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On the Shape of a Charged Drop in the Electrostatic Field of an Extended Spheroid Supported at a Constant Electric Potential

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Abstract—In asymptotic calculations, the equilibrium shape of a charged drop is found in a non-uniform electrostatic field created by an extended spheroid modeling a rod supported at a constant electric potential. It has been found that the size of the small axis of the spheroid (thickness of the rod) that creates the field markedly affects the equilibrium shape of the charged drop. The distortion of the spherical shape of the surface of the charged conducting drop of an ideal incompressible liquid in a non-uniform electrostatic field of the rod can be approximately described by the superposition of the excited second and third modes, the amount of which is determined by the required degree of precision of the description.

Keywords: drop, charge, electrostatic field, rod, creating a field, modeled by extended spheroid

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INTRODUCTION

The problem of calculating the equilibrium shape of a charged drop in non-uniform electrostatic fields is of interest for multiple academic, technical, and technological fields of study. It is particularly fascinating in reference to the problems of the contactless determination of the physicochemical properties of liquids [1] and experimental verification of the Rayleigh-criterion [2] realization of electrostatic instability of a strongly charged drop [3–6], since in most types of electrostatic suspensions, the non-uniform electrostatic fields are used for holding and positioning the charged drop. This problem is also intriguing from the view point of the analysis of the laws of the evolution of a cluster-drop phase of ion beams in liquid-metal sources and liquid mass spectrometers [7, 8]. The critical conditions of the realization of instability of the charged electroconducting drop also depend on the drop's equilibrium shape [9]. This study is devoted to the solution of the above problem.

Let us take the case of an external non-uniform field created with a rod of finite length and thickness that is supported at a constant potential (or a rod that is charged) as an example of what is realized most often in practice (see, e.g., [6]). A similar problem was already studied in [10], however, the non-uniform electrostatic field was modeled with the field induced by a fraction of an extremely thin filament (the analytical expression for the potential of the filament; see, e.g., [11, p. 227, problem 74]).

In practice, the electrostatic field of a rod with finite length and thickness has a complicated configuration, and there is no accurate analytical expression needed for its rigorous solution. Below, we model the electrostatic field of the rod using the field of a strongly prolate charged conducting spheroid [12, p. 40]. The large and small semiaxes of the spheroid [characterize the length and thickness of the rod, correspondingly, which will allow us to analyze the effects of these parameters on the obtained solution.

It should be mentioned that the combination of the rod electrode and a counter electrode in the form of a plane ring is typical for many apparatuses and devices for the electrodispersion of insecticides, gasoline and lubricants, and coating compositions (see, e.g., [13–15] and references therein).

PROBLEM SETTING

Let us examine a spherical drop with radius R of an ideal incompressible, ideally conducting liquid with density ρ, surface tension coefficient σ, and carrying charge Q. Let the drop center be located on the longitudinal symmetry axis of a conducting rod with a length of $2a$ and a diameter of $2b$ at distance L from its butt-end. The rod is supported at a constant potential φ_0.

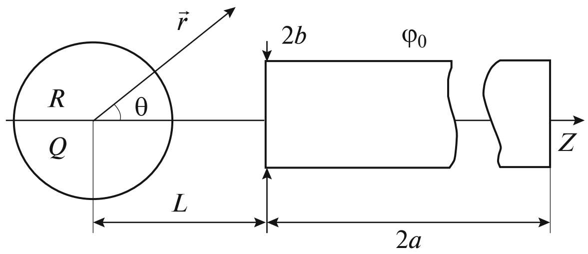

Fig. 1. Relative position of rod and charged drop.

Since the external electrostatic field disturbs the spherical shape of the drop, let us determine the shape of its equilibrium surface. We solve the problem within the spherical coordinates (r, θ, φ) with the coordinate origin in the center of the drop masses. Axis OZ, whose positive direction is the starting point for the calculation of angle θ, we consider to coincide with the longitudinal symmetry axis of the rod, which is directed from the center of the drop mass to the rod (Fig. 1).

The system of calculation is non-inertial, due to the accelerated motion of the charged drop as a unit in the external electric field.

The equilibrium shape of the drop $r(\theta)$ is determined from the condition of a balance of pressures on the surface of the drop:

$$P_0 + P_{\text{in}}(\vec{r}) - P_{\text{atm}} + P_E(\vec{r}) = P_\sigma(\vec{r}). \qquad (1)$$

In (1) P_0 is hydrostatic pressure; P_{atm} is ambient pressure; $P_{\text{in}}(\vec{r})$ is inertial pressure connected with the noninertiality of selected frame of reference; $P_E(\vec{r})$ is electrostatic pressure and $P_\sigma(\vec{r})$ is capillarypressure, which are expressed by the following formulas:

$$P_E(\vec{r}) = \frac{\vec{E}^2(\vec{r})}{8\pi}; \quad P_\sigma(\vec{r}) = \sigma \cdot \operatorname{div}\vec{n}(\vec{r}).$$

Here $\vec{E}^2(\vec{r})$ is the intensity of electrostatic field near the drop; $\vec{n}(\vec{r})$ is a normal vector to the drop surface, which is determined using the following formula:

$$\vec{n}(\vec{r}) = \left.\frac{\nabla(r - r(\theta))}{|\nabla(r - r(\theta))|}\right|_{r=r(\theta)}.$$

To calculate the electric pressure onto the drop surface we need to determine field potential $\Phi(\vec{r})$ in the vicinity of the drop. This potential will be a superposition of potential $\Phi_r(\vec{r})$ external with respect to the drop, which is created by the rod, on potential $\Phi_{\text{d}}(\vec{r})$, created by the drop's self-charge.

$$\Phi(\vec{r}) = \Phi_r(\vec{r}) + \Phi_{\text{d}}(\vec{r}).$$

The analytic expression of the rod potential that is modeled by the potential of the conducting spheroid will be obtained in the following way, namely, in the known [12, p. 40] expression for the potential of the charged conducting spheroid we replace the charge through the potential and electrical capacity of the spheroid. As a result, in the Cartesian system of coordinates, which is connected with the spheroid center, the analytic expression for the potential will take the following form:

$$\Phi_{\text{sph}}(\vec{r}) = \frac{\varphi_0}{\operatorname{arccoth}\sqrt{1 - \dfrac{b^2}{a^2}}} \operatorname{arccoth}\left(2\sqrt{\frac{a^2 - b^2}{\left(\sqrt{(-\sqrt{a^2 - b^2} + z)^2 + x^2 + y^2} + \sqrt{(\sqrt{a^2 - b^2} + z)^2 + x^2 + y^2}\right)^2}}\right), \qquad (2)$$

where a and b are large and small semiaxes of the spheroid.

