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REAL AND COMPLEX ANALYSIS

Meromorphic Functions Sharing Three Values with Their Derivatives in an Angular Domain

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Abstract—In this paper, we investigate the uniqueness of transcendental meromorphic functions sharing three values with their derivatives in an arbitrary small angular domain including a Borel direction. The results extend the corresponding results from Gundersen, Mues and Steinmetz, Zheng, Li et al., and Chen.

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1. INTRODUCTION AND MAIN RESULT

Let $f : C \to \hat{C} = C \bigcup \{\infty\}$ be a meromorphic function, where C is the complex plane. It is assumed that the reader is familiar with the basic results and notations of the Nevanlinna's value distribution theory (see [6, 14, 15]), such as $T(r; f), N(r, f)$ and $m(r, f)$. Meanwhile, the lower order μ and the order λ of a meromorphic function f are in turn defined as follows

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Let f and g be nonconstant meromorphic functions in the domain $D \subseteq C$. If $f - c$ and $g - c$ have the same zeros with the same multiplicities in D, then $c \in C \bigcup \{\infty\}$ is called an CM shared value in a domain $D \subseteq C$ of two meromorphic functions f and g. If $f - c$ and $g - c$ only have the same zeros in D, then $c \in C \bigcup \{\infty\}$ is called an IM shared value in a domain $D \subseteq C$ of two meromorphic functions f and g. The zeros of $f - c$ imply the poles of f when $c = +\infty$.

In 1979, Gundersen [5] and Mues and Steinmetz [10] have considered the uniqueness of a meromorphic function f and its derivative f' and obtained the following result.

Theorem A: *Let f be a nonconstant meromorphic function in C, and let $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers. If f and f' share $a_j (j = 1, 2, 3)$ IM, then $f \equiv f'$.*

Later on, Frank and Schwick [3] generalized the above results and proved the following result.

Theorem B: *Let f be a nonconstant meromorphic function, and let k be a positive integer. If there exist three distinct finite complex numbers a, b, and c such that f and $f^{(k)}$ share a, b, c IM, then $f \equiv f^{(k)}$.*

In 2004, Zheng [16] first considered the uniqueness question of meromorphic functions with shared values in an angular domain and proved the following result (see [16, Theorem 3]):

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Theorem C: *Let f be a transcendental meromorphic function of finite lower order and such that $\delta = \delta(a, f^{(p)}) > 0$ for some $a \in C \bigcup \{\infty\}$ and an integer $p \geq 0$. Let the pairs of real numbers $\{\alpha_j, \beta_j\}(j = 1, ..., q)$ be such that*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq ... \leq \alpha_q < \beta_q \leq \pi,$$

with $\omega = \max\left\{\frac{\pi}{\beta_j - \alpha_j} : 1 \leq j \leq q\right\}$, *and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\delta}\arcsin\sqrt{\delta(a, f^{(p)})/2},$$

where $\delta = \max\{\omega, \mu\}$. For a positive integer k, assume that f and $f^{(k)}$ share three distinct finite complex numbers $a_j(j = 1, 2, 3)$ IM in $X = \bigcup_{l=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f \equiv f^{(k)}$.

In 2015, Li et al. [9] observed that Theorem C is invalid for $q \geq 2$ and proved the following more general result, which extends Theorem C (see [9, p. 443]).

Theorem D: (see [9]). *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in C and such that $\delta(a, f) > 0$ for some $a \in C$. Assume that $q \geq 2$ pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfy the conditions*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq ... \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\frac{\pi}{(\beta_j - \alpha_j)} : 1 \leq j \leq q\}$, *and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\delta}\arcsin\sqrt{\delta(a, f)/2},$$

where $\delta = \max\{\omega, \mu\}$. For a k-th-order linear differential polynomial $L[f]$ in f with constant coefficients given by

$$L[f] = b_k f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_1 f', \tag{1.1}$$

where k is a positive integer, b_k, b_{k-1}, $\cdots$, b_1 are constants and $b_k \neq 0$, assume that f and $L[f]$ share $a_j(j = 1, 2, 3)$ IM in

$$X = \bigcup_{l=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}.$$

where $a_j(j = 1, 2, 3)$ are three distinct finite complex numbers such that $a \neq a_j$ $(j = 1, 2, 3)$. If $\lambda(f) \neq \omega$, then $f = L[f]$.

In 2019, Chen [2] proved the following result.

