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On the Equilibrium Shape of a Charged Drop in an Electrostatic Field

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Abstract—It is found analytically that the equilibrium shape of a charged drop in a uniform electrostatic field in calculations of the first order of smallness for the size of stationary deformation can be considered to be spheroidal, and its eccentricity depends on the charge size and on the intensity of an external electrostatic field. On the threshold of losing stability in relation to the superposition of its own and polarizing charges, the square of the equilibrium eccentricity of a strongly charged drop in a weak field is up to 0.34, and the square equilibrium eccentricity of a weakly charged drop in a strong field is up to 0.78 (in terms of opportunity to realize instability).

Keywords: spheroidal drop, electrostatic field, a charge, equilibrium drop shape

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INTRODUCTION

The problem of an equilibrium shape of a charged drop in an external uniform electrostatic field in the process of its electrostatic disintegration into subsidiary drops is of interest both scientifically and practically, due to the multiple applications of this process, e.g., in atmosphere physics in thunderstorm clouds [1], in engineering and various technologies [2], in instrumentation technology, e.g., in liquid mass-spectrometry as a major object [3], and liquid–metal ion source as a parasite phenomenon [4].

It is known [5–7] that an immobile uncharged drop in an external uniform electrostatic field with intensity E_0 acquires an equilibrium shape of a spheroid elongated along a field with eccentricity $e_0 \equiv \sqrt{9E_0^2R/16\pi\sigma}$, where R is the initial radius of a spherical drop and σ is the coefficient of the surface tension of the liquid. The question is what happens to the form of the drop if the latter has also charge Q? In this case, both free and polarized charges will be on the drop, and the latter changes its sign in the area of the equator of the drop. From common physical concepts it is clear that the drop will become pear-shaped, bulging on the end of the drop where the sign of the polarization charge coincides with that of the free charge and thinning on the opposite end of the drop where the signs of the polarization and intrinsic charges are opposite. The degree of the pear shape will depend on the ratio between the values of the polarization and intrinsic charges; i.e., it will be small at a high value of the intrinsic charge and at a low polarization charge or, vice versa, at a high polarization charge and low intrinsic one. The degree of the pear shape will be great at comparable values of the intrinsic and polarization charges. Note that the free charge on the drop in the presence of an arbitrary small external electrostatic field that determines one direction ($\vec{E}_0$) as preferential will make the two halves of the drop repulse from each other. In the absence of a free charge, only the polarization charge remains on the drop with different signs on different halves, and the field makes the drop elongate. Thus, the physical mechanisms that affect the elongation of a spherical drop into a spheroid are different from the polarized and intrinsic charges

EQUILIBRIAL SHAPE OF THE CHARGED DROP IN THE ELECTROSTATIC FIELD

Let us have a charged drop in a uniform electrostatic field. Assuming that the shape of the drop is close to an elongated spheroid, let us calculate its eccentricity on the basis of a minimality of potential energy as in [5] for an uncharged drop.

First, let us accept that we are given an uncharged drop in an external uniform electrostatic field. The temperatures of the drop and the environment are constant, and the bulk of the liquid phase remains unchanged. According to the aforementioned, the drop takes an equilibrium spheroidal form with e_0

eccentricity. The energy of the forces of surface tension of the drop will be as follows [5]:

$$U_\sigma = 2\pi R^2\sigma(1-e_0^2)^{2/3}\left(1+\frac{\arcsin e_0}{e_0(1-e_0^2)^{1/3}}\right). \qquad (1)$$

The energy of the uncharged immobile drop in E_0 field equals [5]:

$$U_E = -\frac{1}{3}E_0^2R^3\frac{e_0^3}{(1-e_0^2)\left[\ln\frac{1+e_0}{1-e_0}-2e_0\right]}, \qquad (2)$$

where, according to the aforementioned, $e_0 \equiv \sqrt{9E_0^2R/16\pi\sigma}$. The sum of U_σ and U_E gives the total free energy of the system. Taking a derivative on e_0 from this sum and setting it to zero, on the basis of the principle of minimum of free energy, we shall find the above expression of e_0, using E_0, R and σ. This was done in [5] for the first time.

