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Derivation of Hydro- and Electrohydrodynamic Equations by the Dimensional Method

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Abstract—The use of dimensional analysis to solve some problems of hydrodynamics associated with convective transport of a liquid medium is presented. In particular, this forms the basis of deriving equations of continuity, thermal conduction, diffusion, and motion of ideal (Euler) and viscous fluids (Navier–Stokes), with some complements from the field of electrohydrodynamics. In addition, the problem of the presence of two forces of viscous friction is solved, and the problems of sliding and deformation (in compression–extension in a compressible fluid). Formulas for these forces are derived.

Keywords: convective transfer, equations of continuity, motion, heat conduction, diffusion, electrohydrodynamics, viscosity forces

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INTRODUCTION

Problems of hydrodynamics and heat transfer can be complicated by electric fields (*electrohydrodynamics* (EHD)), include the notion of convective transfer [1] when a physical quantity exhibits changes over time at a point in space point (locally) and, alongside this, due to liquid motion. The total change in the quantity under study (let us set it as $F(t, x(t), y(t), z(t))$ per unit time is determined by the total derivative of the composite (scalar or vector) function in time [2]:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial F}{\partial t} + (\vec{v}, \nabla)F = \frac{\partial F}{\partial t} + (\vec{v}, \mathrm{grad})F, \quad (1)$$

where $\vec{v}\left(v_x, v_y, v_z\right)$ is the field of velocities whose components are equal to the time derivatives of the coordinates. Establishing an equation for the functions $F(t, x(t), y(t), z(t))$ based on the physical grounds and its implicit right-hand member remains a problem. Some cases have been presented in which this problem has been solved comparatively simply, using the dimensional method. This derivation of the basic hydrodynamic equations of liquid continuity, convective heat conduction, diffusion, and Navier–Stokes viscous liquid motion is a particularly significant, as it entails some serious difficulties associated with two viscosity forces [2]. We focus our attention on these equations, pre-formulating the problem and the proposed approach briefly.

STATEMENT OF PROBLEMS AND ESSENCE OF THEIR SOLUTION

The questions touched upon are not only of methodological interest but of scientific and educational interest as well, as has been demonstrated many times by the theory of similarity and dimension [3] in the solution of various complex classical problems of fluid mechanics [4–6]. The method of dimension and similarity is used to solve many physical questions and problems [7–10]. In particular, an advanced treatment of the aspects of dimensional analysis and simulation have been conducted [7] including a wide range of examples for different domains of science and technology. Dimensional analysis is a basis for the comparison of systems on different physical scales [8] with the intention of obtaining a key concept for natural phenomena that are too vast to be reproduced under laboratory conditions. It is known that a central position in the theory of dimension and similarity is occupied by the π-theorem considered in work [9] with a focus on the problems typical for heat and mass transfer in solid bodies as well as in laminar and turbulent flows of fluids and gases. The elaborate treatment of some aspects of dimensional analysis and simulation is undertaken in [10] with many examples concerning various transfer processes. The dimensional analysis is used for the physical representation of the hydrodynamic resistance force sustained by a solid body when it moves in a liquid, as well as for the explanation of the regularities of the flow of a fluid through tubes and heat and mass transfer in solid bodies. The study [11] describes the methods and history of the development of this knowledge area, including physical and engineering applications and considers the problems of mechanical science, hydro- and electrodynamics, thermodynamics, and quantum physics, explaining such notions as viscosity and diffusion using specific examples. A connecting link between the hydrodynamic field of velocities $\vec{v}$ and the field of the sought–for quantity F is the operator of differentiation with respect to the nabla coordinates ∇:

$$\nabla \equiv \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z},$$

thus, the sought-for right-hand member of (1) must contain quantities that reflect the dimension of total derivative (1), including this operator, i.e., the solution of equation (1) will be, implicitly,

$$\frac{\partial F}{\partial t} + (\vec{v},\nabla)F = \mu f(\nabla, F, \vec{v}, \gamma, p, \ldots), \tag{2}$$

where γ and p are the mass density and pressure; the ellipsis implies that other independent variables are also available, with allowance taken for the nature of the problem, such as, for instance, the electric field intensity $\vec{E}$ from the EHD region. If the right-hand member of (2) is zero, than the quantity F is constant, as in the case of the isentropic fluid flow [2]:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + (\vec{v},\nabla)S = 0 \Rightarrow S = \text{const.}$$

