ISSN 1068-3623, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2021, Vol. 56, No. 6, pp. 368–374. © Allerton Press, Inc., 2021.
Russian Text © The Author(s), 2021, published in Izvestiya Natsional'noi Akademii Nauk Armenii, Matematika, No. 6, pp. 3–11.

DIFFERENTIAL EQUATIONS

On Complex Schrödinger Type Equations with Solutions in a Given Domain

G. Barsegian[1*] and F. Meng[2]**

[1] *Institute of Mathematics, Armenia*

[2] *School of Mathematics and Information Science, Guangzhou University, China*

Received May 12, 2021; revised May 12, 2021; accepted August 20, 2021

Abstract—Value distribution, particularly the numbers of a-points, weren't studied for meromorphic functions in a given domain which are solutions of some complex differential equations. In fact we have here a "virgin land." A new program of investigations of similar solutions in a given domain was initiated quite recently. In this program some geometric methods were offered to study some standard problems as well as some new type problems related to Gamma-lines and Blaschke characteristic for a-points of the solutions of different equations. In this paper we apply these methods to get bounds for length of Gamma-lines and Blaschke characteristic for a-points for solutions of equations $w'' = gw^{\mu}$ considered in a given domain.

MSC2010 numbers: 30D30, 30D99, 34A26, 34A99, 34M10, 34M99

DOI: 10.3103/S1068362321060029

Keywords: *Schrödinger type equations, solutions of complex equations in a domain, Gamma-lines*

1. INTRODUCTION

There is a huge number of investigations in complex differential equations (CDE) when the solutions are meromorphic in the complex plane or in the unit disk. The main attention was paid to the value distribution type phenomena of the solutions, particularly to the zeros (more generally to the a-points) of these solutions. Meantime, we have very few studies of meromorphic solutions in a given domain, particularly zeros of similar solutions weren't touched at all. In fact, our present situation with the solutions in a given domain is similar to that in the beginning of 20th century when studies of the growth of solutions in the complex plane were started.

Recently, a new program of investigations of CDEs with solutions in a given domain was initiated in [4], where different characteristics of solutions were studied for different CDEs. In this paper we consider two characteristics for the solutions in a given domain of equations $w'' = gw^{\mu}$, where μ is a positive integer number.

2. ON a-POINTS OF SOLUTIONS OF $w'' = gw^{\mu}$

Denote $D_1 = \{z : |z| < 1\}$. Let $w(z)$ be a meromorphic function in D_1. Denote a-points of w by $z_i(a) \in D_1$. The Blaschke sum of zeros of w, i.e., $\sum_i (1 - |z_i(0)|)$, was widely used in the study of meromorphic functions in D_1, particularly in CDEs with solutions in the unit disk. For a given analytic function in D_1, Pommerenke considered in [10] (1982) the equation $w'' = gw$ (one-dimensional complex Schrödinger equation) with solutions w in D_1 and proved for the zeros $z_i(0)$ of w: assumption $\int\int_{D_1} |g(z)|^{1/2} d\sigma < \infty$ implies $\sum_i (1 - |z_i(0)|) < \infty$. A new stage of studies of this equation related to interrelations of g and Blaschke sum for D_1 was started recently by Heittokangas [6] (2005); for further developments see his survey in the book [8].

*E-mail: barsegiangrigor@yahoo.com
**E-mail: mfnfdbx@163.com

As we mentioned above our aim is to study CDEs with solutions in domains D.

Assume that D is a simply connected domain with smooth boundary ∂D of finite length $l(D)$ and area $S(D)$.

We study the following more general equation

$$w'' = gw^{\mu}, \tag{S$^\mu$}$$

where μ is a positive integer number and $g(z)$ is a regular function in $\bar{D} = D \cup \partial D$.

As a characteristic of a-points we consider the following *Blaschke sum of a-points for a given domain D* (considered first in [2, Chapter 1]) which we define as $\mathcal{N}(D, a, w) := \sum_i \text{Dist}(z_i(a), \partial D)$, where $\text{Dist}(x, y)$ stands obviously for the distance between x and y. Notice that in the case when D is the disk D_1 we have $\text{Dist}(z_i(0), \partial D) = 1 - |z_i(0)|$; respectively, the Blaschke sum for D becomes usual Blaschke sum for D_1.

