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STRUCTURAL MECHANICS AND STRENGTH OF FLIGHT VEHICLES

Dynamic Stability of an Orthotropic Cylindrical Shell of Piecewise Constant Thickness under the Action of External Pulsating Pressure

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Abstract—For a shell with three sections, dependences of the critical frequency on the different thicknesses of these sections are obtained. The error of the assumptions accepted at constructing the mathematical model of a shell is evaluated.

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In creating some types of aircraft, cylindrical shells made of composite materials that consist of sections having different thicknesses are used as power elements. Such shells with a piecewise constant thickness are, for example, the engine housings, at the ends of which the docking assemblies are additionally wound, and in the locations of the intermediate supports, the zones of reinforcement are organized, as well as compartments and various protective casings. During the aircraft flight, these thin-walled elements are under the action of acoustic loads (aerodynamic forces, buffeting, atmospheric turbulence) [1] in the form of a pulsating external pressure, which under certain conditions, is able to cause their destruction.

The problems of calculating the variable–thickness shells were investigated in [2, 3]. The problem of dynamic stability of thin-walled structures began to be studied in the middle of the last century in connection with the development of aeronautics and astronautics. Ways to solve it were outlined in a number of monographs and papers [4–12].

Thus, in [8], the dynamic stability of a hinged orthotropic shell supported by elastic bonds (springs) under the action of harmonically changing external pressure was investigated. The springs are arranged in several rows along the length of the shell, and along the circumference—the springs are arranged symmetrically relative to the vertical diameter. The solution of the equation is sought in the form of two trigonometric series with respect to the axial and circumferential coordinates. The problem is reduced to the system of algebraic equations relative to the radial displacements of the shell in the place of installation of the springs. For uniformly spaced similar springs, the solution is obtained explicitly. The effect of the spring stiffness on the boundaries of stability regions was shown.

In [9], the dynamic stability of the orthotropic cylindrical shell discretely supported by annular ribs and loaded by the constant compressive force was considered. The external surface is under the pressure consisting of three components, namely, the static constant component over the entire length, the static variable component over the length, and the harmonic variable one over the entire length. The solution is sought as a series with respect to the axial coordinate. The dependences of the main instability region for different values of the axial force are obtained for three laws of distribution along the axis of the variable component of the static pressure.

In [10], the dynamic stability of the orthotropic cylindrical shell discretely supported by longitudinal ribs and a hollow elastic cylinder under the action of harmonically changing axial forces was studied. The solution of the equations is sought in the form of a trigonometric series with respect to the circumferential coordinate. In solving the Mathieu–Hill type equations, the two-term approximation is used that allows the calculation accuracy to be increased up to 10 %. Solution was obtained in explicit form for uniformly located similar ribs. The change of instability regions depending on the number and geometric parameters of ribs and the cylinder stiffness is shown.

In [11], the dynamic stability of a layered orthotropic shell discretely supported by ring edges and a hollow elastic cylinder loaded with a constant axial force and external pressure changing in time according to the harmonic law was investigated. The equations of motion of the shell take into account the transverse shift that increases the accuracy of determining the boundary of the instability region up to 20 %.

The solution of the problem is sought in the form of a trigonometric series along the axial coordinate. The problem is reduced to a system of three algebraic equations for uniformly spaced equal edges. The dependences of the main instability region on the number of ribs, the shear modulus of the shell, the radius of the cylinder channel and the axial force are established.

In [12], the dynamic stability of the orthotropic shell supported by an elastic hollow cylinder and longitudinal diaphragms under the action of harmonically changing external pressure was considered. Diaphragms have different physical and mechanical properties and are arranged symmetrically with respect to the vertical diameter. The solution of the equation is sought in the form of a trigonometric series with respect to the circumferential coordinate. To improve the accuracy of calculating the Mathieu–Hill equations, the two-term approximation is used. The problem is reduced to a system of two algebraic equations for uniformly spaced similar ribs. In the case of the one-term approximation, the solution was obtained explicitly. Dependences of the stability regions for different stiffnesses of diaphragms, the radii of the cylinder channel, and the magnitude of the axial force were obtained. In [10–12], the cylinder was represented as an elastic Winkler base, the bed coefficient of which was determined from the equations of the three-dimensional theory of elasticity.

In [13], the dynamic stability of the orthotropic layered cylindrical shell under the action of periodically changing external pressure is investigated. Using the Bubnov–Galerkin method, the problem is reduced to the Mathieu–Hill equation. The main instability regions for different shell parameters are constructed.