Now let us locate an electrical charge Q on the drop. As was mentioned above, the elongation of the drop will increase. Assuming the pear-shaped deformation to be small, much smaller than the initial spheroidal shape, let us accept that the drop during the calculations within an accuracy of the order of eccentricity squared will have a spheroidal shape with eccentricity e. Free energy then will increase by [7]:

$$U_Q = -\frac{Q^2}{4R}\frac{(1-e^2)^{1/3}}{e}\ln\frac{1+e}{1-e}, \qquad (3)$$

where $e > e_0$. Further on, in (1) and (2), we approximately assume that $e_0 \approx e$, and add to (1) and (2) the energy of charge Q (3). Then, based on the principle of the minimality of free energy of the new system we shall determine, as was done above, e as follows:

$$e^2 = \frac{9E_0^2R^4}{(16\pi R^3\sigma - Q^2)}. \qquad (4)$$

Introducing the dimensionless parameters of the field Λ^2 and the charge W^2,

$$\Lambda^2 \equiv \frac{E_0^2R}{\sigma}, \quad W^2 \equiv \frac{Q^2}{16\pi\sigma R^3},$$

we shall rewrite (4) as follows:

$$e^2 = \frac{9\Lambda^2}{16\pi(1-W^2)}. \qquad (5)$$

During the derivation of (5) we have made only one assumption, namely, that of the smallness of the pear-shaped deformation. According to what was mentioned above, the pear-shaped deformation is small at low charges and strong fields and, vice versa, large at high charges and weak fields. This occurs in the electrostatic suspension, which is a necessary element of the holding arrangements during the verification of the accuracy of the Rayleigh criterion [8, 9]. The pear-shaped deformation that is proportional to the third degree of eccentricity of the spheroidal component of the drop shape [10] is also small in its absolute value, when the eccentricity itself is small, i.e., at low charges and weak fields.

THE RESULTS AND DISCUSSION

Figures 1a–1c shows dependence (5). The field and charge parameters are assumed to change independently. Figure 1a shows that at high values of the charge and field parameters, the eccentricity rapidly reaches peak values. However, at such values of the parameters the pear-shaped deformation of the drop is considerable and expression (5) is inapplicable. Dependence (5) at small values of the field parameter and during the change of the charge parameter in the formally admissible full range is illustrated in Fig. 1b. The situation with small charges and fields when the pear-shaped deformation is small and expression (5) is valid is shown in Fig. 1c.

However, the charge and field parameters for the limit of stability with respect to the superposition of their own and induced charges are connected between each other by the following relation [11]:

$$W^2 + \frac{\Lambda W}{2\sqrt{\pi}}\left[\frac{e^3}{(1-e^2)^{2/3}(\operatorname{arctanh}e - e)} - 3(1-e^2)^{1/3}\right] + \frac{11\Lambda^2}{21\times 16\pi}\left(\frac{e^6}{(1-e^2)^{4/3}(\operatorname{arctanh}e - e)^2}\right) \geq 1; \qquad (6)$$

where the analytical expression for the eccentricity is determined according to (5). The relation (6) is precisely fulfilled for two terminal points $W^2 = 1$; $\Lambda^2 = 0$ and $W^2 = 0$; $\Lambda^2 = 2.6$, while in the intermediate points it is fulfilled only approximately, for an accuracy of the constant multiplier of ≈1. The graph of this dependence is shown in Fig. 2.

During the derivation of expression (6), the drop was assumed to be immobile in the electrostatic and gravitational fields. The acceleration of the gravitational field $\vec{g}$ was considered to be connected with the charge of the drop and the value of the field:

$$\vec{g} = -\frac{3Q\vec{E}_0}{4\pi R^3\rho},$$

where ρ is the mass density of the liquid.