Theorem E: *Let f be a nonconstant meromorphic function of lower order $\mu(f) > 1/2$ in C, $a_j(j = 1, 2, 3)$ be three distinct finite complex numbers, and let $L[f]$ be given by Theorem D. Then there exists an angular domain $D = \{z : \alpha \leq \arg z \leq \beta\}$, where $0 \leq \beta - \alpha \leq 2\pi$, such that, if f and $L[f]$ share $a_j(j = 1, 2, 3)$ CM in D, then $f = L[f]$.*

In theory of meromorphic functions, a function is uniquely determined by its value on a set with an accumulation point. It is natural to ask if we can prove similar results with the conditions

$$\bar{E}_D(f, a_j) = \bar{E}_D(f', a_j), \quad j = 1, 2, 3$$

for some typical set in C in steads of general angular domain in C, where $\bar{E}_D(a, f) = \{z : z \in D, f(z) = a\}$ (as a set in C). In general, the answer of this question is negative. For $f(z) = e^{2z}$, it is clear that $f(z) \neq f'(z)$, but $|f(z)|$ is bounded by 1 on D being the left half plane. Thus

$$\bar{E}_D(f, n) = \bar{E}_D(f', n) = \emptyset \quad \text{for any} \quad n > 1.$$

This example shows us that, if such angular domain D exists, it must be a region whose image under f should be dense in C.

Based on the theory on singular direction for a meromorphic function (see [14]) and the research results of shared values of a meromorphic function (see [8, 12]), combining with the result of Theorems D and E, we may conjecture that angular domain of the singular direction may be the right. The main result of this paper shows that it is true when D is an angular domain with the Borel direction as the center line for f with order $\lambda > 0$, which extends Theorems D and E.

In order to prove our main results, we introduce some notations about Ahlfors—Shimizu character of meromorphic function in C.

$$T_0(r,f)=\int_0^r \frac{A(t)}{t}dt,\quad A(t)=\frac{1}{\pi}\int_0^{2\pi}\int_0^t\left(\frac{|f'(\rho e^{i\theta})|}{1+|f(\rho e^{i\theta})|^2}\right)^2 d\rho d\theta. \tag{1.2}$$

We recall the Nevanlinna theory on an angular domain.

Let f be a meromorphic function in $D=\{z:\alpha\le\arg z\le\beta\}$, where $0\le\beta-\alpha\le 2\pi$. Nevanlinna [11] defined the following symbols (also see [4]).

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^\omega}-\frac{|b_m|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_m-\alpha),$$

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f)$$

where $\omega=\dfrac{\pi}{(\beta-\alpha)}$, and $b_m=|b_m|e^{i\theta_m}$ are the poles of f in D counting multiplicities.

Throughout the paper, we denote by $R(r,*)$ a quantity satisfying

$$R(r,*)=O\{\log(rT(r,*))\}, r\in E,$$

where E denotes a set of positive real numbers with finite linear measure, which will not necessarily be the same in each occurrence. To state our result, we need the following theorem F and definitions .

Theorem F (see [7]). *Let f be a meromorphic function of infinite order in C. Then there exists a function $\rho(r)$ such that*

(i) *$\rho(r)$ is continuous and nondecreasing for $r\ge r_0$, and $\rho(r)\to+\infty$ as $r\to+\infty$.*

(ii) *$U(r)=r^{\rho(r)}(r\ge r_0)$ satisfies the condition*

$$\lim_{r\to+\infty}\frac{\log U(R)}{\log U(r)}=1,\quad R=r+\frac{r}{\log U(r)}.$$

(iii) $\limsup_{r\to\infty}\dfrac{\log T(r,f)}{\rho(r)\log r}=1.$

The function $\rho(r)$ is also called the precise order of f.

Definition 1.1 (see [13]). Let f be a meromorphic function of finite order $\lambda(f)>0$ in C. A direction $\arg z=\theta_0$ $(0\le\theta_0<2\pi)$ is called a Borel direction of $f(z)$ of order $\lambda(f)$ if for arbitrary small positive ε the following relation holds:

$$\lim_{r\to\infty}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\log r}=\lambda(f)$$

for all $a \in \hat{C} = C \bigcup +\infty$ except at most two exceptional values, where $n(r, \theta_0, \varepsilon, f = a)$ denotes the number of the zeros of $f - a$ counting multiplicities in the sector $|\arg z - \theta_0| < \varepsilon$, $|z| \leq r$.

