}S_{67} - \\ & C_1C_{12}C_{25}C_4C_{41}C_{56}S_{34}S_{67} + C_1C_4C_{41}C_{56}S_{12}S_{25}S_{34}S_{67} + \\ & C_{12}C_{25}C_{56}S_1S_{34}S_4S_{67} - C_{56}S_1S_{12}S_{25}S_{34}S_4S_{67} - \\ & C_1C_{12}C_{25}C_{34}C_{56}S_{41}S_{67} + C_1C_{34}C_{56}S_{12}S_{25}S_{41}S_{67} + \\ & C_{25}C_4C_{56}S_{12}S_{34}S_{41}S_{67} + C_{12}C_4C_{56}S_{25}S_{34}S_{41}S_{67} + \\ & C_{12}C_{25}C_{34}C_{41}S_{56}S_{67} - C_{34}C_{41}S_{12}S_{25}S_{56}S_{67} - \\ & C_1C_{25}C_4C_{41}S_{12}S_{34}S_{56}S_{67} - C_1C_{12}C_4C_{41}S_{25}S_{34}S_{56}S_{67} + \\ & C_{25}S_1S_{12}S_{34}S_4S_{56}S_{67} + C_{12}S_1S_{25}S_{34}S_4S_{56}S_{67} - \\ & C_1C_{25}C_{34}S_{12}S_{41}S_{56}S_{67} - C_1C_{12}C_{34}S_{25}S_{41}S_{56}S_{67} - \\ & C_{12}C_{25}C_4S_{34}S_{41}S_{56}S_{67} + C_4S_{12}S_{25}S_{34}S_{41}S_{56}S_{67}, \end{aligned}$$

$$\begin{aligned} a_2 = {} & 2C_4C_{41}C_{67}S_1S_{34}S_{56} + 2C_1C_{67}S_{34}S_4S_{56} + \\ & 2C_{34}C_{67}S_1S_{41}S_{56} - 2C_4C_{41}C_{56}S_1S_{34}S_{67} - \\ & 2C_1C_{56}S_{34}S_4S_{67} - 2C_{34}C_{56}S_1S_{41}S_{67}, \end{aligned}$$

$$\begin{aligned} a_3 = {} & C_{12}C_{25}C_{34}C_{41}C_{56}C_{67} - C_{73} - C_{34}C_{41}C_{56}C_{67}S_{12}S_{25} - \\ & C_1C_{25}C_4C_{41}C_{56}C_{67}S_{12}S_{34} - C_1C_{12}C_4C_{41}C_{56}C_{67}S_{25}S_{34} + \\ & C_{25}C_{56}C_{67}S_1S_{12}S_{34}S_4 + C_{12}C_{56}C_{67}S_1S_{25}S_{34}S_4 - \\ & C_1C_{25}C_{34}C_{56}C_{67}S_{12}S_{41} - C_1C_{12}C_{34}C_{56}C_{67}S_{25}S_{41} - \\ & C_{12}C_{25}C_4C_{56}C_{67}S_{34}S_{41} + C_4C_{56}C_{67}S_{12}S_{25}S_{34}S_{41} - \\ & C_{25}C_{34}C_{41}C_{67}S_{12}S_{56} - C_{12}C_{34}C_{41}C_{67}S_{25}S_{56} - \\ & C_1C_{12}C_{25}C_4C_{41}C_{67}S_{34}S_{56} + C_1C_4C_{41}C_{67}S_{12}S_{25}S_{34}S_{56} + \\ & C_{12}C_{25}C_{67}S_1S_{34}S_4S_{56} - C_{67}S_1S_{12}S_{25}S_{34}S_4S_{56} - \\ & C_1C_{12}C_{25}C_{34}C_{67}S_{41}S_{56} + C_1C_{34}C_{67}S_{12}S_{25}S_{41}S_{56} + \\ & C_{25}C_4C_{67}S_{12}S_{34}S_{41}S_{56} + C_{12}C_4C_{67}S_{25}S_{34}S_{41}S_{56} + \\ & C_{25}C_{34}C_{41}C_{56}S_{12}S_{67} + C_{12}C_{34}C_{41}C_{56}S_{25}S_{67} + \\ & C_1C_{12}C_{25}C_4C_{41}C_{56}S_{34}S_{67} - C_1C_4C_{41}C_{56}S_{12}S_{25}S_{34}S_{67} - \\ & C_{12}C_{25}C_{56}S_1S_{34}S_4S_{67} + C_{56}S_1S_{12}S_{25}S_{34}S_4S_{67} + \\ & C_1C_{12}C_{25}C_{34}C_{56}S_{41}S_{67} - C_1C_{34}C_{56}S_{12}S_{25}S_{41}S_{67} - \\ & C_{25}C_4C_{56}S_{12}S_{34}S_{41}S_{67} - C_{12}C_4C_{56}S_{25}S_{34}S_{41}S_{67} + \\ & C_{12}C_{25}C_{34}C_{41}S_{56}S_{67} - C_{34}C_{41}S_{12}S_{25}S_{56}S_{67} - \\ & C_1C_{25}C_4C_{41}S_{12}S_{34}S_{56}S_{67} - C_1C_{12}C_4C_{41}S_{25}S_{34}S_{56}S_{67} + \\ & C_{25}S_1S_{12}S_{34}S_4S_{56}S_{67} + C_{12}S_1S_{25}S_{34}S_4S_{56}S_{67} - \\ & C_1C_{25}C_{34}S_{12}S_{41}S_{56}S_{67} - C_1C_{12}C_{34}S_{25}S_{41}S_{56}S_{67} - \\ & C_{12}C_{25}C_4S_{34}S_{41}S_{56}S_{67} + C_4S_{12}S_{25}S_{34}S_{41}S_{56}S_{67}, \end{aligned}$$

