ISSN 8756-6990, Optoelectronics, Instrumentation and Data Processing, 2017, Vol. 53, No. 3, pp. 230–236. © Allerton Press, Inc., 2017.
Original Russian Text © V.M. Artyushenko, V.I. Volovach, 2017, published in Avtometriya, 2017, Vol. 53, No. 3, pp. 36–43.

ANALYSIS AND SYNTHESIS OF SIGNALS AND IMAGES

Identification of the Distribution Parameters of Additive and Multiplicative Non-Gaussian Noise

V. M. Artyushenko[a*] **and V. I. Volovach**[b**]

[a] *University of Technology, ul. Gagarina 42, Korolev, 141070, Moscow region, Russia*
[b] *Volga Region State University of Service, ul. Gagarina 4, Toliatti, 445017 Russia*
[*] *E-mail: artuschenko@mail.ru*
[**] *E-mail: volovach.vi@mail.ru*

Received December 28, 2016

Abstract—This paper considers issues related to the identification of the parameters and form of the probability density function of generally non-Gaussian additive and multiplicative noise affecting the signal. The results of numerical simulation of methods for estimating the information parameters of random processes with a non-Gaussian probability density function for a finite sample.

Keywords: distribution parameters, non-Gaussian probability density function, additive noise, multiplicative noise.

DOI: 10.3103/S8756699017030050

INTRODUCTION

In most cases, the demodulation (filtering) algorithms for the information processes of useful signals have been obtained assuming that the types and parameters of the probability density function (PDF) of additive n or multiplicative η noise are exactly known [1, 2]. In practice, the situation often arises where a priori information on the noise PDF and parameters are known partially or completely absent [3]. Typically, information on noise PDFs have a general nature, e.g., only the class $A_i\{W_{\mathrm{n}}(\mathrm{n})\}$ ($\mathrm{n} = n, \eta$) to which distribution of belongs is known. In addition, during operation of radio systems, not only the characteristics of information processes but also disturbances can change, which leads to unsatisfactory results when using synthesized algorithms.

The aim of this work is to identify the distribution parameters of the non-Gaussian noise affecting the processed signal under a priori uncertainty conditions.

Currently, there are two most widely used approaches to the synthesis of demodulators (meters) under a priori uncertainty: adaptive (self-adjusting) [4] and robust (stable, steady) [5]. The first approach uses iterative techniques to estimate the unknown random parameters of noise during operation of the demodulator. In the second approach, values are specified by a set of possible PDFs $W_{\mathrm{n}}(\mathrm{n})$, and not by a particular density probability function $\{W_{\mathrm{n}}(\mathrm{n})\}_i$. In this case, instead of a particular accuracy characteristic of the synthesized demodulator (for a corresponding PDF), a guaranteed characteristic due to the worst PDF $W^*_{\mathrm{n}}(\mathrm{n}) \in A_i = \{W_{\mathrm{n}}(\mathrm{n})\}$ is used. In this case, the criterion of the minimum of the a posteriori variance σ^2_ε of the estimated parameter guaranteed on the set A_i can serve as an optimality criteria.

Note that there is no general method for solving problems with an arbitrarily specified set A_i. Only particular cases of efficient numerical algorithms based on a coarsened maximum likelihood method have been developed [3].

IDENTIFICATION OF THE NON-GAUSSIAN NOISE DISTRIBUTION FORM

An important issue in the synthesis of demodulators under a priori uncertainty conditions is the identification of the noise distribution form. There are various methods for identifying the distribution of random processes. In particular, the skewness ($K_{\rm s}$) and kurtosis ($K_{\rm k}$) coefficients have been proposed [6] as characteristics of the PDF form. However, if the noise probability density function is symmetric, then $K_{\rm s} = 0$, and $K_{\rm k}$ remains the only information characteristics. In this case, it is efficient to use of the entropy coefficient of the PDF $K_{\rm en} = \sigma^{-1}\Delta_{\rm en} = (2\sigma)^{-1}\exp\{H(n)\}$, where σ is the standard deviation (SD); $\Delta_{\rm en}$ is the entropy value of the error; $H(n) = -\int\limits_{-\infty}^{\infty} W_{\rm n}({\rm n})\ln W_{\rm n}({\rm n})d{\rm n}$ is the entropy of the PDF.