If this is not the case, a solution to the mathematical problem possessing physical meaning corresponds to each function f. Thus, the method of testing functions can be applied, which can be picked out maintaining the dimensions of the right-hand member of (2) in correspondence with the left-hand member. The parameter μ (dimensional and dimensionless) is written for generality, being a corrective parameter, and it is defined more accurately on the physical grounds in the course of the dimensional analysis. Presenting equation (2) according to the theory of dimensions and similarity [3] in the form of power multipliers, among others with respect to the "nabla" operator ∇,

$$\begin{aligned} &\mu f(\nabla, F, \vec{v}, \ldots) = \mu \cdot \nabla^k \cdot F^l \cdot \vec{v}^{\,m} \cdot, \ldots \\ &\Rightarrow [(\vec{v}, \nabla F)] = [\mu] \cdot [\nabla]^k \cdot [F]^l \cdot [\vec{v}]^m \ldots \end{aligned} \tag{3}$$

and equating the powers at the left and right of last equation (3) in view of (2), we obtain a set of equations and, solving it, find an explicit right-hand member of (2). The obtained results and possible products (scalar, vector, etc.) are analyzed, and the conclusions concerning the structure of the right-hand member of equation (2) and its aspect in large are made.

The summary of the proposed method consists in equations (2) and (3) as well as in the above mentioned. Let us consider some examples.

1. Mass Conservation (Continuity) Equation

We search for an equation for the total derivative of the mass density of a homogeneous liquid in the general form (2):

$$\frac{d\gamma}{dt} = \frac{\partial \gamma}{\partial t} + (\vec{v},\nabla)\gamma = \mu f(\nabla, \gamma, \vec{v}, p), \tag{4}$$

assuming that the density does not depend on an external electric or gravitational field (these fields are absent). The problem is initially solved according to general pattern (2), (3) being restricted for the right-hand member of (4) to the variables within the brackets:

$$\begin{aligned} T^{-1}(ML^{=3}) &= \mu f(\nabla, \gamma, \vec{v}, p) = \mu \nabla^k \gamma^l v^m p^n \\ &= \mu L^{-k}(M^l L^{-3l})(L^m T^{-m})(M^n L^{-n} T^{-2n}). \end{aligned} \tag{5}$$

Setting the indexes of power in the left- and right-hand members of this equality as equal, we get the following system:

$$\begin{cases} -3l + m - n = k - 3, \\ l + n = 1, \\ m + 2n = 1. \end{cases} \tag{6}$$

This system is consistent, but it has an innumerable number of solutions for $k = 1$. Consequently, we can assume $k = 1$ in equalities (5), thus transforming system (6) into an indefinite one, and then into a definite one with an acceptable set of unknowns: $l = m = 1$; $n = 0$, as only this set leads to the results reasonable in terms of physics. For this set of variables the right-hand member of the sought-for relationship (5) admits two forms of products structurally matched in dimensions: $\mu\gamma(\nabla\vec{v})$ or $\mu\nabla(\gamma\vec{v})$. The first form at $\mu = -1$ completely satisfies our search resulting in the commonly known mass conservation law:

$$\frac{d\gamma}{dt} = \frac{\partial \gamma}{\partial t} + (\vec{v},\nabla)\gamma = -\gamma(\nabla, \vec{v}). \tag{7}$$

Hence, there automatically follows the continuity equation for its most common standard form:

$$\frac{\partial \gamma}{\partial t} + \nabla \vec{i} = 0,$$

where

$$\vec{i} = \gamma \vec{v}$$

is the mass flux. To be certain that (5) is in fact a continuity equation, it can be sufficiently integrated in terms of the Gauss theorem:

$$\frac{\partial}{\partial t}\int_V \gamma dV = -\oint_S \vec{i} \cdot \vec{dS};$$

the mass conservation law in a clear integral form. The second method helps to test immediately the expression $(-\gamma(\nabla, \vec{v})$ as a trial function making sure that it satisfies all the problem demands. It is easily seen that other solutions, say $\nabla(\gamma\vec{v})$, fail to meet the requirements.