On fulfilling the conditions $W^2 \geq 1$ and $\Lambda^2 = 0$, a strongly charged spherical drop undergoes electrostatic instability. It is the major mode of its oscillations that is exposed to instability, its amplitude starts growing uncontrollably, and the drop elongates into a

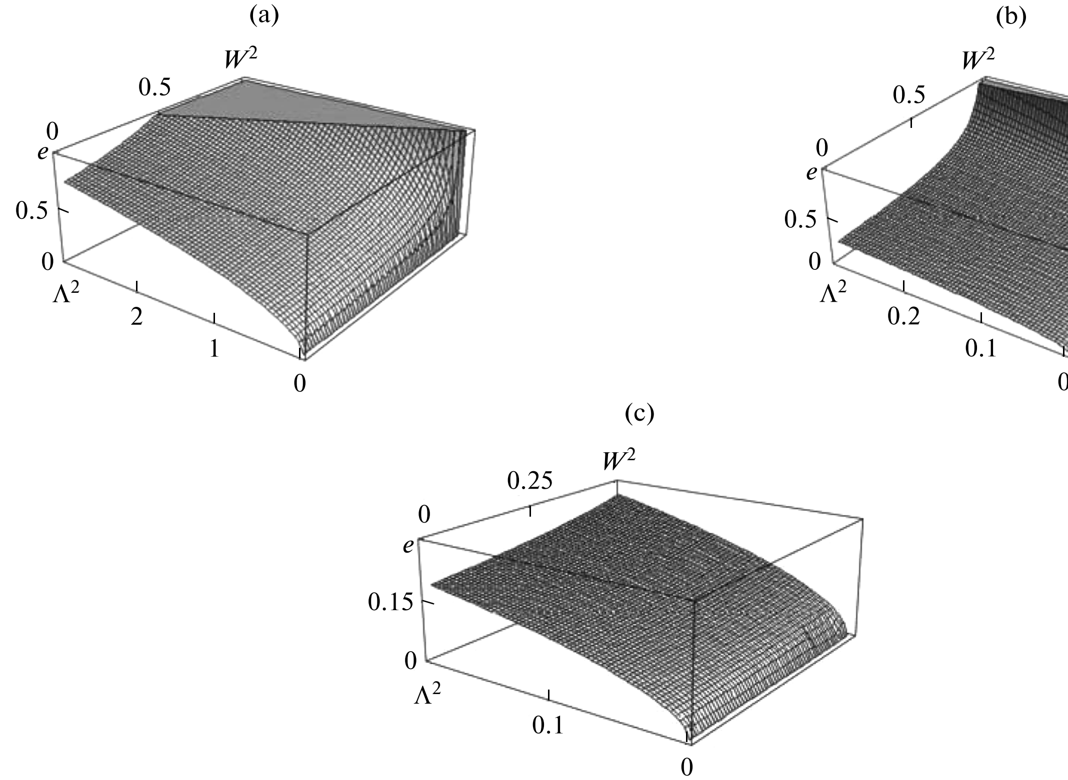

Fig. 1. Dependence of the eccentricity square of a charged drop in the electric field on the value of the dimensionless charge and field parameters built on assumption that they change independently: (a) is the general view; (b) is a strongly charged drop in a weak field; (c) low charges and weak fields when pear-shaped deformation is small.

spheroid, throwing off extra charge from its two ends by way of emission about two hundred (smaller by two orders of magnitude than the initial one) strongly charged drops [12–14].

On the fulfillment of conditions $W^2 = 0$ and $\Lambda^2 \geq 2.6$, the uncharged drop in the uniform electrostatic field undergoes electrostatic instability [15–17]. The patterns of its realization are about the same as of the

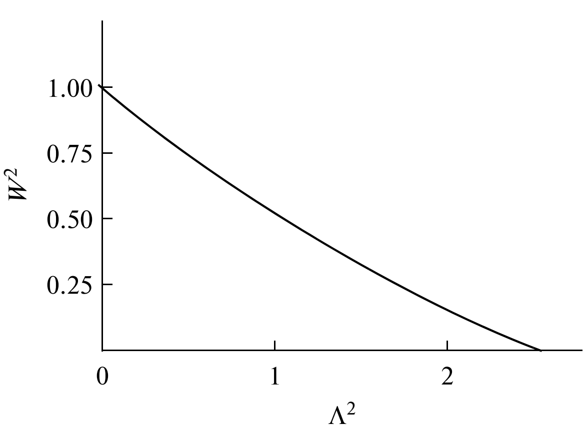

Fig. 2. Connection between charge and field parameters at stability limit.

charged drop [16, 17]; however, the initial equilibrium form is not spherical, but rather spheroidal with eccentricity $e_0 = \dfrac{9\Lambda^2}{16\pi}$ [5, 15–17].