$$\begin{aligned} a_4 = {} & (-2C_4C_{41}\ S_1S_{34}\ S_{67} - 2C_1\ S_{34}\ S_4\ S_{67} - \\ & 2\,C_{34}\ S_1\ S_{41}\ S_{67}), \end{aligned}$$

$$\begin{aligned} a_5 = {} & 4C_{25}C_{34}C_{41}S_{12}S_{67} + 4C_{12}C_{34}C_{41}S_{25}S_{67} + \\ & 4C_1C_{12}C_{25}C_4C_{41}S_{34}S_{67} - 4C_1C_4C_{41}S_{12}S_{25}S_{34}S_{67} - \\ & 4C_{12}C_{25}S_1S_{34}S_4S_{67} + 4S_1S_{12}S_{25}S_{34}S_4S_{67} + \\ & 4C_1C_{12}C_{25}C_{34}S_{41}S_{67} - 4C_1C_{34}S_{12}S_{25}S_{41}S_{67} - \\ & 4C_{25}C_4S_{12}S_{34}S_{41}S_{67} - 4C_{12}C_4S_{25}S_{34}S_{41}S_{67}, \end{aligned}$$

$$a_6 = 2C_4C_{41}S_1S_{34}S_{67} + 2C_1S_{34}S_4S_{67} + 2C_{34}S_1S_{41}S_{67},$$

$$\begin{aligned} a_7 = {} & C_{12}C_{25}C_{34}C_{41}C_{56}C_{67} - C_{73} - C_{34}C_{41}C_{56}C_{67}S_{12}S_{25} - \\ & C_1C_{25}C_4C_{41}C_{56}C_{67}S_{12}S_{34} - C_1C_{12}C_4C_{41}C_{56}C_{67}S_{25}S_{34} + \\ & C_{25}C_{56}C_{67}S_1S_{12}S_{34}S_4 + C_{12}C_{56}C_{67}S_1S_{25}S_{34}S_4 - \\ & C_1C_{25}C_{34}C_{56}C_{67}S_{12}S_{41} - C_1C_{12}C_{34}C_{56}C_{67}S_{25}S_{41} - \\ & C_{12}C_{25}C_4C_{56}C_{67}S_{34}S_{41} + C_4C_{56}C_{67}S_{12}S_{25}S_{34}S_{41} + \\ & C_{25}C_{34}C_{41}C_{67}S_{12}S_{56} + C_{12}C_{34}C_{41}C_{67}S_{25}S_{56} + \\ & C_1C_{12}C_{25}C_4C_{41}C_{67}S_{34}S_{56} - C_1C_4C_{41}C_{67}S_{12}S_{25}S_{34}S_{56} - \\ & C_{12}C_{25}C_{67}S_1S_{34}S_4S_{56} + C_{67}S_1S_{12}S_{25}S_{34}S_4S_{56} + \\ & C_1C_{12}C_{25}C_{34}C_{67}S_{41}S_{56} - C_1C_{34}C_{67}S_{12}S_{25}S_{41}S_{56} - \\ & C_{25}C_4C_{67}S_{12}S_{34}S_{41}S_{56} - C_{12}C_4C_{67}S_{25}S_{34}S_{41}S_{56} + \\ & C_{25}C_{34}C_{41}C_{56}S_{12}S_{67} + C_{12}C_{34}C_{41}C_{56}S_{25}S_{67} + \\ & C_1C_{12}C_{25}C_4C_{41}C_{56}S_{34}S_{67} - C_1C_4C_{41}C_{56}S_{12}S_{25}S_{34}S_{67} - \end{aligned}$$