2. Equation of Convective Heat Conduction

The equation of convective heat conduction is an equation which bounds the variation in the temperature of a liquid volume unit in a time unit with the fields of velocities $\vec{v}$ and temperature T in the vicinity of a selected fluid particle. In the simplified version of derivation the variation in the *heat* amount of the volume unit inside the particle due to the reasons of this change (heat conduction) will be proportional to the heat conduction coefficient λ ($\mu \equiv \lambda$) and some function that contains the operator ∇, temperature T, rate $\vec{v}$, and pressure p. At this, we consider the following force fields as negligible:

$$c_p \gamma \frac{dT}{dt} = \lambda f(\nabla, T, \vec{v}, p), \quad (8)$$

where c_p is the specific heat capacity at the constant pressure, and the complete derivative of the temperature with respect to time at the left of (8) according to (1) is:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + (\vec{v}, \nabla)T. \quad (9)$$

Dividing (8) by the product between the heat capacity coefficient c_p and the density γ in terms of (9) we simplify equation (8) with some other physical meaning:

$$\frac{\partial T}{\partial t} + (\vec{v}, \nabla)T = af(\nabla, T, \vec{v}, p), \quad (10)$$

where there appears a new coefficient a with a dimension of m^2/s which is called the heat conduction coefficient as it characterizes the leveling rate of *temperature* (unlike thermal) fields:

$$a \equiv \lambda/(c_p \gamma). \quad (11)$$

Let us derive a dimensional equation for equation (10), noting that the pressure p falls out of (10) as the corresponding units of measure at the left of (10) are absent. Then, assuming $f(\nabla, T, \vec{v}) = \nabla^m T^n \vec{v}^l$:

$$\begin{gathered}[dT/dt] = [a][\nabla]^m[T]^n[v]^l \Rightarrow \theta T^{-1} \\ = L^2T^{-1}L^{-m}\theta^n L^l T^{-l} \Rightarrow m = 2; \quad n = 1; \quad l = 0,\end{gathered} \quad (12)$$

where a new unit of measure is introduced for the temperature θ. Turning to equations (9) and (10) in terms of (12) we get:

$$\frac{\partial T}{\partial t} + (\vec{v}, \nabla)T = a\nabla^2 T. \quad (13)$$

It is easy to see that there are no other variants for the right-hand member of (13), which follows from the uniqueness of solution for the indexes of power in (12). The variant $(\nabla \times \nabla)T = \mathrm{rot}(\mathrm{grad}T) \equiv 0$, which seems consistent, is actually inconsistent (being of a vector nature) and leads to the null equation. It remains for us to add that if there are internal sources of heat in the liquid, for example, in the form of electric current as in the EHD problems, then, evidently, their power should be added to the right-hand member of equation (13). Thus, in the EHD problems (in the presence of an external electric field) [6], equation (13) takes on the more general form:

$$\frac{\partial T}{\partial t} + (\vec{v}, \nabla)T = a\nabla^2 T + \sigma E^2/(c_p \gamma), \quad (14)$$

where σ is the specific electric conduction of the liquid, and E is the electric field intensity.