Let us assume that the drop ideally conducts, i.e., we imply that the characteristic hydrodynamic time is much longer than the drop charge relaxation time. The surface of the drop will be accepted as equipotential and the boundary problem will be formulated for the determination of the potential near the drop surface:

$$\Delta\Phi(\vec{r}) = 0; \quad \Phi(\vec{r})\big|_{r=r(\Theta)} = \text{const}; \quad r \to \infty: \Phi \to 0.$$

Let us add three more conditions to the problem set: conservation of the drop volume, immobility of its center of masses in the selected system, and conservation of the drop charge:

$$\iiint_V dV = \frac{4}{3}\pi R^3; \quad \iiint_V \vec{r}\,dV = 0; \quad \iint_S \kappa(\vec{r}) \cdot dS = Q;$$

$$\{V\!: 0 \le r \le r(\theta), \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi\};$$

$$\{S\!: r = r(\theta), \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi\},$$

where κ is the density of the charge on the surface of the drop:

$$\kappa(\vec{r}) = \frac{1}{4\pi}\left(\vec{E}(\vec{r}), \vec{n}(\vec{r})\right).$$

For simplicity of calculation let us proceed to dimensionless variables, accepting $R = \rho = \sigma = 1$. We leave the previous designations in the dimensionless physical values.

The values of hydrodynamic and electric potential, as well as the intensities of the electric field and of the field of rates are expressed in the shares of their typical scales:

$$\psi_* = R^{1/2}\rho^{-1/2}\sigma^{1/2};\quad \varphi_* = R^{1/2}\sigma^{1/2};$$
$$E_* = R^{-1/2}\sigma^{1/2};\quad V_* = R^{-1/2}\rho^{-1/2}\sigma^{1/2}.$$

The equilibrium shape of the drop $r(\theta)$ will be sought as a superposition of the spherical shape of the surface and its minor dimensionless distortion $h(\theta)$: $\max|h(\Theta)| \ll 1$. The distortion is presented in the form of an expansion on the Legendre polynomials:

$$r(\theta) = 1 + h(\theta);\quad h(\theta) = \sum_{n=0}^{\infty} a_n P_n(\mu);\quad \mu \equiv \cos\theta. \quad (3)$$

Since the self-charge of the drop in itself does not disturb its sphericity, the distortion of the shape of the equilibrium surface of the drop will be determined by the pressure of the electrostatic field near the drop. Therefore, we can write the following estimates for the orders of values:

$$\left(E_r(\vec{r})\right)^2 \sim \left(\Phi_r(\vec{r})\right)^2 \sim \max|h(\theta)|.$$

In dimensionless variables the estimation of the value of the external field and the potential will resemble the following:

$$E_r(\vec{r}) \sim \Phi_r(\vec{r}) \sim \left(\max|h(\theta)|\right)^{1/2} = \varepsilon^{1/2},$$

where $\varepsilon \equiv \max|h(\theta)|$ is a dimensionless parameter.

We consider small deviations of the drop from a spherical shape, assuming ε to be a minor parameter of the problem and taking into account the terms of sum up to the value of the order of ε^1 inclusive.

Estimating the values for pressure falling under dynamic condition (1), we obtain that hydrostatic pressure P_0 and ambient pressure P_{atm} are of the order of ε^0, since they are independent of the shape of the surface of the drop. Let us accept that the characteristic time of formation of the equilibrium surface of the drop is much less than the time of a significant shift of the center of masses of the drop in an external electric field. Since inertial pressure $P_{\text{in}}(\vec{r},t)$ is induced primarily by the interaction of the drop charge with the external field, it will have the form $P_{\text{in}} \sim QE_r \sim \varepsilon^{1/2}$ (as the presence of the charge on the drop causes no disturbance of its spherical shape, i.e., $Q \sim \varepsilon^0$). We do not consider corrections to the inertial force on the part of other effects, because they are negligibly small in comparison with the above effect. The pressure of the capillary forces is defined by the size of the drop and the shape of distortion of its surface; therefore, it has components of the zeroth and first orders $P_\sigma(\vec{r}) = P_\sigma^0(\vec{r}) + P_\sigma^1(\vec{r})$. From now on in the text, the above indices will designate the orders of size of the values for ε.

The potential $\Phi(\vec{r})$ that designates electric pressure will have three components: $\Phi(\vec{r}) = \Phi^0(\vec{r}) + \Phi^{1/2}(\vec{r}) + \Phi^1(\vec{r})$. Here $\Phi^0(\vec{r})$ is potential, which is created by the self-charge of undistorted drop; $\Phi^{1/2}(\vec{r})$ is potential created by the charges induced by the external electrostatic field in a spherical drop; and $\Phi^1(\vec{r})$ is a component that is connected with the field alteration caused by the distortion of the spherical shape of the drop. Similarly, electric pressure is also presented as the following expansion:

$$P_E(\vec{r}) = P_E^0(\vec{r}) + P_E^{1/2}(\vec{r}) + P_E^1(\vec{r}).$$

PROBLEM OF THE ε^0 ORDER

This problem describes a spherical charged drop in the absence of an external field and is expressed by the following equations:

$$P_0 - P_{\text{atm}} + P_E^0(\vec{r}) = P_\sigma^0;$$
$$P_E^0(\vec{r}) = \frac{1}{8\pi}\left(\nabla\Phi^0(\vec{r})\right)^2\Big|_{r=1};$$
$$P_\sigma^0 = 2;\quad \Delta\Phi^0(\vec{r}) = 0;$$
$$\Phi^0(\vec{r})\Big|_{r=1} = \text{const};\quad r\to\infty:\ \Phi^0(\vec{r}) \to 0;$$
$$\int_0^{2\pi}\int_{-1}^{1} \partial_r\Phi^0(\vec{r})\Big|_{r=1} d\varphi d\mu = -4\pi Q.$$