Figure 2 shows that dependence (6) is almost linear, but the angles of inclination of the tangent line to the curve (6) at $W^2 \sim 1$ and $\Lambda^2 \sim 2.6$ are different.

To estimate on the basis of the order of value, let us substitute dependence (6) for a straight-line one, which passes via two end points:

$$W^2 = 1 - \frac{1}{2.6}\Lambda^2 \equiv 1 - k\Lambda^2, \qquad (7)$$

where k is the tangent of the angle of inclination of the line (7). Let us express Λ^2 from (7) and insert into (5). We shall obtain an expression for the square of eccentricity at the threshold of stability:

$$e^2 = \frac{9\Lambda^2}{16\pi(1-W^2)} \equiv \frac{9(1-W^2)}{16\pi k(1-W^2)} \equiv \frac{9}{16\pi k}.$$

Let us insert into this expression the value of $1/k$ and obtain $e^2 \approx 0.466$. In fact, dependence (6) differs from the line, and the absolute value of the angle tangent of the inclination of the tangent to curve (6) at $W^2 \sim 1$ will exceed the absolute value by more than 1/2.6; at

$\Lambda^2 \sim 2.6$ it will be less. To find the analytical dependence between the absolute value of the tangent angle of inclination of tangent and parameters of W^2 and Λ^2 according to (6) considering (5) is fairly problematic because of the awkwardness of the required mathematical operations. Therefore, let us find the absolute values of the angular tangents of inclination of tangent in two terminal points graphically, using Fig. 2. All other values will be in between.

At $W^2 \approx 1$, the absolute value of the angle tangent of inclination of the tangent to curve (6) is about $(1/1.9) \approx 0.53$ and of the eccentricity square is $e^2 \approx 0.34$. At $\Lambda^2 \sim 2.6$, the absolute value of the angle tangent of inclination of the tangent to curve (6) is about $(0.59/2.6) \approx 0.23$ and the eccentricity square is $e^2 \approx 0.78$. The above speculations and estimations are certainly approximate, but in the absence of other estimations they can be used as the starting points of future investigations.

THE VERIFICATION OF THE RAYLIEGH CRITERION OF THE ELECTROSTATIC INSTABILITY OF A STRONGLY CHARGED SPHERICAL DROP OF AN IDEAL CONDUCTING LIQUID

Critical conditions of the realization of the instability of an isolated drop of an ideal electroconducting incompressible liquid in regard to its own charge was theoretically precisely deduced by Rayleigh in the form of the relation of $W \equiv (Q^2/16\pi\sigma R^3) \geq 1$ at the end of the 19th century. In the 20th century and at the beginning of the 21st century, this criterion was repeatedly experimentally verified using various laboratory devices, namely, a vertical electrostatic field between flat plates (i.e., using an electrostatic suspension of the type that Millikan used in the experiments on determining the charge on the electron) [18]; in a nonuniform electric field periodically changing in time between the electrodes of a complex geometry (the combination of rings, cylindrical and spherical surfaces) [19]; in a combined electric suspension with electrostatic and periodically changing electric fields between three plate electrodes [20]; in an air stream [21]; and in an electrodynamic suspension based on two-ring electrodes [9, 22]. Experiments were carried out on drops in a wide range of sizes: hundreds of micrometers [18, 21, 23], tens of micrometers [19, 20], and units of micrometers [9, 22]. In all cases, the validity of the Rayleigh criterion was supported. It is noteworthy that the highest accuracy of the experiments was reached in [20], where the Rayleigh criterion was confirmed to an accuracy of 4%, and in [9, 22], where the accuracy was about 5%.