Note that for all distribution laws, the value of $K_{\rm en}$ is within 0–2.066, with the maximum value $K_{\rm en} = 2.066$ having a Gaussian distribution.

As a second characteristic of the PDF form, instead of the kurtosis coefficient $K_{\rm k}$, which varies from 1 to ∞, it is convenient to use the counterkurtosis $K_{\rm an} = K_{\rm k}^{-0.5}$, which varies from 0 to 1.

Using the introduced characteristics, any symmetric probability density function can be represented by a point in the system of the coordinates ($K_{\rm en}$ and $K_{\rm an}$). The proposed representation of analytical models of symmetric PDFs as points in the plane of the characteristics ($K_{\rm en}$ and $K_{\rm an}$) provides a fairly accurate and reliable characterization of the closeness of the points corresponding to the extreme PDFs to one model or another.

It should be noted that the parameters $K_{\rm en}$ and $K_{\rm an}$ of the given analytical distribution are found uniquely. The reverse transition is not unique since a sheaf of curves corresponding to PDFs of different classes can pass through a topographic point with given coordinates $(K_{\rm en}, K_{\rm an})$, which is the main disadvantage of the proposed systematization and classification of PDFs according to their form.

In the case of one-sided probability density functions, characteristic of, e.g., the distributions of envelopes of narrowband random processes (multiplicative noise η), the skewness coefficient should be added to the above quantities. In this case, the distribution $W(\eta)$ being estimated is assigned a point (or a region in the case of multiparameter PDFs) in the space of $(K_{\rm s}, K_{\rm en}, K_{\rm an})$, and not in the plane of $(K_{\rm en}, K_{\rm an})$.

Recursive procedures, requiring much less computer memory than non-recursive (a posteriori) algorithms, are widely used to obtain current estimates of numerical characteristics of random processes.

Recursive estimates of the initial moments of the ith order m_i in sampling y_h, $h = \overline{1,H}$, have the form

$$\hat m_h^{(i)} = \hat m_{h-1}^{(i)} + h^{-1}(y_h - \hat m_{h-1}^{(i)}), \quad m_0 = 0, \;\; h = \overline{1,H}. \tag{1}$$

If the expectation of a random process m_y is known, the estimate of the variance (the second central moment M_2) is given by the formula

$$\hat M_{2h} = \hat M_{2h-1} + h^{-1}((y_h - m_y)^2 - \hat M_{2h-1}). \tag{2}$$

In this case, the third and fourth central moments are given by the expressions

$$\begin{aligned} \hat M_{3h} &= \hat M_{3h-1} + h^{-1}((y_h - m_y)^3 - \hat M_{3h-1}); \\ \hat M_{4h} &= \hat M_{4h-1} + h^{-1}((y_h - m_y)^4 - \hat M_{4h-1}). \end{aligned} \tag{3}$$

Using (1)–(3), we obtain the following recursive relations for the skewness and kurtosis, respectively:

$$K_{{\rm s}h} = \hat M_{3h}\hat M_{2h}^{-1.5}; \quad K_{{\rm k}h} = \hat M_{4h}\hat M_{2h}^{-2}. \tag{4}$$

When implementing these algorithms in cases where the maximum number of measurements H is not determined beforehand and is specified by external conditions, it is necessary to formulate a criterion for stopping the estimation procedure and terminating the calculation of these parameters.

In particular, to stop the calculation procedure in accordance with algorithms (1)–(4), the number of measurements H required to provide accuracy can be evaluated at each step and compared with the number of measurements. As the criterion it is common to use the inequality $|\hat\lambda_h - \hat\lambda_{h-1}| \le \delta$, where $\hat\lambda_h$ is the estimate of the measured parameter λ in step h and δ is the permissible measurement error.