The dynamic stability of the cylindrical shell under the action of torques applied at the ends and changing in time according to the harmonic law is considered in [14]. The main instability regions for four types of boundary conditions are determined.

However, at present, due to the widespread introduction of composite materials in structural design of aircraft and the development of fundamentally new structures, actual problems arose related to the use of additive technologies and the design of shell structures, the thickness of which varies according to a certain law.

The study of oscillations and stability of cylindrical shells of variable thickness, as is known, leads to the solution of differential equations with variable coefficients. For shells with piecewise constant thickness, which can be attributed to a special class of shells, these coefficients are the generalized functions (the unit Heaviside function, the Delta function and its derivatives) [15]. The solution of such equations is a fairly complex process, and the desired function has a poor convergence. Therefore, this paper uses a simplified initial equation, in which the terms containing the delta functions and its derivatives are discarded. The solution error is estimated as a result of introducing these assumptions.

Consider an orthotropic cylindrical shell of piecewise constant thickness, the outer surface of which is under the uniformly distributed pressure that varies in time according to the harmonic law. The shell is hinged at the ends and loaded by axial forces. The tangent and axial components of inertial forces are neglected. Structural damping in the shell is not taken into account. The design scheme is shown in Fig. 1.

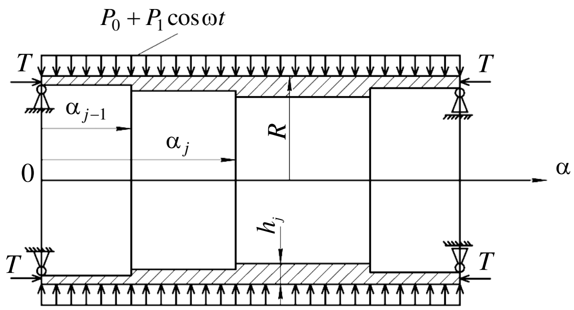

Fig. 1.

We introduce the dimensionless system of cylindrical coordinates, in which the linear dimensions are related to the radius of the outer surface of the shell taken as a coordinate surface. Then the equation of the shell can be represented as [16]

$$\left\{\sum_{j=1}^{N}\left(a_{3j}\nabla^8 + a_{2j}\frac{\partial^4}{\partial\alpha^4} + a_{7j}\nabla^4\frac{\partial^2}{\partial t^2}\right)\left[\sigma_0\left(\alpha-\alpha_{j-1}\right) - \sigma_0\left(\alpha-\alpha_j\right)\right]\right.$$
$$\left. + \nabla^4\left[a_8\left(p_0 + p_1\cos\omega t\right)\left(\frac{\partial^2}{\partial\beta^2}+1\right) + a_9\frac{\partial^2}{\partial\alpha^2}\right]\right\} w = 0, \tag{1}$$

where

$$\nabla^8 = a_1\frac{\partial^8}{\partial\alpha^8} + \left[a_4 + a_6\left(2a_1 - \nu_\beta\right)\right]\frac{\partial^8}{\partial\alpha^6\partial\beta^2} + \left\{a_1a_4\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2 + \left[2a_6\left(a_4 - a_6\nu_\beta\right) + a_1a_4\right]\frac{\partial^4}{\partial\beta^4}\right\}\frac{\partial^4}{\partial\alpha^4} + a_4 \times$$
$$\left[\left(a_4 - a_6\nu_\beta\right)\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2 + 2a_1a_6\frac{\partial^4}{\partial\beta^4}\right]\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + a_1a_4^2\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2\frac{\partial^4}{\partial\beta^4};\quad \nabla^4 = a_1\frac{\partial^4}{\partial\alpha^4} + \left[a_4 - a_6\nu_\beta\right]\frac{\partial^4}{\partial\alpha^2\partial\beta^2} +$$
$$a_1a_4\frac{\partial^4}{\partial\beta^4};\quad a_1 = \frac{G_{\alpha\beta}\left(1-\nu_\alpha\nu_\beta\right)}{E_\alpha};\quad a_{2j} = \frac{a_1a_4h_j}{R};\quad a_{3j} = \frac{h_j^3}{12R^3};\quad a_4 = \frac{E_\beta}{E_\alpha};\quad a_6 = 2a_1 + \nu_\beta;\quad a_{7j} = B_0\rho h_j;$$