For a regular function w in $\bar{D}$ we denote $M(w) := \max_{z \in \partial D} |w(z)|$ [1] and $m(w') := \min_{z \in \bar{D}} |w'(z)|$.

Theorem 2.1. *For an arbitrary regular in $\bar{D}$ solution $w(z)$ of equation (S^μ) and any complex value $a \neq 0$ we have*

$$\mathcal{N}(D, a, w) \leq K_{11} M^{\mu}(w) + K_{12} m(w') + K_{13}, \tag{2.1}$$

where K_{11}, K_{12}, K_{13} are independent of w.

Some comments. Notice that if we know the magnitude $w'(z_0)$ at any point $z_0 \in \bar{D}$ we can substitute $m(w')$ in (2.1) by $|w'(z_0)|$. The coefficients depend on the equation, the value a and the domain D. They are determined in the simple terms:

$$K_{11} = \frac{3\pi + 3\mu}{4|a|} M(g) l(D) S(D), \quad K_{12} = \frac{\pi + \mu}{2|a|} S(D),$$

and

$$K_{13} = \frac{1}{4} \iint\limits_{D} \left| \frac{g'(z)}{g(z)} \right| d\sigma + \frac{\pi + 2}{8} l(D).$$

Thus, K_{11}, K_{12}, and K_{13} are finite when the last double integral is finite so that (2.1) yields, in this case, simply determined bounds for $\mathcal{N}(D, a, w)$.

Finally, we notice that in the case when $g(z)$ is a polynomial of degree n the upper bounds of the double integral can be easily given by n and $S(D)$.

3. GAMMA-LINES OF SOLUTIONS OF $w'' = gw^{\mu}$

3.1. Gamma-Lines, Motivation of Their Studies and the Preceding Results

Let $w(z) := u + iv := \text{Re}w + i\,\text{Im}w$ be a meromorphic function in D. Consider level sets of $u - A$, $-\infty < A < +\infty$, that is, solutions $u(x, y) = A$ (or $\text{Re}w(z) = A$). (By the definition, level sets of real functions $u(x, y)$ are solutions of $u(x, y) = 0$). In turn, level sets are particular cases of Gamma-lines of w which are those curves in D whose w-images belong to a given curve. For instance, when Γ is the real axis, Gamma-lines become level sets of function $u(x, y)$, i.e., solutions of $u(x, y) = 0$,

One can notice a striking similarity between the a-points (which are the solutions $w(z) = a$) and the level sets (which are solutions of $u(x, y) = A$). On the other hand, level sets of $u - A$ admit a lot of interpretations (streaming line, potential line, isobar, isoterm) in different applied fields of engineering, physics, environmental, and other problems. Due to the above arguments (similarity with a-points and applicability), it is pertinent to study largely level sets for different classes of meromorphic functions particularly for the solutions w of different classes of complex differential equations.

[1] Here we may remember that in numerous studies concerning regular functions w in the disks $D(r) := \{z : |z| < r\}$ (instead of the domains D) the magnitude $M(w)$ plays a role of a characteristic. The same is true also for entire functions; in this case we deal usually with $\ln M(w) := \ln \max_{z \in \partial D(r)} |w(z)|$.

We denote the length of Gamma-lines of w lying in D by $L(D,\Gamma,w)$. These lengths were widely studied in [2] for large classes of smooth Jordan curves Γ (bounded or unbounded) in the complex plane. The only restriction for Γ is that $\nu(\Gamma)=\mathrm{Var}_{z\in\Gamma}\alpha_\Gamma(z)<\infty$, where Var means variation and $\alpha_\Gamma(z)$ is the angle between the tangent to Γ at $z\in\Gamma$ and the real axis.

As to Gamma-lines for solutions of equation, they were considered first recently in [1] for solutions in D of equation $w''=gw$, where estimates of $L(D,\Gamma,w)$ were given in terms of Ahlfors–Shimizu classical characteristic.

In this section, we give upper bounds of $L(D,\Gamma,w)$ for solution w of (S^μ). The bounds will be given in terms of $M(w)$, which in application mean often some important physical concepts.

Theorem 3.1. *Let $w(z)$ be a regular function in $\bar{D}$ which is a solution of equation (S^μ) and Γ a smooth Jordan curve with $\nu(\Gamma)<\infty$ which does not pass through zero. Then*

$$L(D,\Gamma,w)\le K_{21}M^\mu(w)+K_{22}m(w')+K_{23}, \tag{3.1}$$

where K_{21}, K_{22}, and K_{23} are independent of w.