Definition 1.2 (see [7]). Let f be a meromorphic function of infinite order in C and let $\rho(r)$ be the precise order of f. A direction $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ is called a Borel direction of $f(z)$ with precise order $\rho(r)$ if for arbitrary small positive ε the following relation holds:

$$\lim_{r\to\infty} \frac{\log n(r, \theta_0, \varepsilon, f = a)}{\rho(r) \log r} = 1$$

for all $a \in \hat{C}$ except at most two exceptional values, where $n(r, \theta_0, \varepsilon, f = a)$ is as in Definition 1.1.

In this paper we will prove the following theorem.

Theorem 1.1. *Let f be a meromorphic function of finite order $\lambda(f) > 0$ in C and ε be an arbitrary small positive number, and a direction $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ be a Borel direction of $f(z)$. Assume that f and f' share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ IM in $A(\theta_0, \varepsilon)$, where $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$. Then $f \equiv f'$.*

Theorem 1.2. *Let f be a meromorphic function of infinite order in C and a direction $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ be a Borel direction of $f(z)$ with precise order $\rho(r)$. Then for arbitrary positive number ε, f and f' share two finite values IM at most in the angular region $\{z : |\arg z - \theta_0| < \varepsilon\}$.*

Theorem 1.3. *Let f be a meromorphic function of infinite order in C and $L[f]$ defined by (1.1), and $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ be a Borel direction of $f(z)$ with precise order $\rho(r)$. Then for arbitrary positive ε, f and $L[f]$ share two finite values CM at most in the angular region $\{z : |\arg z - \theta_0| < \varepsilon\}$.*

2. PRELIMINARY

In this section, we will introduce and prove some lemmas that will be used in the proof of the main result.

Lemma 2.1 ([1, 12]). *Let $\mathcal{F}$ be a family of meromorphic functions such that for every function $f \in \mathcal{F}$ its zeros of multiplicity are at least k. If $\mathcal{F}$ is not a normal family at the origin 0, then for $0 \leq \alpha \leq k$ there exist*

(a) *a real number r $(0 < r < 1)$;*

(b) *a sequence of complex numbers $z_n \to 0, |z_n| < r$;*

(c) *a sequence of functions $f_n \in \mathcal{F}$;*

(d) *a sequence of positive numbers $\rho_n \to 0$;*

such that

$$g_n(z) = {\rho_n}^{-\alpha} f_n(z_n + \rho_n z)$$

*converges locally uniformly with respect to spherical metric to a nonconstant meromorphic function $g(z)$ on **C** and, moreover, g is of order at most two.*

For convenience, we will use the following notation

$$LD(r, f : c_1, c_2) = c_1 \left[m\left(r, \frac{f'}{f}\right) + \sum_{i=1}^{3} m\left(r, \frac{f'}{f - a_i}\right) \right] + c_2 \left[m\left(r, \frac{f''}{f'}\right) + \sum_{i=1}^{3} m\left(r, \frac{f''}{f' - ta_i}\right) \right].$$

Lemma 2.2 ([12]). *Let f be a meromorphic function in a domain $D = \{z : |z| < R\}$ and $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers, and let t be a positive real number and $a \in C$. If*

$$\bar{E}_D(a_j, f) = \bar{E}_D(ta_j, f') \quad for \quad j = 1, 2, 3;$$

and $a \neq a_j$ *and* $f(0) \neq a_j, \infty (j = 1, 2, 3)$, $f'(0) \neq 0$, $f''(0) \neq 0$, $f'(0) \neq tf(0)$, *then for* $0 < r < R$ *we have*

$$T(r, f) \leq LD(r, f : 2, 3) + \log \frac{\prod_{i=1}^{3} |f(0) - a_i|^2 |f'(0) - ta_i|^3}{|tf(0) - f'(0)|^5 |f'(0)|^2}$$
$$+ 3 \log \frac{1}{|f''(0)|} + \left(\log^+ t + m\left(r, \frac{f''}{f' - ta}\right) + 1\right) O(1).$$

where $\bar{E}_D(a, f) = \{z : z \in D, f(z) = a\}$ *(as a set in* $\boldsymbol{C}$*) and* $O(1)$ *is a complex number depending only on* a *and* $a_i (i = 1, 2, 3)$.