$a_8 = \dfrac{B_0}{R}$; $a_9 = \dfrac{B_0T_\alpha}{2\pi R^3}$; $B_0 = \dfrac{R\left(1-\nu_\alpha\nu_\beta\right)}{E_\alpha}$; α,β are the dimensionless coordinates along the generatrix and in the circumferential direction; w is the radial displacement; $E_\alpha, E_\beta, G_{\alpha\beta}$ are the axial and circumferential moduli of elasticity and the shear modulus; ν_α, ν_β are the Poisson's coefficients; R is the radius of the external surface of the shell; h_j is the shell thickness on the *j*th section; T_α is the initial axial force; p_0, p_1 are the constant component and amplitude of a variable component of external pressure; ρ is the shell material density; ω is the pulsation frequency N is the number of sections; $\sigma_0(\alpha)$ is the single function equal to unit at $\alpha > 0$ and to zero at $\alpha < 0$.

A solution of Eq. (1) will be sought in the following form:

$$w = \cos n\beta\sum_{m=1}^{\infty} f_m(t)\sin\gamma_m\alpha, \tag{2}$$

where $\gamma_m = \dfrac{m\pi}{\alpha_0}$; $\alpha_0 = \dfrac{L}{R}$; L is the length of the shell; n is the number of waves in circumferential direction; m is the number of half-waves in axial direction; $f_m(t)$ is the unknown function of time (hereafter, the argument *t* is omitted).

Substituting expression (2) into Eq. (1) and using the Bubnov–Galerkin method, we obtain an infinite system of the Mathieu–Hill differential equations:

$$\sum_{m=1}^{\infty}\left(D_{mk}\frac{\partial f_m}{\partial t^2}+C_{mk}f_m\right)+b_{1k}\cos\omega t f_k+b_{2k}f_k=0,\ k=1,2,3...,\tag{3}$$

where

$$D_{mk}=\sum_{j=1}^{N}a_{7j}\nabla_m^4\varphi_{mk}^{(j)};\quad C_{mk}=\sum_{j=1}^{N}\left(a_{3j}\nabla_m^8+a_{2j}\gamma_m^4\right)\varphi_{mk}^{(j)};$$

$$b_{1k}=a_8p_1\left(1-n^2\right)\nabla_k^4;\quad b_{2k}=\left[a_8p_0\left(1-n^2\right)-a_9\xi_k^2\right]\nabla_k^4;$$

$$\nabla_m^8=a_1\gamma_m^8+\left[a_4+a_6\left(2a_1-\nu_\beta\right)\right]\gamma_m^6n^2+\left\{a_1a_4+2\left[a_6\left(a_4-a_6\nu_\beta\right)+a_1a_4\right]+\left(n^2-1\right)n^2\right\}\gamma_m^4$$

$$+a_4\left[\left[\left(a_4-a_6\nu_\beta\right)\left(1-n^2\right)^2+2a_1a_6n^4\right]\gamma_m^2n^2+a_1a_4^2\left(n^2-1\right)^2n^4;\right.$$

$$\nabla_m^4=a_1\gamma_m^4+\left(a_4-a_6\nu_\beta\right)\gamma_m^2n^2+a_1a_4n^2;$$

$$\nabla_k^4=a_1\xi_k^4+\left(a_4-a_6\nu_\beta\right)\xi_k^2n^2+a_1a_4n^4;$$

$$\xi_k=\frac{k\pi}{\alpha_0};\quad \varphi_{mk}^{(j)}=\begin{cases}F_k^{(j)} & \text{at}\ \ m=k;\\ F_{mk}^{(j)} & \text{at}\ \ m\neq k;\end{cases}$$

$$F_k^{(j)}=\frac{\alpha_j-\alpha_{j-1}}{\alpha_0}-\frac{1}{\pi k}\cos\frac{\pi k\left(\alpha_j+\alpha_{j-1}\right)}{\alpha_0}\sin\frac{\pi k\left(\alpha_j-\alpha_{j-1}\right)}{\alpha_0};$$

$$F_{mk}^{(j)}=\frac{2}{\pi}\left\{\frac{1}{m-k}\cos\frac{\pi(m-k)\left(\alpha_j+\alpha_{j-1}\right)}{2\alpha_0}\sin\frac{\pi(m-k)\left(\alpha_j-\alpha_{j-1}\right)}{2\alpha_0}\right.$$

$$\left.-\frac{1}{m-k}\cos\frac{\pi(m+k)\left(\alpha_j+\alpha_{j-1}\right)}{2\alpha_0}\sin\frac{\pi(m+k)\left(\alpha_j-\alpha_{j-1}\right)}{2\alpha_0}\right\}.$$

Let us seek the solution of Eq. (3) in the following form

$$f_m=\sum_{i=1,3...}^{\infty}\left\{A_{im}\sin\frac{i\omega t}{2}+B_{im}\cos\frac{i\omega t}{2}\right\},\tag{4}$$

where A_{im}, B_{im} are the unknown constants.