Solving this problem, we obtain an expression for the component of the electric potential of a self-charge of the spherical drop:

$$\Phi^0(\vec{r}) = \frac{Q}{r}. \quad (4)$$

PROBLEM OF THE $\varepsilon^{1/2}$ ORDER

Let us proceed to the problem of the $\varepsilon^{1/2}$ order, which describes the redistribution of the drop charge in an external electrostatic field:

$$\Delta\Phi^{1/2}(\vec{r}) = 0;\quad r\to\infty:\ \Phi^{1/2}(\vec{r}) \to 0;$$
$$r = 1:\ P_{\text{in}}(\vec{r}) + P_E^{1/2}(\vec{r}) = 0;$$
$$P_E^{1/2}(\vec{r}) = \frac{1}{4\pi}\left(\nabla\Phi^0\cdot\nabla\Phi^{1/2}(\vec{r})\right);\quad \Phi^{1/2} = \text{const};$$
$$\int_0^{2\pi}\int_{-1}^{1} \partial_r\Phi^{1/2}(\vec{r})\Big|_{r=1} d\varphi d\mu = 0.$$

Here, potential $\Phi^{1/2}(\vec{r})$ consists of the external (for the drop) potential of the rod $\Phi_r(\vec{r})$ and the component, which has a relevant order of smallness, of the drop potential $\Phi_d^{1/2}(\vec{r})$: $\Phi^{1/2}(\vec{r}) = \Phi_r(\vec{r}) + \Phi_d^{1/2}(\vec{r})$. The potential that is created by the, $\Phi_r(\vec{r})$ is known (see (2)); however, in the spherical system of coordinates that is connected with the center of the drop masses. It will be rewritten as follows:

$$\Phi_r(\vec{r}) = \frac{\varphi_0}{\operatorname{arccoth}\sqrt{1-\dfrac{b^2}{a^2}}}\operatorname{arccoth}\sqrt{\frac{4(a^2-b^2)}{\left(\sqrt{(r\mu-\sqrt{a^2-b^2})^2+r^2\left(1-\mu^2\right)}+\sqrt{(r\mu+\sqrt{a^2-b^2})^2+r^2\left(1-\mu^2\right)}\right)^2}}. \quad (5)$$

It is noteworthy that potential $\Phi_r(\vec{r})$ is a solution of the Laplace equation and fulfils the condition of decreasing to zero at infinity. Taking this into account, from the boundary problem obtained above, we obtain a problem for the relevant component of the drop potential $\Phi_d^{1/2}(\vec{r})$:

$$\Delta\Phi_d^{1/2}(\vec{r}) = 0; \quad r \to \infty: \Phi_d^{1/2}(\vec{r}) \to 0, \quad (6)$$

$$\Phi_d^{1/2}(\vec{r})\Big|_{r=1} = \text{const} - \Phi_r(\vec{r})\big|_{r=1}, \quad (7)$$

$$\int_0^{2\pi}\int_{-1}^{1} \partial_r\Phi_d^{1/2}(\vec{r})\Big|_{r=1} d\varphi d\mu = -\int_0^{2\pi}\int_{-1}^{1} (\partial_r\Phi_r(\vec{r}))\big|_{r=1} d\varphi d\mu. \quad (8)$$

A solution of the Laplace equation (6) that decreases with distance for potential $\Phi_d^{1/2}$ resembles a Legendre polynomial expansion $P_k(\mu)$:

$$\Phi_d^{1/2}(\vec{r}) = \sum_{k=1}^{\infty} B_k^{1/2} r^{-(k+1)} P_k(\mu). \quad (9)$$

Substituting the obtained $\Phi_d^{1/2}(\vec{r})$ (9) into the condition of equipotentiality (7), we get:

$$\sum_{k=1}^{\infty} B_k^{1/2} P_k(\mu) = \text{const} - \Phi_r(\vec{r})\big|_{r=1}. \quad (10)$$

Table 1. Coefficients F_k in expansion in a series on the Legendre polynomials of the electric field potential created by the rod on the drop surface, calculated at $L = 2$, $b = 1$, $\varphi_0 = 1$ (in dimensional units $\varphi_0 = 454$ V for $R = 1$ mm, $\sigma = 23$ dyn/cm)

F_k	$a = 10$	$a = 30$	$a = 100$
F_0	0.396	0.418	0.435
F_1	0.739×10^{-1}	0.586×10^{-1}	0.466×10^{-1}
F_2	0.197×10^{-1}	0.150×10^{-1}	0.117×10^{-1}
F_3	0.646×10^{-2}	0.496×10^{-2}	0.390×10^{-2}
F_4	0.236×10^{-2}	0.185×10^{-2}	0.146×10^{-2}
F_5	0.922×10^{-3}	0.732×10^{-3}	0.582×10^{-3}
F_6	0.375×10^{-3}	0.303×10^{-3}	0.242×10^{-3}
F_7	0.157×10^{-3}	0.129×10^{-3}	0.103×10^{-3}

To obey the above condition at any angle θ, we must represent potential $\Phi_r(\vec{r})\big|_{r=1}$ as an expansion of Legendre polynomials:

$$\Phi_r(\vec{r})\big|_{r=1} = \sum_{k=0}^{\infty} F_k P_k(\mu); \quad (11)$$

$$F_k = \frac{2k+1}{2}\int_{-1}^{1} \Phi_r(\vec{r})\big|_{r=1} P_k(\mu) d\mu, \quad (12)$$

where potential $\Phi_r(\vec{r})$ is defined by formula (5), and coefficients F_k depend on parameters of the rod field φ_0, a, b and distance to the rod end L. The analytical calculation of F_k is fairly difficult because of the bulkiness of the potential under study $\Phi_r(\vec{r})$, therefore it is worthwhile to use the numerical estimates of the above coefficients. This approach requires to terminate infinite series (11) on a finite quantity of terms. Table 1 lists the values of coefficients F_k at different rod lengths. It is seen that with increases in the number of modes k at any length of rod, the coefficients rapidly decrease; therefore, the use of a finite quantity of terms in (11), instead of an infinite series, is reasonable. (We can note that with increases in the rod's length, coefficients F_k decrease insignificantly.)

Table 2, shows coefficients F_k calculated at a larger distance from the rod's end to the drop. Comparing the data of the two tables, we can conclude that the rate of decrease of the coefficients is $\dfrac{F_k}{F_{k-1}} \sim \dfrac{1}{L}$; i.e., at a longer distances from the rod end, the series meets even faster.

Let us estimate the value of a relative error δ, which is made during the replacement of the exact value of potential Φ_r on the surface of sphere $r = 1$ through its approximate value, which we write as a finite sum:

$$\Phi_{ap} = \sum_{k=0}^{k_m} F_k P_k(\mu).$$

We determine relative error in accordance with the following expression:

$$\delta = \left\| \frac{\Phi_{ap} - \Phi_r}{\Phi_r} \right\|_{r=1}.$$

Figure 2 shows the dependences of relative error δ from angle θ at different values of k_m, which indicate that the highest discrepancies between the precise and approximate values of the potential are found in points $\theta = 0$ and $\theta = \pi$.