The coefficients depend on the equation, the curve Γ and the domain D. They are determined in the simple terms:

$$K_{21}=K(\Gamma)\frac{3\pi+3\mu}{|a_\Gamma|}M(g)l(D)S(D),\quad K_{22}=K(\Gamma)\frac{2\pi+2\mu}{|a_\Gamma|}S(D),$$

where $K(\Gamma)=3(\nu(\Gamma)+1)$, a_Γ is the closest to the zero point belonging to Γ,[2] and

$$K_{23}=K(\Gamma)\iint\limits_D\left|\frac{g'(z)}{g(z)}\right|d\sigma+K(\Gamma)\frac{\pi+2}{2}l(D).$$

Theorem 3.2. *Assuming in Theorem 3.1 that Γ is a straight line which does not pass through zero, we have*

$$L(D,\Gamma,w)\le K_{31}M^\mu(w)+K_{32}m(w')+K_{33}, \tag{3.2}$$

where K_{31}, K_{32}, and K_{33} are independent of w.

Assuming that a is the closet to zero point on Γ, we have

$$K_{31}=\frac{3\pi+3\mu}{2|a|}M(g)l(D)S(D),\quad K_{32}=\frac{\pi+\mu}{|a|}S(D),$$

and

$$K_{33}=\frac{1}{2}\iint\limits_D\left|\frac{g'(z)}{g(z)}\right|d\sigma+\frac{\pi+2}{4}l(D).$$

4. PROOFS

Proof of Theorem 3.1. We need the following "basic identity for Gamma-lines" (see [2, item 1.1.3, identity (1.1.6)]). We state it as

Lemma 4.1. *For any regular function w in D, we have*

$$\int\limits_0^\infty L(D,\Gamma(R),w)dR=\iint\limits_D|w'|\,d\sigma,$$

where $\Gamma(R)$ is the circumference $\{w:\ |w|=R\}$.

For a given $a\in\mathbb{C}$, $a\neq0$, we denote $D(|a|/2,3|a|/4):=\{z:|a|/2<|w(z)|<3|a|/4\}$. This set consists of some connected components which are simply connected or multiply connected components. Dividing multiply connected components into some simply connected ones, we can consider

[2] If we have more than one similar point we take arbitrary of them.

$D(|a|/2, 3|a|/4)$ as a union of simply connected domains $D_\lambda(|a|/2, 3|a|/4)$, where λ is a counting index of these domains. Applying Lemma 4.1 in each $D_\lambda(|a|/2, 3|a|/4)$ and then summing up for all indexes λ, we obtain

$$\int\limits_{|a|/2}^{3|a|/4} L(D, \Gamma(R), w)dR = \iint\limits_{D(|a|/2, 3|a|/4)} |w'|\, d\sigma.$$

Due to the mean value theorem we conclude that there is $R^* \in (|a|/2, 3|a|/4)$ such that

$$L(D, \Gamma(R^*), w) = \frac{4}{|a|} \iint\limits_{D(|a|/2, 3|a|/4)} |w'|\, d\sigma. \tag{4.1}$$

Denote $D(|w| > c) = \{z : |w(z)| > c > 0\}$. The set $D(|w| > c)$ may consists of one or more domains $D_\eta(|w| > c)$; clearly they can be as simply connected as well as multiply connected. By $\partial D_\eta(|w| > c)$ we denote the union of all boundary components of $D_\eta(|w| > c)$. Notice that the boundary $\partial D_\eta(|w| > R^*)$ should have a (nonempty) common part $\partial D_\eta(|w| > R^*) \cap \partial D$ with ∂D. (Indeed, assume contrary, that $\partial D_\eta(|w| > R^*)$ lies fully inside D. Then w should have a pole inside D which contradicts our assumption that w is regular in $\bar{D}$). Observing that the different common parts (taken for different η) do not overlap we obtain

$$\sum_\eta l\left(\partial D_\eta(|w| > R^*)\right) \leq L(D, \Gamma(R^*), w) + l(\partial D). \tag{4.2}$$

We need also the following "principle of logarithmic derivatives," which was established recently [3] by making use of Gamma-lines technic.