Substitute the first sum of expression (4) into Eq. (3) and equate the coefficients with the same $\sin\frac{i\omega t}{2}$. Let us limit ourselves to the first term of series (4) that defines the main domain of instability and, according to [2], it is sufficient for practical calculations.

As a result, we will have:

$$\theta_{1k}A_{1k}+\sum_{m=1}^{\infty}\theta_{2m}A_{1m}=0,\tag{5}$$

where $\theta_{1k} = b_{2k} \pm \frac{b_{1k}}{2}$; $\theta_{2m} = C_{mk} - \frac{\omega^2}{4} D_{mk}$.

Reducing system (5) and equating the determinant of the matrix obtained to zero, we come to the sought characteristic equation for finding the critical frequency.

Substituting the second sum from expression (4) into Eq. (3), we obtain characteristic equation (5), in which the coefficients A_{1k}, A_{1m} should be replaced by B_{1k}, B_{1m}, and at the coefficient θ_{1k} the sign. "+" should be taken.

As an example, two shells are considered, three sections of which have different thicknesses, but the average integral thicknesses h_{av} and, accordingly, their masses are the same. The basic shell parameters were as follows: $\frac{L}{R} = 6$; $\frac{h_2'}{R} = 0.007$; $\frac{(h_1', h_3')}{R} = 0.014$; $\frac{h_{av}}{R} = 0.0105$; $\frac{h_2''}{R} = 0.014$; $\frac{h_1'', h_3''}{R} = 0.007$; $\frac{E_\beta}{E_\alpha} = 1.5$; $\frac{G_{\alpha\beta}}{E_\alpha} = 0.16$; $\nu_\alpha = 0.15$; $\nu_\beta = 0.23$; $\frac{\rho R g}{E_\alpha} = 0.67 \times 10^{-6}$; $\alpha_1 = 0$; $\alpha_2 = 1.5$; $\alpha_3 = 4.5$; $\alpha_4 = 6$.

Figure 2 shows the instability regions (the shaded part) of the shell with reinforced end sections *1*, the reinforced middle section *2*, and the average integral thickness *3* without action of the axial force ($T = 0$). The ordinate is expressed in terms of the ratio $\left(y = \frac{\omega}{\omega_0} \right)$ of the critical pulsation frequency to the natural frequency of a smooth shell with a base thickness h_2', and the abscissa is expressed in terms of the ratio of the variable component of the external pressure to the constant $\left(x = \frac{p_1}{p_0} \right)$. The constant component was equal to $p_0 = 0.8 p_{cr}$, where p_{cr} is the critical pressure of buckling of a smooth shell with base thickness.

Figure 3 presents the similar dependences for shells additionally loaded by an axial force $T = 0.3 T_{cr}$ (T_{cr} is the axial buckling force of a smooth shell with base thickness).

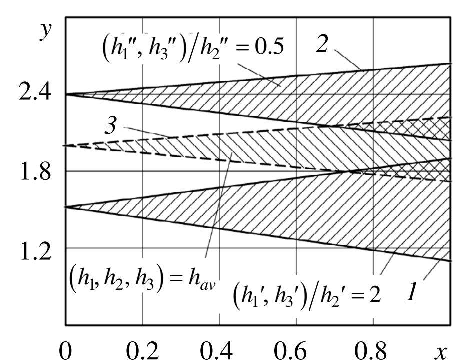

Fig. 2.

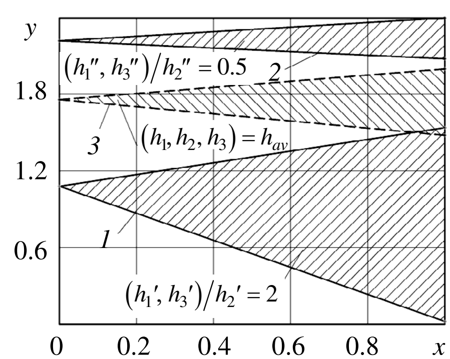

Fig. 3.