Accepting as a criterion of accuracy of expansion the angle maximum value of a relative error and estimating its value in relation to parameters of a, b, and L, makes it possible to define the number of items that are necessary to be taken into account in the potential expansion. According to Fig. 3, to achieve accuracy in the approximate calculations of values ~0.01, it is enough to take into account the first four items in expansion (11), i.e., $k_m = 4$. Thus, the external potential on the spherical surface can be presented as a finite sum:

$$\Phi_r(\vec{r})\big|_{r=1} = \sum_{k=0}^{4} F_k P_k(\mu). \quad (13)$$

From the condition of equipotentiality (10), we can find the coefficients of $B_k^{1/2}$ in expansion (9) in the form of $B_k^{1/2} = -F_k$, and taking into account the condition of charge conservation (8) we can calculate potential $\Phi^{1/2}(\vec{r})$:

$$\Phi^{1/2}(\vec{r}) = \Phi_r(\vec{r}) - \sum_{k=0}^{k_m} F_k r^{-(k+1)} P_k(\mu). \quad (14)$$

The distribution of inertia pressure on the drop surface is determined from the balance of the pressures:

$$r = 1:\ P_{in}(\vec{r}) = -P_E^{1/2}(\vec{r}).$$

Table 2. Coefficients F_k in expansion in series on Legendre polynomials of potentials of electric field that is created by the rod on the drop surface, which are calculated at $L = 5$, $b = 1$, $\varphi_0 = 1$ (in dimensional units $\varphi_0 = 454$ V for $R = 1$ mm, $\sigma = 23$ dyn/cm)

F_k	$a = 10$	$a = 30$	$a = 100$
F_0	0.267	0.313	0.350
F_1	0.264×10^{-1}	0.225×10^{-1}	0.184×10^{-1}
F_2	0.314×10^{-2}	0.241×10^{-2}	0.189×10^{-2}
F_3	0.429×10^{-3}	0.322×10^{-3}	0.251×10^{-3}
F_4	0.641×10^{-4}	0.482×10^{-4}	0.376×10^{-4}
F_5	0.102×10^{-4}	0.769×10^{-5}	0.601×10^{-5}
F_6	0.168×10^{-5}	0.128×10^{-5}	0.100×10^{-5}
F_7	$0.285 \times 10^{--6}$	0.218×10^{-6}	0.171×10^{-6}

In calculations, it is accepted that the characteristic time for the formation of the equilibrium shape of the drop surface (hydrodynamic time) is substantially less than the time for the significant shift of the center of masses of the charged drop under the external electric field (kinematic time); i.e.,

$$\sqrt{\rho R^3/\sigma} < \sqrt{\frac{R}{QE_r m}},$$

when m is the drop mass, which gives:

$$QE_r < \rho^{-2} R^{-5} \sigma.$$

Fulfilling this condition, we assume that the distance L from the drop center to the rod's end remains unchanged.

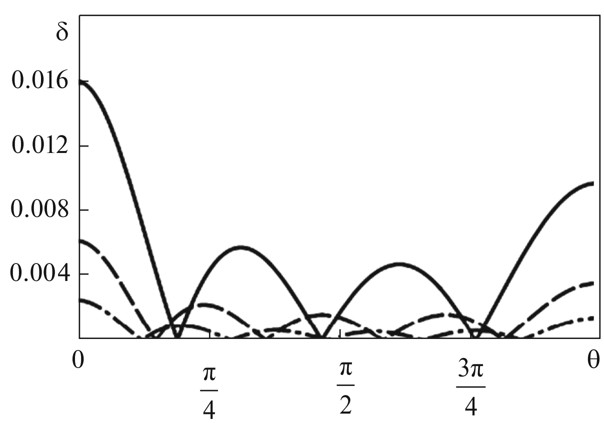

Fig. 2. Dependence of a relative error of potential δ on the angle at different quantity of considered polynomials k_{max} and the following values of parameters: $L = 2$, $a = 30$, $b = 1$. Solid line represents $k_{max} = 2$, dashed line represents $k_{max} = 3$, and dash-and-dot line represents $k_{max} = 4$.

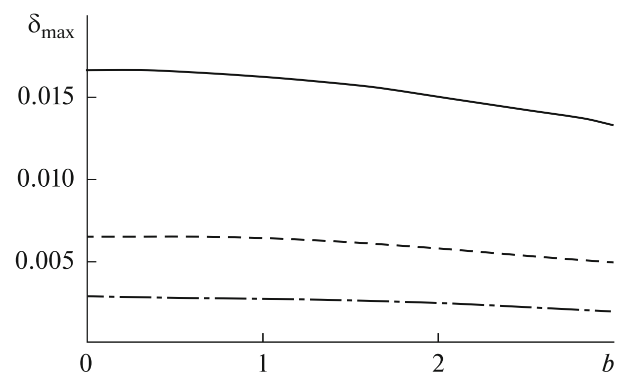

Fig. 3. Dependence of maximal-angle potential error δ on the thickness of rod at a various quantity of k_{max} polynomials taken into account and the following values of parameters $L = 2$, $a = 30$. Solid line represents $k_{max} = 2$, dashed line represents $k_{max} = 3$, and dash-and-dot line represents $k_{max} = 4$.

PROBLEM OF THE ε^1 ORDER

In the first ε order of smallness we obtain the following problem:

$$r = 1:\ P_E^1(\vec{r}) = P_\sigma^1(\vec{r});\quad \Delta\Phi^1(\vec{r}) = 0;$$
$$r \to \infty:\ \Phi^1(\vec{r}) \to 0; \tag{15}$$
$$\Phi^1(\vec{r})\Big|_{r=1} = \mathrm{const} - h(\theta)\partial_r\Phi^0(\vec{r})\Big|_{r=1},$$
$$\int_{-1}^{1} h(\theta)d\mu = 0;\quad \int_{-1}^{1} h(\theta)\mu d\mu = 0, \tag{16}$$
$$\int_0^{2\pi}\int_{-1}^{1}\big(\partial_r\Phi^1(\vec{r}) + (2\partial_r\Phi^0(\vec{r})) + \Phi^0(\vec{r}))h(\theta)\big)\Big|_{r=1} d\varphi d\mu = 0, \tag{17}$$
$$P_E^1(\vec{r}) = \left(\partial_r\frac{\left(\nabla\Phi^0(\vec{r})\right)^2}{8\pi}h(\theta) + \frac{\nabla\Phi^0(\vec{r})\nabla\Phi^1(\vec{r})}{4\pi} + \frac{\left(\nabla\Phi^{1/2}(\vec{r})\right)^2}{8\pi}\right)\Bigg|_{r=1}, \tag{18}$$
$$P_\sigma^1(\theta) = -\left(2h(\theta) + \frac{1}{\sin\theta}\partial_\theta\left(h(\theta)\sin\theta\right)\right). \tag{19}$$