Lemma 4.2. *Let d be a bounded domain with piecewise smooth boundary (d can be also multiply connected); we assume that the intersection of d with any straight line consists of finite number of intervals. Then, for any meromorphic function f in the closure of d and any integer $k \geq 1$, we have*

$$\iint\limits_d \left|\frac{f'(z)}{f(z)}\right| d\sigma \leq \iint\limits_d \left|\frac{f^{(k+1)}(z)}{f^{(k)}(z)}\right| d\sigma + \frac{k\pi}{2} l(\partial d). \tag{4.3}$$

Comment 1. In [3], we assumed that the intersection of d with any straight line consists of finite number of intervals. This restriction on intersection was putted just for simplicity of the proof. To avoid this it is enough to consider a domain d^* ("very close" to d) which satisfies this restriction. Then we can apply (4.3) to d^* and make limit transfer to d. We will come to the above wording of Lemma 4.1.

Assume now that f is our regular function w in $\bar{D}$ and d is one of the domains $D_\eta(|w| > R^*)$. Notice that the part of the boundary $\partial D_\eta(|w| > R^*)$ lying in D consists of piecewise analytic curves with a finite number of possible turning points where $w' = 0$. This implies that the boundary of each $\partial D_\eta(|w| > R^*)$ is piecewise smooth so that we can apply (4.3). Applying it for the derivative w' in a given domain $D_\eta(|w| > R^*)$ with $k \geq 2$ we have

$$\iint\limits_{D_\eta(|w|>R^*)} \left|\frac{w''(z)}{w'(z)}\right| d\sigma \leq \iint\limits_{D_\eta(|w|>R^*)} \left|\frac{w^{(k+1)}(z)}{w^{(k)}(z)}\right| d\sigma + \frac{\pi}{2}(k-1) l\left(\partial D_\eta(|w| > R^*)\right). \tag{4.4}$$

Further, we need the following "tangent variation principle" (see [2, item 1.2.2 inequalities 1.2.8 and 1.2.9]).

Lemma 4.3. *For any meromorphic function $f(z)$ in $\bar{D}$ and any smooth Jordan curve Γ (bounded or unbounded) with $\nu(\Gamma) < \infty$, we have*

$$L(D, \Gamma, f) \leq K(\Gamma) \left\{ \iint\limits_D \left|\frac{f''(z)}{f'(z)}\right| d\sigma + l(\partial D) \right\}, \tag{4.5}$$

where $K(\Gamma) = 3(\nu(\Gamma) + 1)$.

Comment 2. In particular case when Γ is a straight line, the above formula can be improved. Due to Theorem 1 in [5] we have in this case

$$L(D, \Gamma, f) \le \frac{1}{2} \iint\limits_{D} \left| \frac{f''(z)}{f'(z)} \right| d\sigma + \frac{1}{2} l(\partial D). \tag{4.6}$$

Applying (4.4) to the regular function w in any of the domains $D_\eta(|w| > R^*)$ and combining with (4.4), we obtain: for any smooth Jordan curve Γ with $\nu(\Gamma) < \infty$,

$$L(D_\eta(|w| > R^*), \Gamma, w) \le K(\Gamma) \left\{ \iint\limits_{D_\eta(|w|>R^*)} \left| \frac{w^{(k+1)}(z)}{w^{(k)}(z)} \right| d\sigma + \frac{\pi}{2} (k-1)\, l\left(\partial D_\eta(|w| > R^*)\right) + l(D) \right\}.$$

Summing up this inequality by η, we get the following formula for $D(|w| > R^*)$:

$$L(D(|w| > R^*), \Gamma, w) \le$$

$$K(\Gamma) \left\{ \iint\limits_{D(|w|>R^*)} \left| \frac{w^{(k+1)}(z)}{w^{(k)}(z)} \right| d\sigma + \frac{\pi}{2} (k-1)\, l\left(\partial D(|w| > R^*)\right) + l(D) \right\},$$

where

$$l\left(\partial D(|w| > R^*)\right) = \sum_\eta l\left(\partial D_\eta(|w| > R^*)\right).$$

Applying (4.2) to the last inequality, we obtain

$$L(D(|w| > R^*), \Gamma, w) \le K(\Gamma)$$

$$\times \left\{ \iint\limits_{D(|w|>R^*)} \left| \frac{w^{(k+1)}(z)}{w^{(k)}(z)} \right| d\sigma + \frac{\pi(k-1)}{2} L(D, \Gamma(R^*), w) + \left(\frac{\pi\,(k-1)}{2} + 1 \right) l(D) \right\}. \tag{4.7}$$