The following example shows that

—the boundaries of the instability region of a shell with strengthened middle section are 1.6 times higher than the shells with strengthened sections;

—the boundaries of the instability region of the shell with the average integral thickness are 20% lower than for the shell with the strengthened middle section and 30% higher than for the shells with strengthened end sections;

—the presence of an axial compressive force reduces the boundaries of the instability region of the shell with strengthened end sections by 40% and increases the area of the region by two times;

—for shells with average-integral thickness and reinforced middle section, the presence of an axial force reduces the boundaries of the instability regions by 10%;

—the definition of the boundaries of the instability regions using the average integral thickness in the range of 10% error can be carried out for shells, in which the difference of thickness at different sections does not exceed 20%.

To assess the impact of the assumptions made, the results of calculating the standard design by the base (used by different authors) technique and the technique being proposed were compared. A smooth orthotropic shell supported by a rectangular frame was used as a computational model. The material of the shell and the frame was the same and its characteristics corresponded to the above example. In this technique, the total height of the frame and smooth shell was formed by the thickness of the middle section, and the length of this section was equal to the width of the frame. The critical values of the static external pressure and the axial compressive force causing the loss of stability, and the natural frequency of the shell at different values of the height of the frame were determined.

In the base technique, the equation of motion of the shell supported by the frame was used in the form [7]

$$\left\{a_3\nabla^8 + a_1 a_4\frac{\partial^4}{\partial\alpha^4} + \nabla^4\left[a_8 p_1\left(\frac{\partial^2}{\partial\beta^2}+1\right) + a_5\frac{\partial^2}{\partial\alpha^2} + a_7\frac{\partial^2}{\partial t^2}\right]\right\}w$$

$$+\nabla^4 b_1\left[a_{10}\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2 w_i + a_9\frac{\partial^2 w_i}{\partial t^2}\right]\delta\left(\alpha-\alpha_i\right)=0,$$

where $a_3=\dfrac{h^2}{12R^2}$; $a_5=\dfrac{T_1B}{2\pi R^3}$; $a_7=B\rho h$; $a_8=\dfrac{B}{R}$; $a_9=\dfrac{Bh_1\rho}{R}$; $a_{10}=\dfrac{BE_\beta h_1^3}{12R^5}$; $B=\dfrac{R^2\left(1-\nu_\alpha\nu_\beta\right)}{E_\alpha h}$; b_1, h_1 are the width and height of the frame; α_i is the coordinate of the frame location. The other notation corresponds to Eq. (1).

The geometric parameters of the shell and frame were as follows: $\dfrac{L}{R}=6$; $\dfrac{h}{R}=0.007$; $\dfrac{b_1}{R}=0.06$; $\alpha_1=0$; $\alpha_2=\alpha_i=1.3$; $\alpha_3=1.36$; $\alpha_4=6$.

The calculation results for both techniques are presented in the table, where p_k, T_k, f_k are the critical external pressure, the critical axial force and the natural frequency of the shell, respectively; $\Delta_p, \Delta_T, \Delta_f$ is the difference of calculation results between the methods in percentage; $k=0$—results of the proposed technique; $k=1$—the results of the basic technique.

Frame height	Crritical external pressure p_k			Critical axial force T_k			Natural frequency f_k		
$\frac{h_1}{R}\times10^2$	$p_0\times10^2$, MPa	$p_1\times10^2$, MPa	Δ_p, %	$T_0\times10^{-5}$, N	$T_1\times10^{-5}$, N	Δ_T, %	f_0, Hz	f_1, Hz	Δ_f, %
0	3.7	3.7	0	9.0	9.0	0	22	26	15
2	4.3	4.1	5	9.5	9.0	5	23	27	15
4	4.8	4.8	0	10.5	11.0	5	26	30	13
6	4.9	5.1	4	11.0	12.0	8	28	35	20
8	5.0	5.1	2	11.5	12.0	4	32	38	15

Analysis of the results shows that the error of the technique proposed in calculating the stability of the shells of piecewise constant thickness under the action of external pressure and axial force is about

5% that is a fairly good indicator. The error in determining the natural frequencies is about 15% that is quite acceptable in carrying out design work. Note that the stiffness of the shell with an artificial frame is lower than of the shell with a classic frame for all types of load.

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