Lemma 2.3 ([14]). *Let* $f(z)$ *be a meromorphic function with finite order* $\lambda > 0$ *and* $\arg z = \theta_0$ *is a Borel direction of* f. *Then there exist a series of circles*

$$\Gamma_j = \{z : |z - z_j| < \epsilon_j |z_j|\},$$

where $z_j = |z_j| e^{i\theta_0}$, $\lim_{j\to\infty} |z_j| = +\infty$, $\lim_{j\to\infty} \epsilon_j = 0 (j = 1, 2, \cdots)$, *such that* f *takes any complex number at least* $|z_j|^{\lambda - \delta_j}$ *times in every circle* Γ_j *with at most some exceptional values contained in two circles with spherical radius* 2^{-j}, *where* $\lim_{j\to\infty} |\delta_j| = 0$.

Lemma 2.4 ([14]). *Let* $\mathcal{F}$ *be a family of meromorphic function on domain* D, *then* $\mathcal{F}$ *is normal on* D *if and only if, for every bounded closed domain* $K \subseteq D$, *there exists a positive number* M *such that every* $f \in \mathcal{F}$

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq M.$$

Lemma 2.5 ([6, 17]). *Let* m *be the normalized area measure on the Riemann sphere* S. *Then we have*

$$A(r, f) = \int_{\hat{C}} n(r, f = a) dm(a),$$

where $\hat{C} = C \bigcup \{\infty\}$.

Lemma 2.6 ([6, 17]). *Let* $f(z)$ *be a meromorphic function in a domain* $D = \{z : |z| < R\}$. *If* $f(0) \neq \infty$, *then for* $0 < r < R$ *we have*

$$|T(t, f) - T_0(t, f) - \log^+ |f(0)|| \leq \frac{1}{2} \log 2,$$

where $\log^+ |f(0)|$ *will be replace by* $\log |c(0)|$ *when* $f(0) = \infty$, *and* $c(0)$ *is the coefficient of the Laurent series of* $f(z)$ *at 0, and* $T_0(t, f)$ *is defined as (1.2).*

Lemma 2.7 ([8]). *Let* $f(z)$ *be a nonconstant meromorphic function in the complex plane, and* a_1, a_2, a_3 *are three distinct finite complex numbers. Assume that* f *and* f' *share the* $a_i (i = 1, 2, 3)$ *IM in* $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ *with* $0 \leq \alpha < \beta < 2\pi$. *Then one of the following two cases holds: (i)* $f \equiv f'$, *or (ii)* $S_{\alpha,\beta}(r, f) = Q(r, f)$, *where* $Q(r, f)$ *is such a quantity that if* $f(z)$ *is of finite order, then* $Q(r, f) = O(1)$ *as* $r \to \infty$, *and if* $f(z)$ *is of infinite order, then* $Q(r, f) = O(\log(rT(r, f))$ *for* $r \notin E$ *and* $r \to \infty$ *and* E *denotes a set of positive real numbers with finite linear measure.*

Lemma 2.8 ([4, 9]). *Let* f *be a meromorphic function on* $\overline{\Omega}(\alpha, \beta)$. *If* $S_{\alpha,\beta}(r, f) = O(1)$, *then*

$$\log |f(re^{i\phi})| = r^{\omega} c \sin(\omega(\phi - \alpha)) + o(r^{\omega})$$

uniformly for $\alpha \leq \phi \leq \beta$ *as* $r \notin F$ *and* $r \to \infty$, *where* c *is a positive constant,* $\omega = \dfrac{\pi}{\beta - \alpha}$, *and* F *is a set of finite logarithmic measure, and* $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$.

Lemma 2.9 ([13]). *Let f be a meromorphic function of infinite order in C, and let $\rho(r)$ be a precise order of f. Then a direction* $\arg z = \theta_0$ *is a Borel direction of precise order $\rho(r)$ of f if and only if for arbitrarily small $\varepsilon > 0$ we have*

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f)}{\rho(r)\log r} = 1.$$

Lemma 2.10 ([2]). *Let f be a meromorphic function of infinite order in C, $a_j (j = 1, 2, 3)$ be three distinct finite complex numbers and let $L[f]$ be given by (1.1). Suppose that f and $L[f]$ share $a_j (j = 1, 2, 3)$ CM in $D = \{z : \alpha \leq \arg z \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$. If $f \not\equiv L[f]$, then $S_{\alpha,\beta}(r, f) = R(r, f)$.*

Lemma 2.11 ([14]). *Let $f(z)$ be a meromorphic function in disk $D(0, R)$ centered at 0 with radius R. If $f(0) \neq 0, \infty$, then we have for $0 < r < \rho < R$*

$$m(r, \frac{f^{(k)}}{f}) < c_k \left\{1 + \log^+\log^+\left|\frac{1}{f(0)}\right| + \log^+\frac{1}{r} + \log^+\frac{1}{\rho - r} + \log^+\rho + \log^+ T(\rho, f)\right\},$$

where k is a positive integer and c_k is a constant depending only on k.