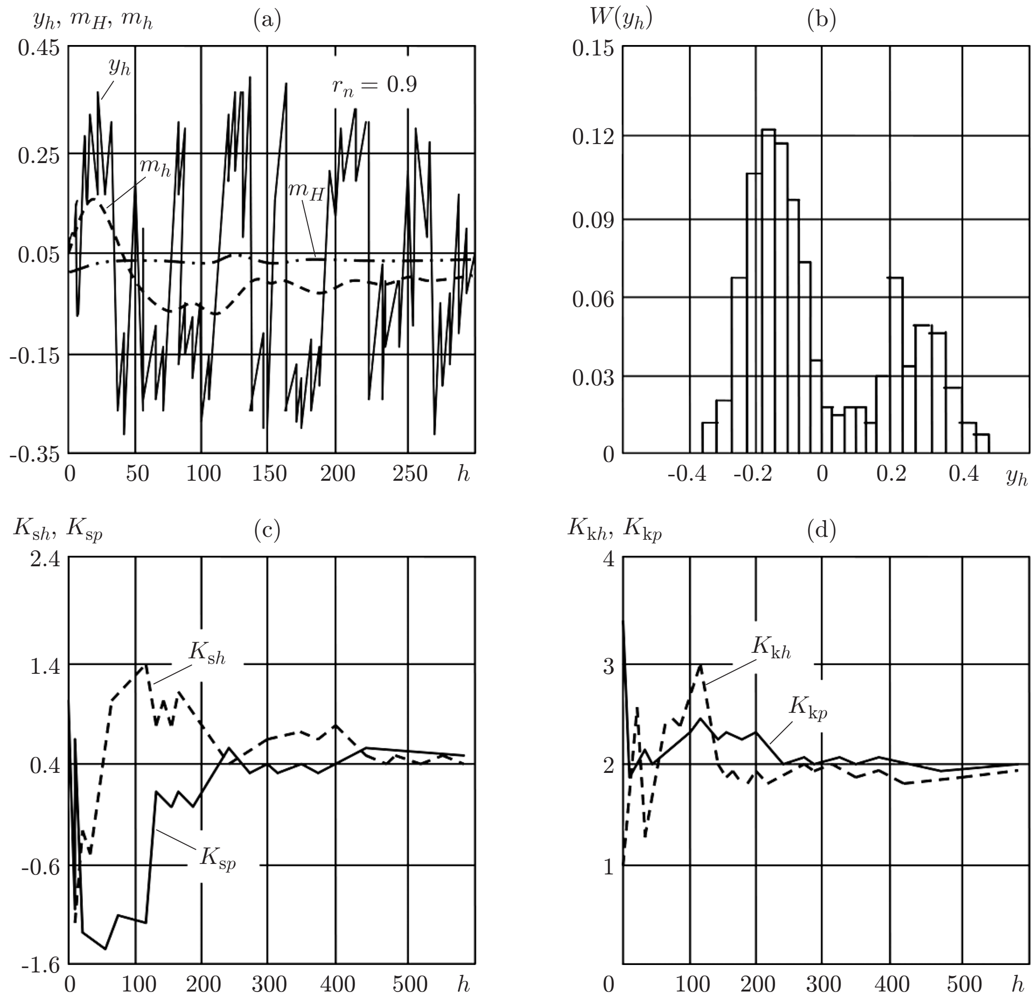

Fig. 1. Numerical simulation results of the statistical characteristics of a random process with a bimodal PDF: (a) is the expectation and variance of the fragment of the simulated process; (b) is the PDF histogram of instantaneous values of the simulated process; (c) is the dependence of the skewness coefficient on the iteration step; (d) is the dependence of the kurtosis coefficient on the iteration step.

Figure 1 shows the results of numerical simulation of the statistical characteristics of a random process y_h with a bimodal PDF of instantaneous values depending on the iteration step h, where $m_H = h^{-1}\Big(\sum\limits_{h=1}^{H} y_h\Big)$. The solid curves in the figure are calculated by formulas (4), and the dashed curves were obtained by non-recursive formulas using a finite sample.

As can be seen from the graphs in Figs. 1c and 1d, the solid and dashed curves converge with increasing number of samples. This indicates the identity of the estimation procedures using the recursive and non-recursive algorithms.