Consideration of (16) allows us to specify the low value of a summation index in (3) for the distortion of $h(\theta)$:

$$r(\theta) \equiv 1 + h(\theta) = 1 + \sum_{n=2}^{\infty} a_n P_n(\mu), \tag{20}$$

and write down (19), taking into account (20) via amplitude a_n, in the following form:

$$P_\sigma^1 = \sum_{n=2}^{\infty} a_n(n-1)(n+2)P_n(\mu). \tag{21}$$

The boundary problem for finding potential $\Phi^1(\vec{r})$ is solved similarly to that considered above; i.e., we present the potential from the Laplace equation as an expansion according to Legendre polynomials:

$$\Phi^1(\vec{r}) = \sum_{k=1}^{\infty} B_k^1 r^{-(k+1)}P_k(\mu). \tag{22}$$

Substituting the general view of potential $\Phi^1(\vec{r})$ (22) in conditions (15) and (17) taking into account (20) for distortion of equilibrium surface, we determine coefficients B_k^1 and get the expressions for potential $\Phi^1(\vec{r})$ via amplitudes of distortions a_n:

$$\Phi^1(\vec{r}) = \sum_{n=2}^{\infty} Qa_n r^{-(n+1)}P_n(\mu). \tag{23}$$

The total potential of the electric field near the drop, in an approximation of up to the first order of magnitude up to ε inclusive, will look like the following form:

$$\Phi(\vec{r}) = \frac{Q}{r} + \Phi_r(\vec{r}) - \sum_{k=0}^{k_m} F_k r^{-(k+1)}P_k(\mu) + \sum_{n=2}^{\infty} Qa_n r^{-(n+1)}P_n(\mu).$$

Electrostatic pressure $P_E^1(\vec{r})$ is calculated with regard to expressions (4), (5), (14), (23), using formula (18). The absence of the angle dependence in potential $\Phi^0(\vec{r})$ (see (4)) and the condition of the equipotentiality of drop surface in (7) result in a simplified expression for pressure $P_E^1(\vec{r})$:

$$r = 1:\ P_E^1(\vec{r}) = \partial_r\frac{(\partial_r\Phi^0(\vec{r}))^2}{8\pi}h(\theta) + \frac{\partial_r\Phi^0(\vec{r})\partial_r\Phi^1(\vec{r})}{4\pi} + \frac{1}{8\pi}\left(\partial_r\Phi^{1/2}(\vec{r})\right)^2. \tag{24}$$

In the last item, which can be written as follows:

$$\frac{1}{8\pi}\left(\partial_r\Phi^{1/2}(\vec{r})\right)^2\Big|_{r=1} = \frac{1}{8\pi}\left(\partial_r\Phi_{\mathrm{d}}^{1/2}(\vec{r}) + \partial_r\Phi_r(\vec{r})\right)^2\Big|_{r=1},$$

the need arises to present a normal component of the external electric field intensity on the spherical surface in the form of a series of Legendre polynomials:

$$\partial_r\Phi_r(\vec{r})\Big|_{r=1} = -(\vec{E}_r(\vec{r}), \vec{n}(\vec{r}))\Big|_{r=1} = -E_{n,r}(\vec{r})\Big|_{r=1} = \sum_{l=0}^{\infty} S_l P_l(\mu), \tag{25}$$

$$S_l = \frac{2l+1}{2}\int_{-1}^{1}\partial_r\Phi_r(\vec{r})\Big|_{r=1} P_l(\mu)d\mu. \tag{26}$$

Since the analytical calculation of coefficients S_l is difficult due to their bulk, we must use numerical estimations, and, hence, restrict expansion (25) using a finite number of polynomials:

$$E_{n,r}(\vec{r})\Big|_{r=1} = -\partial_r\Phi_r(\vec{r})\Big|_{r=1} \approx E_{n,\mathrm{ap}}(\vec{r})\Big|_{r=1} = -\sum_{l=0}^{l_m} S_l P_l(\mu), \tag{27}$$

where $E_{n,\mathrm{ap}}(\vec{r})$ is an approximate value of field intensity.

The dependences of the relative error $\gamma = \left\|\frac{E_{n,\mathrm{ap}} - E_{n,r}}{E_{n,r}}\right\|_{r=1}$ of the approximate presentation of the normal component of intensity on angle θ are quantitatively similar to those presented in Figs. 2, 3.

The calculations show that to achieve the value of the relative error $\gamma < 0.01$, we must take into account $l_m = 8$ of polynomials in expansion (27). This value exceeds $k_m = 4$, which was obtained in the estimation of the error of expansion (13). Therefore, for uniformity, we accept that $k_m = l_m = 8$, with the accuracy of expansion (13) increasing even more.

Using (27), we can bring expression (24) for the electric field pressure, considering expressions (4), (14), (23), to the following form:

$$P_E^1(\theta) = \sum_{n=1}^{\infty}\left(\left(\frac{Q^2}{4\pi}\right)(n-1)a_n + \frac{1}{8\pi}\sum_{k=1}^{k_m}\sum_{m=-k}^{k}\left(S_k - (k+1)F_k\right) \times \left(S_{n+m} - (n+m+1)F_{n+m}\right)U_{m,n+m,n}\right)P_n(\mu); \tag{28}$$

$$U_{a,b,c} = \left(C_{a,0b,0}^{c,0}\right)^2,$$

where $C_{l1,ml2,m2}^{l,m}$ are coefficients of Clebsch–Gordan [16], and F_k and S_k are numerical coefficients, which are determined by formulas (12) and (26), using (5).