$$
\begin{aligned}
&C_{12}C_{25}C_{56}S_1S_{34}S_4S_{67}+C_{56}S_1S_{12}S_{25}S_{34}S_4S_{67}+\\
&C_1C_{12}C_{25}C_{34}C_{56}S_{41}S_{67}-C_1C_{34}C_{56}S_{12}S_{25}S_{41}S_{67}-\\
&C_{25}C_4C_{56}S_{12}S_{34}S_{41}S_{67}-C_{12}C_4C_{56}S_{25}S_{34}S_{41}S_{67}-\\
&C_{12}C_{25}C_{34}C_{41}S_{56}S_{67}+C_{34}C_{41}S_{12}S_{25}S_{56}S_{67}+\\
&C_1C_{25}C_4C_{41}S_{12}S_{34}S_{56}S_{67}+C_1C_{12}C_4C_{41}S_{25}S_{34}S_{56}S_{67}-\\
&C_{25}S_1S_{12}S_{34}S_4S_{56}S_{67}-C_{12}S_1S_{25}S_{34}S_4S_{56}S_{67}+\\
&C_1C_{25}C_{34}S_{12}S_{41}S_{56}S_{67}+C_1C_{12}C_{34}S_{25}S_{41}S_{56}S_{67}+\\
&C_{12}C_{25}C_4S_{34}S_{41}S_{56}S_{67}-C_4S_{12}S_{25}S_{34}S_{41}S_{56}S_{67},
\end{aligned}
$$

$$
\begin{aligned}
a_8=\;&2C_4C_{41}C_{67}S_1S_{34}S_{56}+2C_1C_{67}S_{34}S_4S_{56}+\\
&2C_{34}C_{67}S_1S_{41}S_{56}+2C_4C_{41}C_{56}S_1S_{34}S_{67}+\\
&2C_1C_{56}S_{34}S_4S_{67}+2C_{34}C_{56}S_1S_{41}S_{67},
\end{aligned}
$$

$$
\begin{aligned}
a_9=\;&C_{12}C_{25}C_{34}C_{41}C_{56}C_{67}-C_{73}-C_{34}C_{41}C_{56}C_{67}S_{12}S_{25}-\\
&C_1C_{25}C_4C_{41}C_{56}C_{67}S_{12}S_{34}-C_1C_{12}C_4C_{41}C_{56}C_{67}S_{25}S_{34}+\\
&C_{25}C_{56}C_{67}S_1S_{12}S_{34}S_4+C_{12}C_{56}C_{67}S_1S_{25}S_{34}S_4-\\
&C_1C_{25}C_{34}C_{56}C_{67}S_{12}S_{41}-C_1C_{12}C_{34}C_{56}C_{67}S_{25}S_{41}-\\
&C_{12}C_{25}C_4C_{56}C_{67}S_{34}S_{41}+C_4C_{56}C_{67}S_{12}S_{25}S_{34}S_{41}-\\
&C_{25}C_{34}C_{41}C_{67}S_{12}S_{56}-C_{12}C_{34}C_{41}C_{67}S_{25}S_{56}-\\
&C_1C_{12}C_{25}C_4C_{41}C_{67}S_{34}S_{56}+C_1C_4C_{41}C_{67}S_{12}S_{25}S_{34}S_{56}+\\
&C_{12}C_{25}C_{67}S_1S_{34}S_4S_{56}-C_{67}S_1S_{12}S_{25}S_{34}S_4S_{56}-\\
&C_1C_{12}C_{25}C_{34}C_{67}S_{41}S_{56}+C_1C_{34}C_{67}S_{12}S_{25}S_{41}S_{56}+\\
&C_{25}C_4C_{67}S_{12}S_{34}S_{41}S_{56}+C_{12}C_4C_{67}S_{25}S_{34}S_{41}S_{56}-\\
&C_{25}C_{34}C_{41}C_{56}S_{12}S_{67}-C_{12}C_{34}C_{41}C_{56}S_{25}S_{67}-\\
&C_1C_{12}C_{25}C_4C_{41}C_{56}S_{34}S_{67}+C_1C_4C_{41}C_{56}S_{12}S_{25}S_{34}S_{67}+\\
&C_{12}C_{25}C_{56}S_1S_{34}S_4S_{67}-C_{56}S_1S_{12}S_{25}S_{34}S_4S_{67}-\\
&C_1C_{12}C_{25}C_{34}C_{56}S_{41}S_{67}+C_1C_{34}C_{56}S_{12}S_{25}S_{41}S_{67}+\\
&C_{25}C_4C_{56}S_{12}S_{34}S_{41}S_{67}+C_{12}C_4C_{56}S_{25}S_{34}S_{41}S_{67}-\\
&C_{12}C_{25}C_{34}C_{41}S_{56}S_{67}+C_{34}C_{41}S_{12}S_{25}S_{56}S_{67}+\\
&C_1C_{25}C_4C_{41}S_{12}S_{34}S_{56}S_{67}+C_1C_{12}C_4C_{41}S_{25}S_{34}S_{56}S_{67}-\\
&C_{25}S_1S_{12}S_{34}S_4S_{56}S_{67}-C_{12}S_1S_{25}S_{34}S_4S_{56}S_{67}+\\
&C_1C_{25}C_{34}S_{12}S_{41}S_{56}S_{67}+C_1C_{12}C_{34}S_{25}S_{41}S_{56}S_{67}+\\
&C_{12}C_{25}C_4S_{34}S_{41}S_{56}S_{67}-C_4S_{12}S_{25}S_{34}S_{41}S_{56}S_{67};
\end{aligned}
$$