Note that the PDF histogram of instantaneous values can be used to calculate the entropy coefficient $K_{\text{en}} = (2\sigma)^{-1}(bN)10^{-\frac{1}{N}\sum\limits_{i=1}^{m} n_i \ln n_i}$. Here b is the width of the histogram column, N is the sample size, m is the number of histogram columns, n_i is the number of samples in the ith column.

Below, we give formulas for estimating the spread of the SD estimates of the counterkurtosis and entropy coefficient as a function of the sample size and kurtosis of the distribution.

The sample variance and standard deviation for $N > 20$ with an error of 10 % can be defined as $D(D^*) = N^{-1}(M_4 - \sigma^4)$, $\sigma(\sigma^*) = (2\sigma N)^{-1}(M_4 - \sigma^4)^{0.5}$; here σ^2 and M_4 are the second and fourth central moments of the population.

The relative rms error of the SD estimate depends on the sample size and kurtosis of the PDF $\delta(\sigma^*) = \sigma^{-1}(\sigma(\sigma^*)) = (2N^{0.5})^{-1}(\varepsilon - 1)^{0.5}$, where $\varepsilon = \lambda_h - \hat{\lambda}_h$. The spread of the counterkurtosis estimate for any PDFs with an error of no more than 8–10 % is defined as $\delta(K_{\rm an}) = K_{\rm an}^{-1}(\sigma(K_{\rm an}))^{-1} = ((29N)^{0.5})^{-1}(\varepsilon^2 - 1)^{3/4}$.

The spread of the estimates of the entropy coefficient $K_{\rm en}$ and the entropy value of the error $\Delta_{\rm en}$ can be found from the relations

$$\sigma(K_{\rm en}) = 0.9(K_{\rm en}K_{\rm en}(K_{\rm en}N)^{0.5})^{-1},$$

$$\delta(K_{\rm en}) = \frac{\sigma(K_{\rm en})}{K_{\rm en}} = \frac{0.9}{K_{\rm en}K_{\rm en}^2(K_{\rm en}N)^{0.5}}, \quad \delta(\Delta_{\rm en}) = \frac{\sigma(\Delta_{\rm en})}{\Delta_{\rm en}} \approx \Big[\frac{9.15 \cdot 10^{-4}}{(1 - K_{\rm en})^3} + 5.1(1 - K_{\rm en})^3\Big]^{0.5}.$$

Taking into account the expression for $H(n)$, we represent the entropy $I_{\rm Sh}$ (information according to Shannon, Shannon entropy) by the true PDF

$$I_{\rm Sh} = I_{\rm un} - \Delta I, \tag{5}$$

where $I_{\rm un}$ is the entropy of the uniform distribution; ΔI is the deviation of the true entropy of the random process with the PDF $W(n)$ from the entropy of the uniform distribution.

It has been shown [6] that the entropy of the uniform distribution $I_{\rm un}$ depends only on the measurement range (scale) of the random process, with the uncertainty interval in this case lying between $n_{\min}$ and $n_{\max}$, and the amount of Shannon information is a logarithmic measure of the length of this interval:

$$I_{\rm Sh} = -\int\limits_{n_{\min}}^{n_{\max}} n_{\rm Sh}^{-1} \log(n_{\rm Sh})^{-1} dn = \log n_{\rm Sh}, \quad n_{\rm Sh} = n_{\max} - n_{\min}.$$

Note that the entropy of any PDF does not depend on mathematical expectations i.e., does not change with transfer of the origin of the random variable.

A real-time estimate of the entropy of random processes with a uniform distributions can be calculated either via the sample mean $\hat{m}_h$ and sample variance $\hat{M}_2$, or by using order statistics $\{y_h\}$.

In the first case,

$$\hat{n}_{\min} = \hat{m}_h - \sqrt{3}\hat{M}_{2h}; \qquad \hat{n}_{\max} = \hat{m}_h + \sqrt{3}\hat{M}_{2h}. \tag{6}$$

In the second case,

$$\hat{n}_{\min} = y_1' - (h-1)^{-1}(y_h' - y_1'); \qquad \hat{n}_{\max} = y_1' + (h-1)^{-1}(y_h' - y_1'). \tag{7}$$

Here $\{y_h'\}$, $h = \overline{1, H}$ is a variational series whose elements $y_h - y_{h-1}'$ are used to obtain order statistics or to group (systematize) experimental data.