Lemma 2.12 ([14]). *Let $T(r)$ be a continuous, nondecreasing, nonnegative function and $a(r)$ be a nonincreasing, nonnegative function on $[r_0, R](0 < r_0 < R < \infty)$. If there exist constants b, c such that*

$$T(r) < a(r) + b\log^+\frac{1}{\rho - r} + c\log^+ T(\rho),$$

for $r_0 < r < \rho < R$, then

$$T(r) < 2a(r) + B\log^+\frac{2}{R - r} + C,$$

where B, C are two constants depending only on b, c.

Lemma 2.13. *Let $f(z)$ be a meromorphic function with finite order $\lambda > 0$ and $\arg z = \theta_0$ be a Borel direction of f, and $\Gamma_j = \{z : |z - z_j| < \epsilon_j|z_j|\}$ be a series of circles, where $z_j = |z_j|e^{i\theta_0}$, and $\lim_{j\to\infty} |z_j| = +\infty$, $\lim_{j\to\infty} \epsilon_j = 0 (j = 1, 2, \cdots)$. Suppose that f and f' share three distinct finite complex numbers $a_j (j = 1, 2, 3)$ IM in $A(\theta_0, \varepsilon)$, where $A(\theta_0, \varepsilon) = \{z : |\arg z - \theta_0| < \varepsilon\}$. If $f \not\equiv f'$, then for every sufficiently large $n(n \geq n_0)$,*

$$A(\varepsilon_n, z_n, f) \leq O(1)(1 + \log^+ |z_n|), \tag{2.1}$$

where $\varepsilon_n = |z_n|\epsilon_n$.

Proof. Set $f_n(z) = f(z_n + \varepsilon_n z)$. We consider two cases:

Case 1. Assume that $f_n(z)$ be normal at $|z| \leq 1$, by Lemma 2.4, implying that

$$\frac{|f_n'(z)|}{1 + |f_n(z)|^2} = \frac{\varepsilon_n|f'(z_n + \varepsilon_n z)|}{1 + |f(z_n + \varepsilon_n z)|^2} \leq M \quad (n = 1, 2, ...)$$

in $|z| \leq 1$, where M is a positive numbers. Then we have

$$A(\varepsilon_n, z_n, f) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon_n} \left(\frac{|f'(z_n + \rho e^{i\theta})|}{1 + |f(z_n + \rho e^{i\theta})|^2}\right)^2 \rho d\rho d\theta \leq 2M^2.$$

So, (2.1) holds.

Case 2. Assume that $f_n(z)$ is not normal at $|z| \leq 1$.

According to Lemma 2.1, there exist

(1) a sequence of point $\{z_n'\} \subset \{|z| < 1\}$;

(2) a subsequence of $\{f_n(z)\}_1^\infty$, without loss of generality, we still denote it by $\{f_n(z)\}$;

(3) positive numbers ρ_n with $\rho_n \to 0 (n \to \infty)$; such that

$$h_n(z) = f_n(z_n' + \rho_n z) \to g(z) \tag{2.2}$$

in spherical metric uniformly on a compact subset of **C** as $n \to \infty$, where $g(z)$ is a nonconstant meromorphic function. Thus, for any positive integer k, we have

$$h_n^{(k)}(\xi) = {\rho_n}^k f_n^{(k)}(z_n' + \rho_n \xi) \to g^{(k)}(\xi).$$

We claim $g''(\xi) \not\equiv 0$. Otherwise, $g(z) = cz + d$, $(c, d \in \mathbf{C}$ and $c \neq 0)$. We can choose ξ_0, with $g(\xi_0) = a_1$. By Hurwitz's theorem, there exists a sequence $\xi_n \to \xi_0$ such that

$$h_n(\xi_n) = f_n(z_n' + \rho_n \xi_n) = g(\xi_0) = a_1.$$

Notice that f and f' share a_1 IM in $\{z : |\arg z - \theta_0| < \varepsilon\}$, we have

$$c = g'(\xi_0) = \lim_{n\to\infty} h_n'(\xi_n) = \lim_{n\to\infty} \rho_n \varepsilon_n f'(z_n + \varepsilon_n(z_n' + \rho_n \xi_n))$$
$$= \lim_{n\to\infty} \rho_n \varepsilon_n f(z_n + \varepsilon_n(z_n' + \rho_n \xi_n)) = \lim_{n\to\infty} \rho_n \varepsilon_n a_1.$$