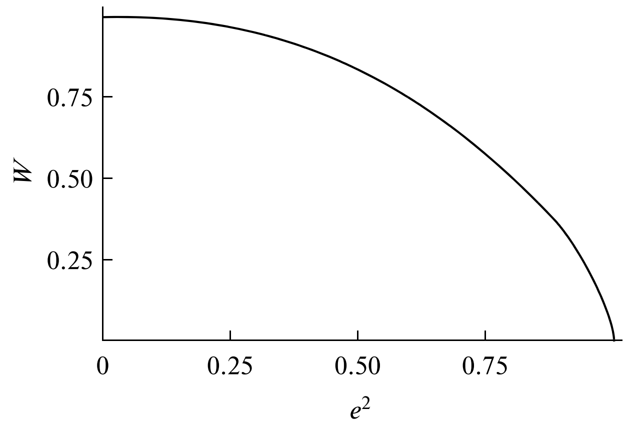

Fig. 3. Dependence of critical value of charge parameter of spheroidal drop on square of its eccentricity.

ELECTROSTATIC SUSPENSION

Let us assume that $W^2 \sim 1$ is in the electrostatic field of a vanishing low intensity, the action of whose force on the drop can be neglected, and it can be considered that the field simply gives direction. The situation is typical for electrostatic suspensions that are used to verify the validity of the Rayleigh criterion, and according to the obtained results it is possible to estimate admissible error during their use.

If at the initial moment of time, the charged drop is virtually given a spheroidal form, then the value of the critical charge for realization of the electrostatic instability will depend on the eccentricity square of the drop [13]. The relevant dependence of the charge parameter on the eccentricity square is shown in Fig. 3 [13].

For the ease of further calculations, so as not to depend on the drop sizes and the value of the surface tension, let us proceed to the dimensionless variables in which $R = \sigma = \rho = 1$. In this case, the criterion of the electrostatic instability with respect to its own charge is as follows:

$$W^2 \equiv \frac{Q^2}{16\pi} \geq 1.$$

Let us now consider the situation of $W^2 \sim 1$ and $\Lambda^2 \sim 1 - W^2$. According to the aforementioned, the eccentricity square of a strongly charged drop in a weak electrostatic field is $e^2 \approx 0.34$. Using Fig. 3, let us find the critical value of the charge parameter at such a value of the eccentricity square $W^2 = 0.92$. Hence, it is easy to find the value of the critical charge of the drop, which can be measured in the electrostatic suspension $Q \approx 0.959$. That is, the error with respect to the true critical charge will be ~4%. It is this very error that is observed during accurate measurements of [9, 18, 22].

It is noteworthy that the offered model of disintegration of a strongly charged drop explains the fact of decrease in the critical charge of the drop during several successive disintegrations [9]. Indeed, during

Rayleigh disintegration the drop loses ≈23% of its charge and ≈5% of its mass [14, 20]. With the remaining charge, the drop returns to a stable spheroidal shape, since it is in the electrostatic field and has an electric charge; however, its eccentricity will be smaller than that of the initial drop due to the lower charge. In order that the drop comes in view of a microscope (of an experimentalist) one must increase the intensity of the electrostatic field. In accordance with the abovementioned (according to Fig. 2), it will shift down the curve in Fig. 2. Its eccentricity will then increase, and a smaller charge will be required for the drop disintegration, according to Fig. 3. This can be repeated several times. In [9], this experimental fact was explained by a decrease in the value of the surface tension of the liquid. This explanation should be recognized to be fairly feeble.

It is noteworthy that after several successive disintegrations the charge of the drop will decrease considerably, and the external electric field that is necessary to keep the drop immobile together with the induced charge will increase. Hence, pear-shaped deformation will become substantial, and the above speculations will turn out to be invalid.

CONCLUSIONS

The present calculations show that the equilibrium shape of a charged drop in a uniform electrostatic field in calculations that are linear with respect to the stationary deviation from a sphere can be considered as an elongated spheroid in two extreme cases, i.e., when a drop with a strong charge is in a weak field and a drop with a weak charge is in a strong field. The eccentricity in these two cases differs greatly. On the threshold of a stability loss of a strongly charged drop in a uniform electrostatical field, the eccentricity of the drop remains terminal.

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