If we set $I_{\rm un} \gg \Delta I$ in (5), algorithms (6) or (7) can be used to estimate the entropy of non-Gaussian processes.

Figure 2 shows numerical values for the entropy depending on the sample for a random process with a bimodal PDF. It is evident from the figure that the estimation of the entropy requires a fairly large sample size ($H > 100$).

IDENTIFICATION OF DISTRIBUTION PARAMETERS FOR MULTIPLICATIVE NON-GAUSSIAN NOISE AS AN EXAMPLE

As an example, we determine the main characteristics for the PDF of the envelope of a narrowband signal (multiplicative noise) described by the Nakagami distribution:

$$W(\eta) = (2/\Gamma(m))(m/\Omega)^m \eta^{2m-1} \exp\{m\eta^2/\Omega\}, \quad \eta \geq 0, \ \ m = \Omega^2/\langle(\eta^2 - \Omega^2)^2\rangle \geq 0.5,$$

where $\Omega = \langle\eta^2\rangle$ are the distribution parameters and $\Gamma(\cdot)$ is the gamma function. Taking into account the expression for the initial υ moments of the Nakagami PDF $m_\eta^\upsilon = \Gamma(m+\upsilon/2)/(\Gamma(m)(\Omega/m)^{-\upsilon/2})$, we express the central moments as

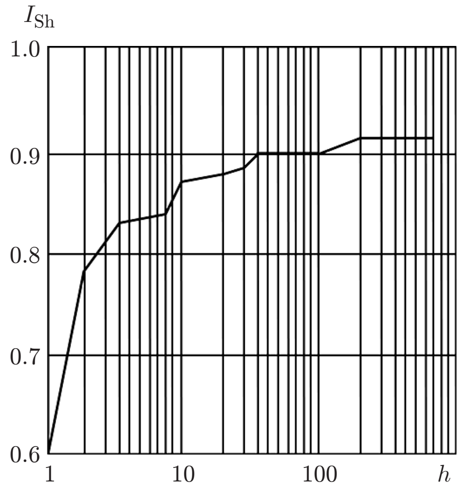

Fig. 2. Functional dependence of $I_{\rm Sh}$ on h.

$$M_2 = \Omega^2\Big[1 - \frac{\Gamma^2(m+0.5)}{m\Gamma^2(m)}\Big], \qquad M_3 = \frac{\Omega^3}{m^{1.5}}\Big[\frac{(0.5-2m)\Gamma(m+0.5)}{\Gamma(m)} + \frac{2\Gamma^3(m+0.5)}{\Gamma^3(m)}\Big],$$

$$M_4 = \frac{\Omega^4}{m}\Big[m + 1 + \frac{(2m-2)\Gamma^2(m+0.5)}{m\Gamma^2(m)} - \frac{3\Gamma^4(m+0.5)}{m\Gamma^4(m)}\Big].$$

In this case, the skewness and kurtosis coefficients are obtained in the form

$$K_{\rm s} = \frac{M_3}{M_2^{1.5}} = \frac{(0.5-2m)\Gamma^2(m)\Gamma(m+0.5) + 2\Gamma^3(m+0.5)}{[m\Gamma^2(m) - \Gamma^2(m+0.5)]^{1.5}},$$

$$K_{\rm k} = \frac{M_4}{M_2^2} = \frac{m(m+1)\Gamma^4(m) + 2(m-1)\Gamma^2(m+0.5)\Gamma^2(m) - 3\Gamma^4(m+0.5)}{[m\Gamma^2(m) - \Gamma^2(m+0.5)]^2}.$$