Thus, we have

$$\lim_{n\to\infty} \rho_n \varepsilon_n = \frac{c}{a_1}.$$

For finite complex number a_2, we can choose η_0 with $g(\eta_0) = a_2$. By Hurwitz's theorem, there exists a sequence $\eta_n \to \eta_0$ such that

$$h_n(\eta_n) = f_n(z_n' + \rho_n \eta_n) = g(\eta_0) = a_2.$$

Likewise, we get

$$\lim_{n\to\infty} \rho_n \varepsilon_n = \frac{c}{a_2},$$

this gives a contradiction.

For a sequence of positive numbers $\rho_n \varepsilon_n$, it is easy to know that there exists a subsequence, we still denoted by $\rho_n \varepsilon_n$, such that $\lim_{n\to\infty} \rho_n \varepsilon_n = a_0$, where $a_0 \in [0, +\infty) \bigcup \{+\infty\}$. Now, we consider two cases: $a_0 = 0$ or $+\infty$ and $0 < a_0 < +\infty$.

Case 2.1. Assume that $\lim_{n\to\infty} \rho_n \varepsilon_n = 0$ or ∞.

We choose $\xi_0 \in C$ such that

$$g(\xi_0) \neq 0, a_1, a_2, a_3, \infty, g'(\xi_0) \neq 0, \infty, g''(\xi_0) \neq 0, \infty.$$

Let $p_n(z) = f_n(z_n' + \rho_n \xi_0 + z)$ for arbitrary small $\varepsilon > 0$, in view of

$$\overline{E}_{A(\theta_0,\varepsilon)}(a_j, f) = \overline{E}_{A(\theta_0,\varepsilon)}(a_j, f'), \quad j = 1, 2, 3,$$

and $\lim_{n\to\infty} \epsilon_n = 0$, and for sufficiently large n,

$$\Gamma_n = \{z | z - z_n| < \epsilon_n |z_n|, z_n = |z_n| e^{i\theta_0}\} \subseteq A(\theta_0, \varepsilon/2).$$

Therefore, for every sufficiently large $n (n \geq n_0)$, we have

$$\bar{E}_D(a_i, p_n(z)) = \bar{E}_D(\varepsilon_n a_i, p_n'(z)) \quad (i = 1, 2, 3),$$

where $D = \{z : |z| < 4\}$. Note that

$$p_n(0) = f_n(z_n' + \rho_n \xi_0) = h_n(\xi_0) \to g(\xi_0) \neq a_1, a_2, a_3, \infty,$$

$$p_n'(0) = f_n'(z_n' + \rho_n \xi_0) = \frac{h_n'(\xi_0)}{\rho_n}, \quad h_n'(\xi_0) \to g'(\xi_0),$$

$$p_n''(0) = f_n''(z_n' + \rho_n \xi_0) = \frac{h_n''(\xi_0)}{\rho_n^2}, \quad h_n''(\xi_0) \to g''(\xi_0),$$

$$\varepsilon_n p_n(0) - p_n'(0) = \frac{\varepsilon_n \rho_n h_n(\xi_0) - h_n'(\xi_0)}{\rho_n}.$$

Thus, we have

$$\log\frac{\prod_{i=1}^{3}|p_n(0)-a_i|^2|p_n'(0)-\varepsilon_n a_i|^3}{|\varepsilon_n p_n(0)-p_n'(0)|^5|p_n'(0)|^2}+3\log\frac{1}{|p_n''(0)|}$$
$$=\log\frac{\prod\limits_{i=1}^{3}|p_n(0)-a_i|^2|p_n'(0)-\varepsilon_n a_i|^3}{|\varepsilon_n p_n(0)-p_n'(0)|^5|p_n'(0)|^2|p_n''(0)|^3}$$
$$=4\log\rho_n+\log\frac{\prod\limits_{i=1}^{3}|h_n(\xi_0)-a_i|^2|h_n'(\xi_0)-\rho_n\varepsilon_n a_i|^3}{|\rho_n\varepsilon_n h_n(\xi_0)-h_n'(\xi_0)|^5|h_n'(\xi_0)|^2|h_n''(\xi_0)|^3}. \tag{2.3}$$