3. Convective Diffusion Equation

The mass concentration of the diffusing substance c in the carrying liquid phase plays the role of a sought-for field, and on the analogy of (10), we can write:

$$\frac{\partial c}{\partial t} + (\vec{v}, \nabla)c = Df(\nabla, c, \vec{v}), \quad (15)$$

where the diffusion coefficient $\mu \equiv D$, also measured in m^2/s, is the transfer coefficient (analogous to the coefficient a). It should be pointed out that in the problems on the electrical cleaning of a dielectric liquid from semiconducting and conducting impurities using an electric field [12] the notions of electrical diffusion and the corresponding electrical diffusion coefficient D_e were introduced, which should be accounted for in (15) in the case of electrical cleaning. Pressure p does not occur in expression (15) for the same reason as it falls out of (10). Assuming in (16) the concentration dimension θ and deriving the dimensional equation for equation (19), we get:

$$\begin{gathered}[dc/dt] = [D][\nabla]^m[c]^n[v]^l \Rightarrow \tilde{k}T^{-1} \\ = L^2T^{-1}L^{-m}\tilde{k}^n L^l T^{-l} \Rightarrow m = 2; \quad n = 1; \quad l = 0,\end{gathered} \quad (16)$$

where k denotes the measurement unit for the concentration c. Thus, convective diffusion equation (15) in terms of (16) takes the commonly known form:

$$\frac{\partial c}{\partial t} + (\vec{v}, \nabla)c = D\nabla^2 c. \quad (17)$$

4. Liquid Motion Equation

The equation of liquid motion is initially based on Newton's second law:

$$m\vec{a} = \vec{F}. \quad (18)$$

Replacing the particle mass m with its density γ, i. e., with the volume unit mass, and the acceleration with the total acceleration according to formula (1), we obtain:

$$\gamma\left[\partial\vec{v}/\partial t + (\vec{v}, \nabla)\vec{v}\right] = \vec{f}, \quad (19)$$

where in the general case by $\vec{f}$ should be meant the vector sum of all the densities of forces acting on this liquid particle including the bulk forces of the external fields of forces acting on the liquid. The essence of the derivation of the motion equations consists in the search for the explicit form of these forces. Thus, looking ahead we shall indicate that in the field of pressure forces, their bulk density is

$$\vec{f}_p = -\nabla p, \tag{20}$$

the only vector combination of the *nabla* operator and pressure and the required dimension N/m^3. The sign "–" has a clear physical meaning: the pressure force is directed towards the decrease in pressure. Please note one more specific property of the density of pressure forces: it is formed in the course of liquid motion, so it is determined by the hydrodynamic and force factors, being a sought quantity in the hydrodynamic equations, alongside with rate and temperature. For the sake of the consequence, formula (20) is derived below according to general rules (3). The density of the gravity field forces is defined by an obvious formula that needs no comment:

$$\vec{f}_g = \gamma\vec{g}. \tag{21}$$

In recent decades, the electric hydrodynamics that studies the hydromechanical behavior of dielectric and low-conductivity liquids in electric fields has become well understood. In the case of EHD, the medium–electric field interaction is ordinarily performed by merely Coulomb forces similar to (21):

$$\vec{f}_E = \rho\vec{E}, \tag{22}$$

where ρ is the volume density of the free electric charges in the medium, and $\vec{E}$ is the vector of the electric field intensity. The coulomb force (22) appears particularly simple in the case of corona discharge, i.e., under the conditions of unipolar electric conduction, when

$$\vec{j} = k\rho\vec{E} \Rightarrow \vec{f}_E = \rho\vec{E} = \vec{j}/k, \tag{23}$$

where $\vec{j}$ is the current density, and k is the mobility of ions of the corona electrode sign, that is, of the constituent of carriers which supply current. Expressions (20)–(23) can appear separately or together on the right hand of equation (19), supplemented by the electric field equations (Maxwell equations) in the case of electric forces. These forces are not to be derived. In the simplest case of an ideal liquid, the equation of motion is derived, replacing the right-hand member of (19) by expression (20), and so on. The main difficulty is that the viscosity forces should be taken into account in the case of a viscous liquid. Our aim is to solve this problem with the help of the dimensional method, and though for the ideal liquid there is solution (19) with automated right-hand member of (20), we will also derive equations for this case using the proposed procedure.