Substituting (28) and (21) into the condition of balance of pressures, we obtain the expressions for amplitudes a_n in the expansion of equilibrium shape of the drop (20):

$$a_n = \frac{1}{8\pi}\frac{1}{(n+2-Q^2/4\pi)(n-1)} \times \sum_{k=1}^{k_m}\sum_{m=-k}^{k}\left(S_k - (k+1)F_k\right) \times \left(S_{n+m} - (n+m+1)F_{n+m}\right)U_{m,n+m,n}. \tag{29}$$

The amplitude of distortion $a_n = 0$ at $n \geq k_m$ and $n < 2$. Taking this into account, the equilibrium form of the drop surface $r(\theta)$ (20) will be written as follows:

$$r(\theta) = 1 + \sum_{n=2}^{2k_m} a_n P_n(\mu). \tag{30}$$

ANALYSIS OF EQUILIBRIUM FORMS

Let us proceed to the analysis of the equilibrium form of the drop. The range of possible values of self-charges on the surface of a stable spherical charged drop will be determined based on the Rayleigh criterion. It is known [2] that for the stability of the drop with respect to its self-charge, condition $Q^2/4\pi\sigma R^3 < 4$ must be fulfilled. This gives us $0 \leq |Q| < |Q_{\mathrm{cr}}| \equiv 4\sqrt{\pi\sigma R^3}$ or in the dimensionless form $|Q_{\mathrm{cr}}| = 4\sqrt{\pi}$. The presence of the external electrostatic field will certainly affect the drop's stability [13–14, 17–18]. However, for construction of the equilibrium shapes of the drop, we use, e.g., the value of the charge $Q = \dfrac{Q_{\mathrm{cr}}}{2}$ as a typical value.

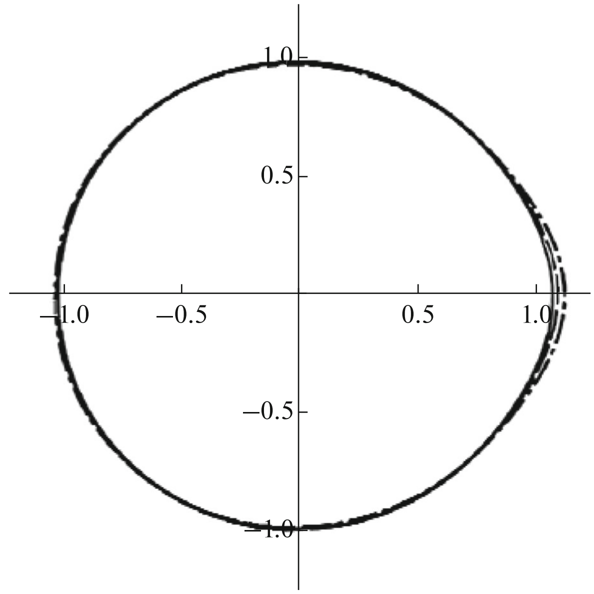

Fig. 4. Equilibrium shapes of drops in the fields of rods with different thicknesses calculated at $L = 2$, $a = 30$. $Q = Q_{\mathrm{cr}}/2$, $\varphi_0 = 9$ (in dimensional units $\varphi_0 = 4.1$ kV for $R = 1$ mm, $\sigma = 23$ dyn/cm). Solid line represents $b = 0.3$, dashed line represents $b = 0.5$, and dash-and-dot line represents $b = 1$.

We set the value of the rod potential φ_0 (which characterizes the value of the external non-uniform field) in dimensionless variables at the maximum possible (at the accepted geometrical dimensions of the rod), so that they simultaneously remain within the scope of assumption of the smallness of distortion. Accepting that $\max|h(\theta)| \leq 0,2$ for distances L which are used in calculations, as well as for the values of length and thickness of the rod (of large and small spheroid semiaxes a and b) we obtain an estimation of the value of the rod potential $\varphi_0 \approx 9$. Considering $R =$ 1 mm ethanol drop with the surface tension coefficient of $\sigma = 23$ dyn/cm, this value of the dimensionless potential is relevant to the dimension value of the potential of 4.1 kV.

Figure 4 shows the equilibrium shapes of the drop in the field of rods with various thicknesses. The calculations show that examining thin $b \leq R$ long rods reveals that the distortion value increases with the rod extension.

Figure 5 shows the equilibrium shapes of the drop surface in the field of rods with different lengths. Note that the total value of the distortion decreases with an increase in the rod's length.

To explain this tendency, Fig. 6 shows the dependences of the value of field intensity of the rod at distance H from its end at rod's length a; the dependences are calculated according to the precise expression of potential (5). It is seen that at values of H of the order

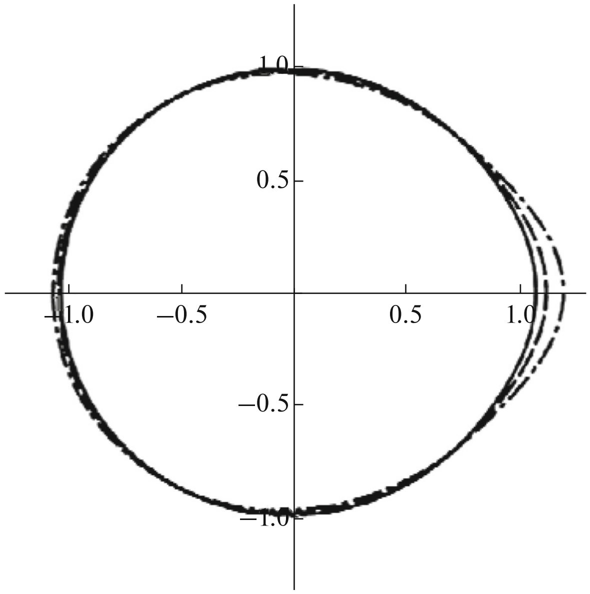

Fig. 5. Equilibrium forms of drop in the fields of rods with various lengths calculated at $L = 2$, $b = 1$, $Q = Q_{cr}/2$, $\varphi_0 = 9$ (in dimension units $\varphi_0 = 4.1$ kV for $R = 1$ mm, $\sigma = 23$ dyn/cm). Solid line represents $a = 100$, dashed line represents $a = 30$, and dash-and-dot line represents $a = 10$.