$$
\begin{aligned}
b_1=\;&C_{12}C_{25}C_{41}C_{56}C_{67}-C_{34}C_{73}-C_{41}C_{56}C_{67}S_{12}S_{25}-\\
&C_1C_{25}C_{56}C_{67}S_{12}S_{41}-C_1C_{12}C_{56}C_{67}S_{25}S_{41}+\\
&C_{25}C_{41}C_{67}S_{12}S_{56}+C_{12}C_{41}C_{67}S_{25}S_{56}+\\
&C_1C_{12}C_{25}C_{67}S_{41}S_{56}-C_1C_{67}S_{12}S_{25}S_{41}S_{56}-\\
&C_{25}C_{41}C_{56}S_{12}S_{67}-C_{12}C_{41}C_{56}S_{25}S_{67}-\\
&C_1C_{12}C_{25}C_{56}S_{41}S_{67}+C_1S_{12}S_{25}S_{41}S_{67}+\\
&C_{12}C_{25}C_{41}S_{56}S_{67}-C_{41}S_{12}S_{25}S_{56}S_{67}-\\
&C_1C_{25}S_{12}S_{41}S_{56}S_{67}-C_1C_{12}S_{25}S_{41}S_{56}S_{67}-C_\sigma S_{34}S_{73},
\end{aligned}
$$

$$
b_2=2C_{67}S_1S_{41}S_{56}-2C_{56}S_1S_{41}S_{67},
$$

$$
\begin{aligned}
b_3=\;&C_{12}C_{25}C_{41}C_{56}C_{67}-C_{34}C_{73}-C_{41}C_{56}C_{67}S_{12}S_{25}-\\
&C_1C_{25}C_{56}C_{67}S_{12}S_{41}-C_1C_{12}C_{56}C_{67}S_{25}S_{41}-\\
&C_{25}C_{41}C_{67}S_{12}S_{56}-C_{12}C_{41}C_{67}S_{25}S_{56}-\\
&C_1C_{12}C_{25}C_{67}S_{41}S_{56}+C_1C_{67}S_{12}S_{25}S_{41}S_{56}+\\
&C_{25}C_{41}C_{56}S_{12}S_{67}+C_{12}C_{41}C_{56}S_{25}S_{67}+\\
&C_1C_{12}C_{25}C_{56}S_{41}S_{67}-C_1S_{12}S_{25}S_{41}S_{67}+\\
&C_{12}C_{25}C_{41}S_{56}S_{67}-C_{41}S_{12}S_{25}S_{56}S_{67}-\\
&C_1C_{25}S_{12}S_{41}S_{56}S_{67}-C_1C_{12}S_{25}S_{41}S_{56}S_{67}-C_\sigma S_{34}S_{73},
\end{aligned}
$$

$$
b_4=-2S_1S_{41}S_{67},
$$

$$
\begin{aligned}
b_5=\;&4C_{25}C_{41}S_{12}S_{67}+4C_{12}C_{41}S_{25}S_{67}+4C_1C_{12}C_{25}S_{41}S_{67}-\\
&4C_1S_{12}S_{25}S_{41}S_{67},
\end{aligned}
$$