4.1. Equations of the euler ideal liquid motion. It is supposed that the liquid is under the action of the field of pressure forces, thus, to write the motion equation one should find the density of the pressure forces *f* in (19) acting on the volume unit. We obtain a general equation of liquid motion:

$$\gamma\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v}\right] = \vec{f}_p.$$

The force of pressure from the liquid around the particle is searched in form (the partial derivative dimension is omitted):

$$\begin{aligned}[\gamma(\vec{v}\ \nabla)\vec{v}] &= [f_p = f_p(\nabla, p, \vec{v})] \\ &= \mu\nabla^m p^n \nu^1 \to -\nabla p.\end{aligned} \tag{24}$$

Substituting the corresponding dimensions in this formula one can easily obtain: $m = 1$, $n = 1$, and $l = 0$ to arrive at a formula [20] for $\mu = -1$, which is easily found by integrating the formula with respect to the volume and using the Gauss theorem. We derive Euler's equation for the ideal liquid motion from (24):

$$\gamma\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v}\right] = -\nabla p. \tag{25}$$

In electrical and gravitational fields the right-hand member in (25) should be supplemented with the density of the gravity forces $\gamma\vec{g}$ [2] and electric forces $\rho\vec{E}$ [6]:

$$\gamma\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v}\right] = -\nabla p + \gamma\vec{g} + \rho\vec{E}. \tag{26}$$

4.2. Viscous liquid motion equation. two viscosity forces. We are derive the Navier–Stokes equation, whose essence consists in the fact that it accounts for the force of viscous friction between the liquid layers in the following form:

$$\vec{f}' = \vec{f}'(\eta, \nabla, \vec{v}, p) = \eta\nabla^m\vec{v}^n p^k, \tag{27}$$

where on the right-hand side, only the quantities that are de facto inherent to the right-hand member of (2 appear); $\mu \equiv \lambda$ is the dynamic viscosity coefficient. According to the laws of dimensions we apply the dimensions to (27) and find *m, n*, and, *k*, equal to the corresponding indexes of power on the both sides of the equality:

$$[\vec{f}] = ML^{-2}T^{-2};\ \ [\eta] = ML^{-1}T^{-1};$$
$$[\nabla] = L^{-1};\ \ [\vec{v}] = LT^{-1};$$
$$[p] = M^kL^{-k}T^{-2k},$$

and obtain:

$$\begin{aligned}ML^{-2}T^{-2} &= ML^{-1}T^{-1}L^{-m}\left(LT^{-1}\right)^n, \\ M^k\left(L^{-1}T^{-2}\right)^K &\Rightarrow m = 2;\ \ n = 1;\ \ k = 0.\end{aligned} \tag{28}$$

Thus, according to (27) the solution of the problem is as follows:

$$\vec{f}' = \eta(\nabla,\nabla)\vec{v} = \eta\nabla^2\vec{v}, \tag{29}$$

the known expression for the liquid viscosity force. It is easy to see that substituting pressure with density in (27), we obtain the same result (29), which apparently points to a small dependence of viscosity forces on pressure and density on the whole. However, to formula (27) there corresponds another expression for another twofold application of the operator ∇ leading to the notion of *the second viscosity* [2, 4, 5] with the other transfer coefficient ζ but with the same dimension as η:

$$\vec{f}'' = \zeta\nabla(\nabla,\vec{v}) \equiv \zeta\mathrm{grad}(\mathrm{div}\vec{v}). \quad (30)$$

The double vector product has the same dimension; it, however, falls into two previous products according to the known identical equation of vector analysis:

$$\nabla(\nabla\vec{v}) = \mathrm{rot}(\mathrm{rot}\vec{v}) \equiv \nabla(\nabla,\vec{v}) - \nabla^2\vec{v}, \quad (31)$$

with no contribution to the motion equation, as they are already included in formulas (29) and (30). Thus, the problem is settled, and the resulting friction force is determined according to formulas (29) and (30) by the following:

$$\vec{f}_{tr} = \eta\nabla^2\vec{v} + \zeta\nabla(\nabla,\vec{v}), \quad (32)$$

and, thus, there is solved the question concerning two viscosity forces: one is associated with the sliding friction between the liquid layers with respect to each other and described by the velocity Laplacian, and the other is caused by the friction appeared in the process of the compression and extension of the liquid and expressed through the velocity divergence, and, therefore, it is vanishing for an incompressible liquid.