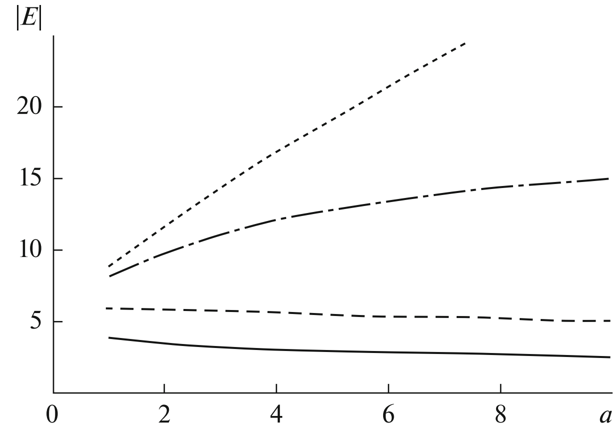

Fig. 6. Dependences of the value of electrostatic field intensity near the rod's end on its length, calculated at $b = 1$, $\varphi_0 = 9$ (in dimensional units $\varphi_0 = 4.1$ kV for $R = 1$ mm, $\sigma = 23$ dyn/cm). Solid line represents $H = 0.5$, dashed line represents $H = 0.25$, and dash-and-dot line represents $H = 0.05$, dotted line to $H = 0$.

of or greater than b, the longer rods correspond to smaller values of the field intensities at a preset distance H. It would seem natural to expect that since with an increase in the length of spheroid (which models the rod) its electric capacity increases, meaning that to maintain the rod potential constant involves an increase in its charge and, as a consequence, a growth in intensity of the field created. However, the above statement is valid just for the area near the end of the spheroid which can approximately be described by the ratio $H \ll b$. As to the region of $H \geq b$, the opposite tendency is observed, which is a consequence of the redistribution of the self-charge of the rod during its extension.

Tables 3 and 4 list the values of coefficients a_n in expansion (30). It is easy to see that a_n monotonically decreases with an increase in the number of mode, and comparing the data of Tables 3 and 4 we can see that the rate of decrease enhances with an increase in H. Thus, if we pursue a certain degree of precision in describing an equilibrium shape of the drop, we can decrease a quantity of items in the sum (30) by limiting its polynomials up to the order of $p < 2k_m$, inclusive.

Let us estimate what values of the drop charges make this possible. For this, we introduce a parameter, which characterizes the ratio of the amplitudes' sum of rejected modes to a similar sum of the modes taken into account:

$$\lambda(p) = \sum_{n=p+1}^{2k_m} |a_n| \Bigg/ \sum_{n=2}^{p} |a_n|.$$

Figure 7 shows the dependences of value λ on the charge on the drop at different amounts of p polynomials taken into account in expansion (30). It is seen that for the description of the equilibrium shape with a certain preset accuracy, it is possible to restrict the amount of polynomials that are taken into account by value $p < 2k_m$ (remember it has been accepted above that $k_m = 8$). For example, in case we need a description of the equilibrium shape of a drop surface with an

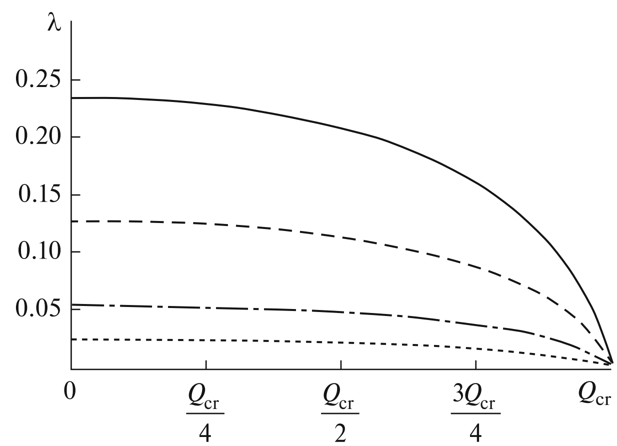

Fig. 7. Dependences of parameter λ on the value of charge on the drop at different amounts p of polynomials taken into account in expansion of equilibrium shape of the drop calculated at $L = 2$, $a = 10$. $b = 1$, $\varphi_0 = 9$ (in dimensional units $\varphi_0 = 4{,}1$ kV for $R = 1$ mm, $\sigma = 23$ dyn/cm). Solid line represents $p = 5$, dashed line represents $p = 6$, dash-and-dot line represents $p = 7$, abd dotted line represents $p = 8$.

accuracy of ≤5% for the rod parameters presented in Fig. 7, we can be restricted in (30) by taking into account the items of up to $P_8(\cos\theta)$, inclusive.

The calculations show that in a non-uniform field of the rod, the contribution to the equilibrium shape of the drop of the two or three first modes substantially exceeds the contribution of all the remaining modes only when the charges of the drop are close to a critical value. However, this case is beyond the scope of applicability of the calculation performed, since at those values of the charge, the assumption of the smallness of the value of distortion $\varepsilon \equiv \max|h(\theta)|$ becomes invalid. If we consider the dependences of ε on the value of the charge on the drop, we shall obtain that the application area of the solutions for (29) and (30), as the values of parameters of the system accepted in calculation will be limited by the value of the self-charge of the drop: $|Q| < \frac{3}{4}|Q_{cr}|$.

Thus, we can infer that the equilibrium shape of the surface of a conducting drop of an ideal liquid in a non-uniform electrostatic field of the rod can be approximately described by the superposition of several first Legendre polynomials (whose amount depends on the required precision of the description) with amplitudes determined by (29).

MODELING OF EXPERIMENTAL DATA

In [6], the disintegration of an ethanol drop was observed experimentally in a non-uniform electrostatic field of a thin corona pin and plane counter-electrode. Initially, an uncharged drop fell through the ion cloud created by the corona discharge from the pin point and was charged in it, being subjected to electrostatic instability. The unstable drop emitted a liquid jet that disintegrated into separate droplets with lengths much longer than the diameter of the drop. It was possible to observe the realization of typical instabilities of several first modes of oscillations of the jet surface. In particular, we could observe axisymmetrical, whip-shaped, and electrostatic [19–21] modes of jet disintegration [22, 23]. Note that the jet was directed not to the side of the maximum non-uniformity of the field, i.e., towards the rod, but instead, to the opposite side, to the plane counter-electrode.

In experiments [6], the charge of the drop was not controlled, however, in accordance with the experimental conditions the charge could reach the rate of the order of hundredths of the critical value by Rayleigh [2]. The non-uniform electrostatic field was created by a $d \approx 0.57$ mm pin and with the length of the order of 10 cm. The distance between the pin's end and the falling drop with diameter of $d \approx 2$ mm was 10 mm. The potential fed to the pin varied from 33 to 41 kV. In the dimensional variables used in this work, the values of the experimental parameters are as follows: $\varphi_0 = 7.25$–90.1, $a = 50$, $b = 0.29$. The equilibrium shape of the uncharged drop in such a field, according to the model under study appears to be an egg-shaped distorted sphere (Fig. 8).