Since $\lim\limits_{n\to\infty}\rho_n\varepsilon_n=0$ or ∞, by simple calculation we can deduce for sufficiently large $n(n\geq n_0)$

$$\log\frac{\prod_{i=1}^{3}|h_n(\xi_0)-a_i|^2|h_n^{(k)}(\xi_0)-\rho_n^k\varepsilon_n a_i|^3}{|\rho_n^k\varepsilon_n h_n(\xi_0)-h_n^{(k)}(\xi_0)|^5|h_n^{(k)}(\xi_0)|^2|h_n^{(k+1)}(\xi_0)|^3}\leq O(1)\log^+|z_n|. \tag{2.4}$$

Applying Lemma 2.2 to $p_n(z)$ with properties (2.3), (2.4), we have

$$T(r,p_n)\leq LD(r,p_n;2,3)+O(1)\left(\log^+|z_n|+m\left(r,\frac{p_n''}{p_n'-\varepsilon_n a}\right)+1\right)$$

for $0<r\leq 3$ and sufficiently large n, where $a\neq a_j$ $(j=1,2,3)$ and $a\in C$.

By Lemmas 2.11 and 2.12, we have

$$T(r,p_n)\leq O(1)(1+\log^+|z_n|).$$

In view of Lemma 2.6, we obtain

$$T_0(r,p_n)\leq O(1)(1+\log^+|z_n|).$$

Thus, we get

$$T_0(3\varepsilon_n,z_n+\varepsilon_n(z_n'+\rho_n\xi_0),f)\leq O(1)(1+\log^+|z_n|).$$

It follows that

$$A(2\varepsilon_n,z_n+\varepsilon_n(z_n'+\rho_n\xi_0),f)\leq O(1)(1+\log^+|z_n|).$$

Note that $z_n'+\rho_n\xi_0\to 0$, we get

$$\{z:|z-z_n|<\varepsilon_n\}\subseteq\{z:|z-z_n-\varepsilon_n(z_n'-\rho_n\xi_0)|<2\varepsilon_n\}.$$

Therefore, we have

$$A(\varepsilon_n,z_n,f)\leq O(1)(1+\log^+|z_n|).$$

Case 2.2. Assume that $\lim\limits_{n\to\infty}\rho_n\varepsilon_n=a_0, a_0\neq 0,\infty$. Now, we distinguish two subcases $a_0g(z)\not\equiv g'(z)$ and $a_0g(z)\equiv g'(z)$.

Case 2.2.1. $a_0g(z)\not\equiv g'(z)$. We can choose $\xi_0\in C$ such that

$$g(\xi_0)\neq 0,a_1,a_2,a_3,\infty,g'(\xi_0)\neq 0,\infty,g''(\xi_0)\neq 0,\infty,a_0g(\xi_0)-g'(\xi_0)\neq 0,\infty.$$

Let

$$p_n(z)=f_n(z_n'+\rho_n\xi_0+z).$$

By the same arguments as in case 2.1, we can get

$$A(\varepsilon_n,z_n,f)\leq O(1)(1+\log^+|z_n|).$$

Case 2.2.2. $a_0g(z)\equiv g'(z)$. We can derive that $g(z)=e^{a_0z+b_0}$, where $b_0\in C$. From (2.2), we obtain

$$h_n(z)=f_n(z_n'+\rho_n z)=f(z_n+\varepsilon_n(z_n'+\rho_n z))=f(z_n+\varepsilon_n z_n'+\varepsilon_n\rho_n z)\to g(z). \tag{2.5}$$

On the other hand, noting that f and f' share $a_i, i = 1, 2, 3$ in $A(\theta_0, \varepsilon)$, by Lemma 2.7, we have $S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) = O(1)$. Therefore, applying Lemma 2.8 to f in $A(\theta_0, \varepsilon)$, we obtain

$$\log|f(re^{i\phi})| = r^{\omega} c\sin(\omega(\phi - \alpha)) + o(r^{\omega})$$

uniformly for $\theta_0 - \varepsilon = \alpha \le \phi \le \beta = \theta_0 + \varepsilon$ as $r \notin F$ and $r \to \infty$, where c is a positive constant, $\omega = \frac{\pi}{\beta - \alpha} = \frac{\pi}{2\varepsilon}$, and F is a set of finite logarithmic measure.