$$
b_6=2S_1S_{41}S_{67},
$$

$$
\begin{aligned}
b_7=\;&C_{12}C_{25}C_{41}C_{56}C_{67}-C_{34}C_{73}-C_{41}C_{56}C_{67}S_{12}S_{25}-\\
&C_1C_{25}C_{56}C_{67}S_{12}S_{41}-C_1C_{12}C_{56}C_{67}S_{25}S_{41}+\\
&C_{25}C_{41}C_{67}S_{12}S_{56}+C_{12}C_{41}C_{67}S_{25}S_{56}+\\
&C_1C_{12}C_{25}C_{67}S_{41}S_{56}-C_1C_{67}S_{12}S_{25}S_{41}S_{56}+\\
&C_{25}C_{41}C_{56}S_{12}S_{67}+C_{12}C_{41}C_{56}S_{25}S_{67}+\\
&C_1C_{12}C_{25}C_{56}S_{41}S_{67}-C_1S_{12}S_{25}S_{41}S_{67}-C_{12}C_{25}C_{41}S_{56}S_{67}+\\
&C_{41}S_{12}S_{25}S_{56}S_{67}+C_1C_{25}S_{12}S_{41}S_{56}S_{67}+\\
&C_1C_{12}S_{25}S_{41}S_{56}S_{67}-C_\sigma S_{34}S_{73},
\end{aligned}
$$

$$
b_8=2C_{67}S_1S_{41}S_{56}+2C_{56}S_1S_{41}S_{67},
$$

$$
\begin{aligned}
b_9=\;&C_{12}C_{25}C_{41}C_{56}C_{67}-C_{34}C_{73}-C_{41}C_{56}C_{67}S_{12}S_{25}-\\
&C_1C_{25}C_{56}C_{67}S_{12}S_{41}-C_1C_{12}C_{56}C_{67}S_{25}S_{41}-\\
&C_{25}C_{41}C_{67}S_{12}S_{56}-C_{12}C_{41}C_{67}S_{25}S_{56}-\\
&C_1C_{12}C_{25}C_{67}S_{41}S_{56}+C_1C_{67}S_{12}S_{25}S_{41}S_{56}-\\
&C_{25}C_{41}C_{56}S_{12}S_{67}-C_{12}C_{41}C_{56}S_{25}S_{67}-\\
&C_1C_{12}C_{25}C_{56}S_{41}S_{67}+C_1S_{12}S_{25}S_{41}S_{67}-\\
&C_{12}C_{25}C_{41}S_{56}S_{67}+C_{41}S_{12}S_{25}S_{56}S_{67}+\\
&C_1C_{25}S_{12}S_{41}S_{56}S_{67}+C_1C_{12}S_{25}S_{41}S_{56}S_{67}-C_\sigma S_{34}S_{73};
\end{aligned}
$$

$$
\begin{aligned}
d_1=\;&a_7^2b_1^2-a_4a_7b_1b_4+a_1a_7b_4^2+a_4^2b_1b_7-\\
&2\,a_1a_7b_1b_7-a_1a_4b_4b_7+a_1^2b_7^2,
\end{aligned}
$$

$$
\begin{aligned}
d_2=\;&2\,a_7a_8b_1^2+2\,a_7^2b_1b_2-a_5a_7b_1b_4-a_4a_8b_1b_4-\\
&a_4a_7b_2b_4+a_2a_7b_4^2+a_1a_8b_4^2-a_4a_7b_1b_5+\\
&2\,a_1a_7b_4b_5+2\,a_4a_5b_1b_7-2\,a_2a_7b_1b_7-2\,a_1a_8b_1b_7+\\
&a_4^2b_2b_7-2\,a_1a_7b_2b_7-a_2a_4b_4b_7-a_1a_5b_4b_7-\\
&a_1a_4b_5b_7+2\,a_1a_2b_7^2+a_4^2b_1b_8-2\,a_1a_7b_1b_8-\\
&a_1a_4b_4b_8+2\,a_1^2b_7b_8,
\end{aligned}
$$

$$
\begin{aligned}
d_3=\;&a_8^2b_1^2+2\,a_7a_9b_1^2+4\,a_7a_8b_1b_2+a_7^2b_2^2+\\
&2\,a_7^2b_1b_3-a_6a_7b_1b_4-a_5a_8b_1b_4-a_4a_9b_1b_4-\\
&a_5a_7b_2b_4-a_4a_8b_2b_4-a_4a_7b_3b_4+a_3a_7b_4^2+\\
&a_2a_8b_4^2+a_1a_9b_4^2-a_5a_7b_1b_5-a_4a_8b_1b_5-a_4a_7b_2b_5+\\
&2\,a_2a_7b_4b_5+2\,a_1a_8b_4b_5+a_1a_7b_5^2-a_4a_7b_1b_6+\\
&2\,a_1a_7b_4b_6+a_5^2b_1b_7+2\,a_4a_6b_1b_7-2\,a_3a_7b_1b_7-
\end{aligned}
$$