This result qualitatively agrees with the photos presented in [6]. However, those photos also show that the drop has its own non-compensated charge of the same polarity with the pin electrode. Experiments [6] illustrate that the jets emitted from the drop at the electrostatic instability are directed not towards the most non-uniform electrostatic field (to the rod), as it

Table 3. Amplitude of distortion modes of equilibrium drop shape in the field of rods with different lengths calculated at $L = 2$, $b = 1$, $Q = Q_{cr}/2$, $\varphi_0 = 9$ (in dimensional units $\varphi_0 = 4.1$ kV for $R = 1$ mm and $\sigma = 23$ dyn/cm)

a_k	$a = 10$	$a = 30$	$a = 100$
a_2	0.821×10^{-1}	0.512×10^{-1}	0.323×10^{-1}
a_3	0.393×10^{-1}	0.244×10^{-1}	0.154×10^{-1}
a_4	0.249×10^{-1}	0.154×10^{-1}	0.976×10^{-2}
a_5	0.180×10^{-1}	0.112×10^{-1}	0.706×10^{-2}
a_6	0.139×10^{-1}	0.869×10^{-2}	0.547×10^{-2}
a_7	0.109×10^{-1}	0.680×10^{-2}	0.429×10^{-2}
a_8	0.479×10^{-2}	0.298×10^{-2}	0.187×10^{-2}
a_9	0.391×10^{-2}	0.244×10^{-2}	0.154×10^{-2}
a_{10}	0.341×10^{-3}	0.199×10^{-3}	0.122×10^{-3}
a_{11}	0.418×10^{-4}	0.249×10^{-4}	0.154×10^{-4}
a_{12}	0.611×10^{-5}	0.376×10^{-5}	0.236×10^{-5}
a_{13}	0.990×10^{-6}	0.630×10^{-6}	0.400×10^{-6}
a_{14}	0.170×10^{-6}	0.112×10^{-6}	0.716×10^{-7}
a_{15}	0.292×10^{-7}	0.197×10^{-7}	0.128×10^{-7}
a_{16}	0.450×10^{-8}	0.313×10^{-8}	0.205×10^{-8}

Table 4. Amplitudes of distortion modes of equilibrium shape of drop in the field of rods with different lengths calculated at $L = 5$, $b = 1$, $Q = Q_{cr}/2$, $\varphi_0 = 9$ (in dimensional units $\varphi_0 = 4.1$ kV for $R = 1$ mm and $\sigma = 23$ dyn/cm)

a_k	$a = 10$	$a = 30$	$a = 100$
a_2	0.948×10^{-2}	0.685×10^{-2}	0.459×10^{-2}
a_3	0.440×10^{-2}	0.317×10^{-2}	0.211×10^{-2}
a_4	0.279×10^{-2}	0.201×10^{-2}	0.135×10^{-2}
a_5	0.203×10^{-2}	0.146×10^{-2}	0.978×10^{-3}
a_6	0.158×10^{-2}	0.114×10^{-2}	0.765×10^{-3}
a_7	0.128×10^{-2}	0.929×10^{-3}	0.622×10^{-3}
a_8	0.530×10^{-3}	0.382×10^{-3}	0.255×10^{-3}
a_9	0.454×10^{-3}	0.328×10^{-3}	0.220×10^{-3}
a_{10}	0.756×10^{-5}	0.445×10^{-5}	0.271×10^{-5}
a_{11}	0.156×10^{-6}	0.879×10^{-7}	0.533×10^{-7}
a_{12}	0.372×10^{-8}	0.211×10^{-8}	0.128×10^{-8}
a_{13}	0.984×10^{-10}	0.563×10^{-10}	0.344×10^{-10}
a_{14}	0.278×10^{-11}	0.161×10^{-11}	0.990×10^{-12}
a_{15}	0.825×10^{-13}	0.484×10^{-13}	0.299×10^{-13}
a_{16}	0.245×10^{-14}	0.146×10^{-14}	0.903×10^{-15}

should be in terms of general physical concepts, but, instead, to the opposite side, to the region of the weak field. This means that on the half of the drop that is turned to the rod, the sign of the charge is opposite to the self-charge of the drop, whereas on the other half of the drop, the charge has a similar sign. Instability is realized in the area where the surface density of the net charge is maximum. As a result, it is where the emission protrusion and the jet form.

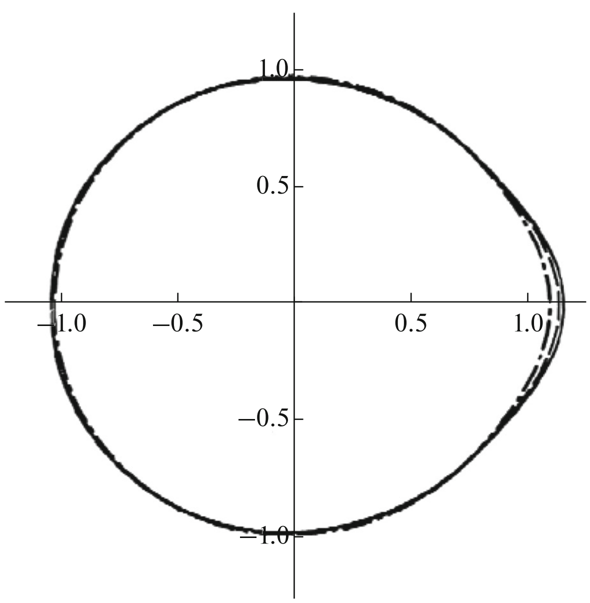

Fig. 8. Equilibrium shapes of the drop surface calculated at $L = 10$, $b = 0.29$, $a = 50$, $Q = 0$ (in dimensional units $\varphi_0 = 33$, 39, and 41 kV, correspondingly, for $R = 1$ mm, $\sigma = 23$ dyn/cm). Solid line represents $\varphi_0 = 72.5$, dashed line represents $\varphi_0 = 85.7$, and dash-and-dot line represents $\varphi_0 = 90.1$.

An important distinction ought to be noted. The present calculation deals with the equilibrium drops, and in [6] the drops are certainly unstable: the electrostatic field intensity in [6] is beyond the critical point for realization of electrostatic instability of both a drop and a jet emitted by the latter [14, 19, 20, 22–23].

CONCLUSIONS

It was found that with the thickness of the rod (which serves as an electrode and creates a potential near the charged drop) taken into account, a marked distortion of the equilibrium shape of the charged drop results. The distortion of the equilibrium shape of the surface of the conducting charged drop of ideal liquid in a non-uniform electrostatic field of the rod is described by a superposition of several first polynomials of Legendre with amplitudes, which are described by the expressions derived in this study.

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