A detailed mathematical derivation of this equation on the basis of tensor analysis (see, for example, [4]) shows that in (30) ,

$$\zeta = \xi + \eta/3, \quad (33)$$

where ξ is the specific friction coefficient of the volume deformation at the absence of sliding (at $\eta = 0$) and is called the *second* viscosity coefficient [2]. However, for the incompressible liquid, i.e., in the most frequent case in practice the coefficient ξ is not important, as it gets out of the common set of equations due to $\nabla\vec{v} \equiv \mathrm{div}\vec{v} = 0$. In the general case there is valid the Navier–Stokes equation in form [2]:

$$\gamma\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v}\right] = -\nabla p + \vec{f} + \eta\nabla^2\vec{v} + (\xi + \eta/3)\nabla(\nabla,\vec{v}), \quad (34)$$

where the basic difficulties were overcome owing to the dimensions. In (34) $\vec{f}$ is the volume density of the external forces, for example, the gravitational or electrostatic field.

CONCLUSIONS

(1) A general idea of applying the dimensional analysis to derive equations of hydro- and electrohydrodynamics is presented.

(2) The equations of continuity, heat conduction, diffusion, motion of ideal (Euler) and viscous (Navier–Stokes) liquids are derived on this basis.

(3) The dimensional method was used to solve the problem concerning the presence of two viscous friction forces: sliding and deformation (at compression extension in the compressible liquid). The compact formulas for these forces are derived.

CONFLICT OF INTEREST

The authors report no conflict of interest.

REFERENCES

1. Lykov, A.V., *Teplomassoobmen. Spravochnik* (Heat and Mass Transfer: Handbook), Moscow: Energiya, 1978.
2. Landau, L.D. and Lifshitz, E.M., *Course of Theoretical Physics*, Vol. 6: *Fluid Mechanics*, Oxford: Pergamon, 1987.
3. Sedov, L.I., *Metody podobiya i razmerennosti v mekhanike* (Methods of Similarity and Dimension in Mechanics), Moscow: Nauka, 1977.
4. Kochin, N.E., Kibel', I.A., Roze, N.V., *Teoreticheskaya gidromekhanika* (Theoretical Hydromechanics), Moscow: Fizmatgiz, 1963, part 2.
5. Loitsyanskii, L.G., *Mekhanika zhidkosti i gaza* (Mechanics of Fluid and Gas), Moscow: Drofa, 2003.
6. Bologa, M.K., Grosu, F.P., and Kozhukhar', I.A., *Elektrokonvektsiya i teploobmen* (Electroconvection and Heat Exchange), Chisinau: Shtiintsa, 1977.
7. Szirtes, T. and Rozsa, P., *Applied Dimensional Analysis and Modeling*, Amsterdam: Elsevier, 2006.
8. Bolster, D., Hershberger, R.E., and Donnelly, R.J., *Phys. Today*, 2011, vol. 64, no. 9, pp. 242–247.
9. Yarin, L.P., *The Pi-Theorem: Applications to Fluid Mechanics and Heat and Mass Transfer*, Berlin: Springer-Verlag, 2012.
10. Jensen, J.H., *Am. J. Phys.*, 2013, vol. 81, pp. 688–694. https://doi.org/10.1119/1.4813064
11. Lemons, D.S., *A Student's Guide to Dimensional Analysis*, Cambridge: Cambridge Univ. Press, 2017.
12. Grosu, F.P., Bologa, M.K., Leu, V.I., and Bologa, Al.M., *Surf. Eng. Appl. Electrochem.*, 2012, vol. 48, no. 3, pp. 253–259.

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