$$
\begin{aligned}
&2\,a_2a_8b_1b_7-2\,a_1a_9b_1b_7+2\,a_4a_5b_2b_7-2\,a_2a_7b_2b_7-\\
&2\,a_1a_8b_2b_7+a_4^2b_3b_7-2\,a_1a_7b_3b_7-a_3a_4b_4b_7-\\
&a_2a_5b_4b_7-a_1a_6b_4b_7-a_2a_4b_5b_7-a_1a_5b_5b_7-a_1a_4b_6b_7+\\
&a_2^2b_7^2+2\,a_1a_3b_7^2+2\,a_4a_5b_1b_8-2\,a_2a_7b_1b_8-\\
&2\,a_1a_8b_1b_8+a_4^2b_2b_8-2\,a_1a_7b_2b_8-a_2a_4b_4b_8-a_1a_5b_4b_8-\\
&a_1a_4b_5b_8+4\,a_1a_2b_7b_8+a_1^2b_8^2+a_4^2b_1b_9-2\,a_1a_7b_1b_9-\\
&a_1a_4b_4b_9+2\,a_1^2b_7b_9,
\end{aligned}
$$

$$
\begin{aligned}
d_4=\;&2\,a_8a_9b_1^2+2\,a_8^2b_1b_2+4a_7a_9b_1b_2+2\,a_7a_8b_2^2+\\
&4a_7a_8b_1b_3+2\,a_7^2b_2b_3-a_6a_8b_1b_4-a_5a_9b_1b_4-\\
&a_6a_7b_2b_4-a_5a_8b_2b_4-a_4a_9b_2b_4-a_5a_7b_3b_4-\\
&a_4a_8b_3b_4+a_3a_8b_4^2+a_2a_9b_4^2-a_6a_7b_1b_5-a_5a_8b_1b_5-\\
&2\,a_3a_7b_4b_5+a_4a_9b_1b_5-a_5a_7b_2b_5-a_4a_8b_2b_5-a_4a_7b_3b_5+\\
&2a_2a_8b_4b_5+2a_1a_9b_4b_5+a_2a_7b_5^2+a_1a_8b_5^2-a_5a_7b_1b_6-\\
&a_4a_8b_1b_6-a_4a_7b_2b_6+2\,a_2a_7b_4b_6+2a_1a_8b_4b_6+\\
&2a_1a_7b_5b_6+2\,a_5a_6b_1b_7-2\,a_3a_8b_1b_7-2\,a_2a_9b_1b_7+\\
&a_5^2b_2b_7+2\,a_4a_6b_2b_7-2\,a_3a_7b_2b_7-2a_2a_8b_2b_7-\\
&2\,a_1a_9b_2b_7+2\,a_4a_5b_3b_7-2\,a_2a_7b_3b_7-2\,a_1a_8b_3b_7-\\
&a_3a_5b_4b_7-a_2a_6b_4b_7-a_3a_4b_5b_7-a_2a_5b_5b_7-a_1a_6b_5b_7-\\
&a_2a_4b_6b_7-a_1a_5b_6b_7+2a_2a_3b_7^2+a_5^2b_1b_8+2a_4a_6b_1b_8-\\
&2a_3a_7b_1b_8-2\,a_2a_8b_1b_8-2\,a_1a_9b_1b_8+2\,a_4a_5b_2b_8-\\
&2\,a_2a_7b_2b_8-2\,a_1a_8b_2b_8+a_4^2b_3b_8-2\,a_1a_7b_3b_8-\\
&a_3a_4b_4b_8-a_2a_5b_4b_8-a_1a_6b_4b_8-a_2a_4b_5b_8-a_1a_5b_5b_8-\\
&a_1a_4b_6b_8+2\,a_2^2b_7b_8+4a_1a_3b_7b_8+2a_1a_2b_8^2+\\
&2\,a_4a_5b_1b_9-2\,a_2a_7b_1b_9-2\,a_1a_8b_1b_9+a_4^2b_2b_9-\\
&2\,a_1a_7b_2b_9-a_2a_4b_4b_9-a_1a_5b_4b_9-a_1a_4b_5b_9+\\
&4\,a_1a_2b_7b_9+2\,{a_1}^2\,b_8b_9,
\end{aligned}
$$