The entropy and entropy coefficient of the multiplicative noise will be determined based on the relations

$$H\{\eta\} = \ln\Big\{\frac{\Gamma(m)\Omega\exp(m)}{2m^{0.5}}\Big\} - \frac{2m-1}{2}\Psi(m), \quad K_{\rm en} = \frac{\Gamma^2(m){\rm e}^m \cdot \exp\{-(m-0.5)\Psi(m)\}}{4\{m\Gamma^2(m) - \Gamma^2(m+0.5)\}^{0.5}},$$

where

$$\Psi(m) = \frac{\Gamma'(m)}{\Gamma(m)}, \qquad \Gamma'(m) = \int\limits_0^\infty t^{m-1}\exp\{-t\}\ln t dt.$$

Curves of the skewness, kurtosis, counterkurtosis, entropy, and entropy coefficient on the parameter m are shown in Fig. 3. It is evident from these curves that as the parameter m increases, the values of $K_{\rm a}$, $H\{\eta\}$ decrease and the values of the coefficients $K_{\rm k}$, $K_{\rm en}$, and $K_{\rm an}$ remain practically unchanged.

Consider the methods of estimating the information parameters of the multiplicative noise PDF described by the Nakagami distribution.

In approximating the distribution of the envelopes of narrowband useful signals using the Nakagami PDF, the distribution parameters m and Ω need to be determined from statistical data. It is not difficult to estimate the parameter Ω which characterizes the average power of the multiplicative noise with known m. It is much more difficult to choose the parameter m which specifies the distribution form [7].

Using the results obtained in [8], we can show that the estimate of the parameter Ω is effective and can be found from the formula

$$\hat{\Omega} = H^{-1}\sum_{h=1}^{H}\eta_h^2.$$

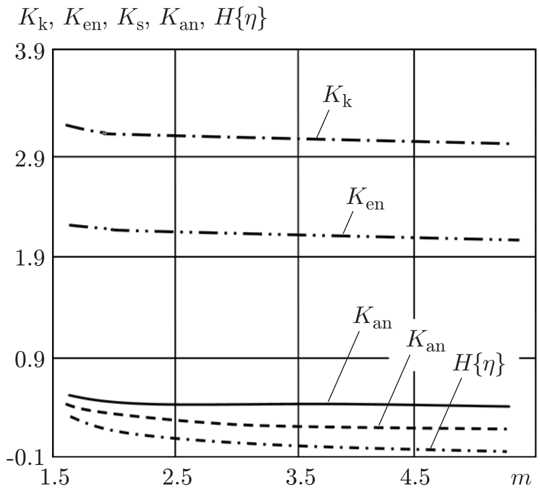

Fig. 3. Coefficients K_{a}, K_{k}, K_{en}, K_{an}, and $H\{\eta\}$ versus the PDF parameter m.

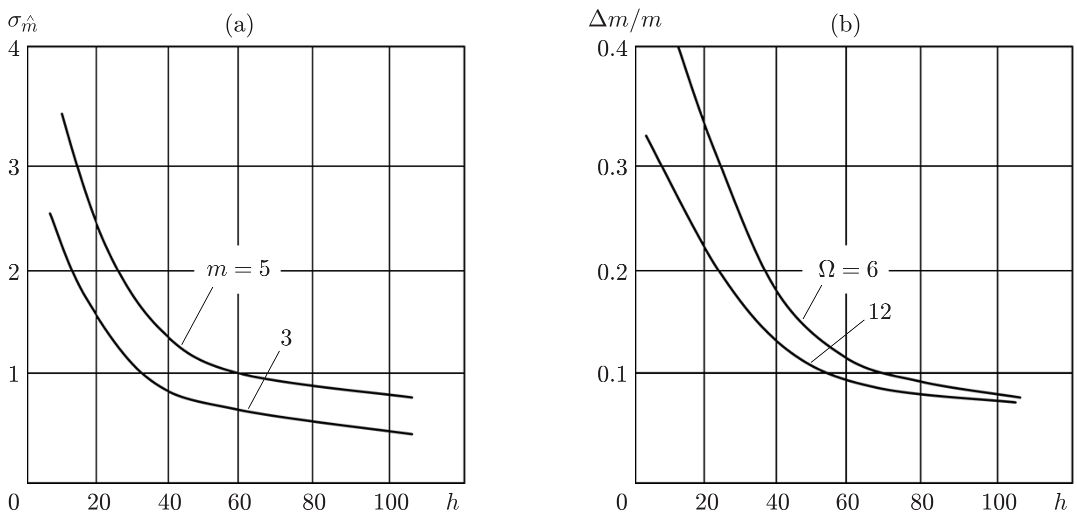

Fig. 4. Curves of the standard deviation $\sigma_{\hat{m}}$ (a) and the relative bias $\Delta m/m$ (b) versus the sample parameter h.