Noting that F is a set of finite logarithmic measure. Therefore, there exist a real number $R, 0 < R < \infty$, and a sequence of complex numbers $u_n, 0 < |u_n| < R$, for every sufficiently large n such that

$$\log|f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)| = r_n^{\omega} c\sin(\omega(\phi - \alpha)) + o(r_n^{\omega}), \tag{2.6}$$

where $r_n = |z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n| \notin F$, $\phi_n = \arg(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)$, $\theta_0 - \varepsilon/2 \le \phi_n \le \theta_0 + \varepsilon/2$, and $\alpha = \theta_0 - \varepsilon$.

From (2.5), we get $\lim_{n\to\infty}(f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n) - g(u_n)) = 0$. Noting that u_n is a bounded sequence, there exists convergent subsequence, we still denote it by u_n and set $u_n \to u_0 (n \to \infty)$. We have $\lim_{n\to\infty} g(u_n) = \lim_{n\to\infty} e^{a_0 u_n + b_0} = e^{a_0 u_0 + b_0}$, and it follows that

$$\lim_{n\to\infty} \frac{\log|f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)|}{r_n^{\omega}} = 0.$$

On the other hand, by (2.6) we obtain that

$$\lim_{n\to\infty} \frac{\log|f(z_n + \varepsilon_n z_n' + \varepsilon_n \rho_n u_n)|}{r_n^{\omega}} = \lim_{n\to\infty} c\sin\omega(\phi - \alpha) \ge c\sin\frac{\pi}{4} > 0$$

we obtain a contradiction, and so case 2.2 is false. This completes the proof of Lemma 2.13.

3. PROOF OF THEOREMS

Proof of Theorem 1.1. Suppose that $f \not\equiv f'$, since $\arg z = \theta_0$ is a Borel direction of f, by Lemma 2.3, there exists a series of circles

$$\Gamma_j = \{z : |z - z_j| < \epsilon_j |z_j|\},$$

where $z_j = |z_j|e^{i\theta_0}$, and $\lim_{j\to\infty}|z_j| = +\infty$, $\lim_{j\to\infty}\epsilon_j = 0 (j = 1, 2, \cdots)$, such that f takes any complex number at least $|z_j|^{\lambda-\delta_j}$ times in every circle Γ_j with at most some exceptional values contained in two circles with spherical radius 2^{-j}, where $\lim_{j\to\infty}|\delta_j| = 0$. We denote the two circles by Δ_{j1} and Δ_{j2}.

Therefore, by Lemma 2.5, we have

$$A(\epsilon_j|z_j|, z_j, f) = \int_{\hat{C}} n(\epsilon_j|z_j|, z_j, f = a)dm(a)$$

$$\ge \int_{\hat{C}-\Delta_{j1}-\Delta_{j2}} n(\epsilon_j|z_j|, z_j, f = a)dm(a) \ge \frac{1}{2}|z_j|^{\lambda-\delta_j}. \tag{3.1}$$

On the other hand, from Lemma 2.13, the following inequality holds

$$A(\varepsilon_n, z_n, f) \le O(1)(1 + \log^+|z_n|), \tag{3.2}$$

where $|z| \le 1$ and $\varepsilon_n = |z_n|\epsilon_n$.

Combining with (3.1) and (3.2), we get

$$\frac{1}{2}|z_n|^{\lambda-\delta_n} \le A(\varepsilon_n, z_n, f) \le O(1)(1 + \log^+|z_n|).$$

Noting that $\lambda > 0$ and $\lim_{n\to\infty}\delta_n = 0$, this contradicts with $\lim_{n\to\infty}|z_n| = +\infty$. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Suppose that f and f' share three distinct finite complex numbers $a_j(j=1,2,3)$ IM in $A(\theta_0,\varepsilon)$, by Lemma 2.7, in view of f with infinite order and $f\not\equiv f'$, we have $S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f)=R(r,f)$, implying that

$$S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f)=O(\log U(r)), U(r)=r^{\rho(r)}.$$

On the other hand, $\arg z=\theta_0$ is a Borel direction of f with precise order $\rho(r)$. By Lemma 2.9, for arbitrarily small $\varepsilon>0$, we have

$$\limsup_{r\to+\infty}\frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f)}{\rho(r)\log r}=1.$$

Thus, we arrive at a contradiction. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Suppose that f and $L[f]$ share three distinct finite complex numbers $a_j(j=1,2,3)$ CM in $A(\theta_0,\varepsilon)$. Using Lemmas 2.10 and 2.9 in $A(\theta_0,\varepsilon)$, similar to proof of Theorem 1.2, we can conclude a contradiction. This completes the proof of Theorem 1.3.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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