$$
\begin{aligned}
d_5=\;&a_9^3b_1^2+4\,a_8a_9b_1b_2+a_8^2b_2^2+2\,a_7a_9b_2^2+\\
&2\,a_8^2b_1b_3+4\,a_7a_9b_1b_3+4\,a_7a_8b_2b_3+a_7^2b_3^2-\\
&a_6a_9b_1b_4-a_6a_8b_2b_4-a_5a_9b_2b_4-a_6a_7b_3b_4-a_5a_8b_3b_4-\\
&a_4a_9b_3b_4+a_3a_9b_4^2-a_6a_8b_1b_5-a_5a_9b_1b_5-a_6a_7b_2b_5-\\
&a_5a_8b_2b_5-a_4a_9b_2b_5-a_5a_7b_3b_5-a_4a_8b_3b_5+2\,a_3a_8b_4b_5+\\
&2\,a_2a_9b_4b_5+a_3a_7b_5^2+a_2a_8b_5^2+a_1a_9b_5^2-a_6a_7b_1b_6-\\
&a_5a_8b_1b_6-a_4a_9b_1b_6-a_5a_7b_2b_6-a_4a_8b_2b_6-a_4a_7b_3b_6+\\
&2a_3a_7b_4b_6+2a_2a_8b_4b_6+2a_1a_9b_4b_6+2a_2a_7b_5b_6+\\
&2a_1a_8b_5b_6+a_1a_7b_6^2+a_6^2b_1b_7-2\,a_3a_9b_1b_7+2\,a_5a_6b_2b_7-\\
&2\,a_3a_8b_2b_7-2\,a_2a_9b_2b_7+a_5^2b_3b_7+2\,a_4a_6b_3b_7-\\
&2\,a_3a_7b_3b_7-2\,a_2a_8b_3b_7-2a_1a_9b_3b_7-a_3a_6b_4b_7-\\
&a_3a_5b_5b_7-a_2a_6b_5b_7-a_3a_4b_6b_7-a_2a_5b_6b_7-a_1a_6b_6b_7+\\
&a_3^2b_7^2+2a_5a_6b_1b_8-2\,a_3a_8b_1b_8-2\,a_2a_9b_1b_8+{a_5}^2b_2b_8+\\
&2a_4a_6b_2b_8-2a_3a_7b_2b_8-2a_2a_8b_2b_8-2a_1a_9b_2b_8+\\
&2a_4a_5b_3b_8-2\,a_2a_7b_3b_8-2a_1a_8b_3b_8-a_3a_5b_4b_8-\\
&a_2a_6b_4b_8-a_3a_4b_5b_8-a_2a_5b_5b_8-a_1a_6b_5b_8-a_2a_4b_6b_8-\\
&a_1a_5b_6b_8+4\,a_2a_3b_7b_8+a_2^2b_8^2+2\,a_1a_3b_8^2+a_5^2b_1b_9+\\
&2\,a_4a_6b_1b_9-2\,a_3a_7b_1b_9-2\,a_2a_8b_1b_9-2\,a_1a_9b_1b_9+
\end{aligned}
$$

$$
\begin{aligned}
&2\,a_4a_5b_2b_9-2a_2a_7b_2b_9-2a_1a_8b_2b_9+a_4^2b_3b_9-\\
&2a_1a_7b_3b_9-a_3a_4b_4b_9-a_2a_5b_4b_9-a_1a_6b_4b_9-a_2a_4b_5b_9-\\
&a_1a_5b_5b_9-a_1a_4b_6b_9+2\,a_2^2b_7b_9+4\,a_1a_3b_7b_9+4\,a_1a_2b_8b_9\\
&+{a_1}^2b_9^2,
\end{aligned}
$$

$$
\begin{aligned}
d_6=\;&2\,a_9^2b_1b_2+2\,a_8a_9b_2^2+4\,a_8a_9b_1b_3+2\,a_8^2b_2b_3+\\
&4\,a_7a_9b_2b_3+2\,a_7a_8b_3^2-a_6a_9b_2b_4-a_6a_8b_3b_4-\\
&a_5a_9b_3b_4-a_6a_9b_1b_5-a_6a_8b_2b_5-a_5a_9b_2b_5-a_6a_7b_3b_5-\\
&a_5a_8b_3b_5-a_4a_9b_3b_5+2\,a_3a_9b_4b_5+a_3a_8b_5^2+a_2a_9b_5^2-\\
&a_6a_8b_1b_6-a_5a_9b_1b_6-a_6a_7b_2b_6-a_5a_8b_2b_6-a_4a_9b_2b_6-\\
&a_5a_7b_3b_6-a_4a_8b_3b_6+2\,a_3a_8b_4b_6+2\,a_2a_9b_4b_6+\\
&2\,a_3a_7b_5b_6+2\,a_2a_8b_5b_6+2\,a_1a_9b_5b_6+a_2a_7b_6^2+\\
&a_1a_8b_6^2+a_6^2b_2b_7-2\,a_3a_9b_2b_7+2a_5a_6b_3b_7-2\,a_3a_8b_3b_7-\\
&2a_2a_9b_3b_7-a_3a_6b_5b_7-a_3a_5b_6b_7-a_2a_6b_6b_7+a_6^2b_1b_8-\\
&2a_3a_9b_1b_8+2\,a_5a_6b_2b_8-2\,a_3a_8b_2b_8-2\,a_2a_9b_2b_8+\\
&a_5^2b_3b_8+2a_4a_6b_3b_8-2a_3a_7b_3b_8-2a_2a_8b_3b_8-\\
&2a_1a_9b_3b_8-a_3a_6b_4b_8-a_3a_5b_5b_8-a_2a_6b_5b_8-a_3a_4b_6b_8-\\
&a_2a_5b_6b_8-a_1a_6b_6b_8+2a_3^2b_7b_8+2\,a_2a_3b_8^2+2\,a_5a_6b_1b_9-\\
&2\,a_3a_8b_1b_9-2\,a_2a_9b_1b_9+a_5^2b_2b_9+2\,a_4a_6b_2b_9-\\
&2\,a_3a_7b_2b_9-2\,a_2a_8b_2b_9-2\,a_1a_9b_2b_9+2\,a_4a_5b_3b_9-\\
&2a_2a_7b_3b_9-2\,a_1a_8b_3b_9-a_3a_5b_4b_9-a_2a_6b_4b_9-\\
&a_3a_4b_5b_9-a_2a_5b_5b_9-a_1a_6b_5b_9-a_2a_4b_6b_9-a_1a_5b_6b_9+\\
&4a_2a_3b_7b_9+2\,a_2^2b_8b_9+4\,a_1a_3b_8b_9+2\,a_1a_2{b_9}^2,
\end{aligned}
$$