The expectation and variance of the estimate of the parameter Ω can be determined from the expressions

$$\langle\hat{\Omega}\rangle = H^{-1}\sum_{h=1}^{H}\langle\eta_i^2\rangle = \Omega, \qquad \sigma^2_{\hat{\Omega}} = \frac{\Omega^2}{mH}.$$

Note that the estimates of Ω obtained by the method of matching moments and the maximum likelihood method coincide. The estimate obtained by the method of matching moments is

$$\hat{m} = \hat{\Omega}^2\Big[H^{-1}\sum_{h=1}^{H}\eta_h^4 - \hat{\Omega}^2\Big].$$

The variance of the estimate $\hat{m}$ is difficult to investigate analytically; therefore, statistical simulation is used as a rule. For practical calculations, it is advisable to use the expressions

$$\hat{m} = 0.504\Big[\ln H^{-1}\sum_{h=1}^{H}\eta_h^2 + 2H^{-1}\sum_{h=1}^{H}\eta_h\Big]^{-1} + 0.126, \quad \sigma^2_{\hat{m}} = H^{-0.5}(1.6m - 0.36).$$

The linear approximation error in this case does not exceed one percent.

Curves of the standard deviation $\hat{m}$ and the relative bias $\Delta m/m$ versus sample size are presented in Fig. 4. It can be seen from the graphs that the estimate $\hat{m}$ is due not only to the sample size, but also to the parameter Ω. The bias decreases with increasing Ω. The estimate of the parameter $\hat{m}$ is effective only for large sample sizes ($H \geq 100$).

CONCLUSIONS

The identification of non-Gaussian noise parameters under a priori uncertainty was discussed. It is shown that the counterkurtosis K_{an} and the entropy coefficient K_{en} can be used for the current identification of the PDF form parameters of additive non-Gaussian noise having a symmetric distribution. For the current identification of non-Gaussian noise with a one-sided PDF (of multiplicative noise), the skewness coefficient K_{s} should be added to the specified quantities. The results of numerical simulation of methods for estimating the information parameters of random processes with a non-Gaussian PDF for a finite sample show that the estimates obtained by recursive and asymptotic methods converge with increasing number of samples.

REFERENCES

1. V. I. Tikhonov and V. N. Kharisov, *Statistical Analysis and Synthesis of Wireless Devices and Systems* (Radio i Svyaz, Moscow, 1991) [in Russian].
2. V. M. Artyushenko, V. I. Volovach, "Measuring Information Signal Parameters under Additive Non-Gaussian Correlated Noise," Avtometriya **52** (6), 22–28 (2016) [Optoelectron., Instrum. Data Process. **52** (6), 546–551 (2016)].
3. V. M. Artyushenko and V. I. Volovach, "Quasi-Optimal Processing of Signals with Additive and Multiplicative Non-Gaussian Noise," Radiotekhnika, No. 1, 124–130 (2016).
4. K. K. Vasil'ev, *The Reception of Signals with Multiplicative Noise* (Izd. Saratov Univ., Saratov, 1983) [in Russian].
5. Ya. Z. Tsypkin, *Principles of the Information Theory of Identification* (Nauka, Moscow, 1984) [in Russian].
6. P. V. Novitskii and I. A. Zograf, *Estimating the Errors of Measurement Results* (Energoatomisdat, Moscow, 1991) [in Russian].
7. V. M. Artyushenko, V. I. Volovach, and V. V. Ivanov, "Statistical Properties of Signals and Noise in Radio Engineering Devices of Short-Range Detection," Izv. Vuzov. Priborostroenie **57** (7), 46–50 (2014).
8. G. V. Gusinskii, "Estimation of m-Distribution Parameters," Izv. Vuzov. Radioelektronika **22** (1), 129–133 (1977).