$$
\begin{aligned}
d_7=\;&a_9^2b_2^2+2\,a_9^2b_1b_3+4\,a_8a_9b_2b_3+a_8^2b_3^2+\\
&2\,a_7a_9b_3^2-a_6a_9b_3b_4-a_6a_9b_2b_5-a_6a_8b_3b_5-\\
&a_5a_9b_3b_5+a_3a_9b_5^2-a_6a_9b_1b_6-a_6a_8b_2b_6-a_5a_9b_2b_6-\\
&a_6a_7b_3b_6-a_5a_8b_3b_6-a_4a_9b_3b_6+2a_3a_9b_4b_6+\\
&2\,a_3a_8b_5b_6+2\,a_2a_9b_5b_6+a_3a_7b_6^2+a_2a_8b_6^2+\\
&a_1a_9b_6^2+a_6^2b_3b_7-2\,a_3a_9b_3b_7-a_3a_6b_6b_7+a_6^2b_2b_8-\\
&2a_3a_9b_2b_8+2a_5a_6b_3b_8-2\,a_3a_8b_3b_8-2\,a_2a_9b_3b_8-\\
&a_3a_6b_5b_8-a_3a_5b_6b_8-a_2a_6b_6b_8+a_3^2b_3^2+a_6^2b_1b_9-\\
&2\,a_3a_9b_1b_9+2\,a_5a_6b_2b_9-2\,a_3a_8b_2b_9-2\,a_2a_9b_2b_9+\\
&a_5^2b_3b_9+2a_4a_6b_3b_9-2a_3a_7b_3b_9-2\,a_2a_8b_3b_9-\\
&2\,a_1a_9b_3b_9-a_3a_6b_4b_9-a_3a_5b_5b_9-a_2a_6b_5b_9-\\
&a_3a_4b_6b_9-a_2a_5b_6b_9-a_1a_6b_6b_9+2\,a_3^2b_7b_9+\\
&4\,a_2a_3b_8b_9+a_2^2b_9^2+2\,a_1a_3b_9^2,
\end{aligned}
$$

$$
\begin{aligned}
d_8=\;&2a_9^2b_2b_3+2a_8a_9b_3^2-a_6a_9b_3b_5-a_6a_9b_2b_6-\\
&a_6a_8b_3b_6-a_5a_9b_3b_6+2a_3a_9b_5b_6+a_3a_8b_6^2+\\
&a_2a_9b_6^2+a_6^2b_3b_8-2a_3a_9b_3b_8-a_3a_6b_6b_8+\\
&a_6^2b_2b_9-2\,a_3a_9b_2b_9+2\,a_5a_6b_3b_9-2\,a_3a_8b_3b_9-\\
&2\,a_2a_9b_3b_9-a_3a_6b_5b_9-a_3a_5b_6b_9-a_2a_6b_6b_9+\\
&2\,a_3^2b_8b_9+2\,a_2a_3b_9^2,
\end{aligned}
$$

$$
\begin{aligned}
d_9=\;&a_9^2b_3^2-a_6a_9b_3b_6+a_3a_9b_6^2+a_6^2b_3b_9-\\
&2\,a_3a_9b_3b_9-a_3a_6b_6b_9+a_3^2b_9^2.
\end{aligned}
$$