Comment 3. For a straight line Γ we can apply (4.6) instead of (4.5). Respectively instead of (4.7) we get

$$L(D(|w| > R^*), \Gamma, w) \le \frac{1}{2} \iint\limits_{D(|w|>R^*)} \left| \frac{w^{(k+1)}(z)}{w^{(k)}(z)} \right| d\sigma$$

$$+ \frac{\pi}{4} (k-1)\, L(D, \Gamma(R^*), w) + \frac{1}{2} \left(\frac{\pi\,(k-1)}{2} + 1 \right) l(D). \tag{4.8}$$

Now, we consider a curve Γ in Theorem 3.1 which does not pass through zero. Assume a_Γ is the point on Γ which is the closest to the point 0; if we have more than one similar points, we take arbitrary one of them. With this value a_Γ we define as above corresponding value $R^*_\Gamma \in (|a_\Gamma|/2, 3|a_\Gamma|/4)$ and notice that the curve Γ (which we consider in w-plane) lies fully in the set $D(|w| > R^*_\Gamma)$. Respectively, Gamma-lines of this Γ lie fully in the set $D\left(|w| > R^*_\Gamma\right)$ so that we have $L(D(|w| > R^*_\Gamma), \Gamma, w) = L(D, \Gamma, w)$ and (4.7) yields

$$L(D, \Gamma, w) \le K(\Gamma)$$

$$\times \left\{ \iint\limits_{D(|w|>R^*_\Gamma)} \left| \frac{w^{(k+1)}(z)}{w^{(k)}(z)} \right| d\sigma + \frac{\pi}{2} (k-1)\, L(D, \Gamma(R^*_\Gamma), w) + \left(\frac{\pi\,(k-1)}{2} + 1 \right) l(D) \right\}. \tag{4.9}$$

Now, we apply the last inequality to our solution $w(z)$ of equation (S^μ) for $\mu = 2$, we have for any $z \in \bar{D}$

$$\left|\frac{w'''(z)}{w''(z)}\right| = \left|\frac{g'(z)\,(w(z))^\mu + \mu g(z)\,(w(z))^{\mu-1}\,w'(z)}{g(z)\,(w(z))^\mu}\right| \leq \left|\frac{g'(z)}{g(z)}\right| + \mu\left|\frac{w'(z)}{w(z)}\right|.$$

Thus, due to definition of R^*_Γ, for any $z \in D(|w| > R^*_\Gamma)$ we have $|w(z)| > |a_\Gamma|/2$; consequently,

$$\left|\frac{w'''(z)}{w''(z)}\right| \leq \left|\frac{g'(z)}{g(z)}\right| + \frac{2\mu}{|a_\Gamma|}\left|w'(z)\right|,$$

and taking into account that $D(|w| > R^*_\Gamma) \subset D$, we get

$$\iint\limits_{D(|w|>R^*_\Gamma)} \left|\frac{w'''(z)}{w''(z)}\right| d\sigma \leq \iint\limits_{D(|w|>R^*_\Gamma)} \left\{\left|\frac{g'(z)}{g(z)}\right| + \frac{2\mu}{|a_\Gamma|}\left|w'(z)\right|\right\} d\sigma \leq \iint\limits_{D} \left|\frac{g'(z)}{g(z)}\right| d\sigma + \frac{2\mu}{|a_\Gamma|}\iint\limits_{D}\left|w'(z)\right| d\sigma.$$

Due to (4.1) we also have

$$L(D, \Gamma(R^*_\Gamma), w) \leq \frac{4}{|a_\Gamma|}\iint\limits_{D}\left|w'\right| d\sigma$$

so that, applying the last two inequalities to (4.9) applied for $\mu = 2$, we obtain

$$L(D, \Gamma, w) \leq K(\Gamma)\left\{\iint\limits_{D}\left|\frac{g'(z)}{g(z)}\right| d\sigma + \frac{2\pi + 2\mu}{|a_\Gamma|}\iint\limits_{D}\left|w'(z)\right| d\sigma + \left(\frac{\pi}{2} + 1\right) l(D)\right\}. \tag{4.10}$$

Since w and g are regular functions and μ is an integer, we conclude that gw^μ is a regular function so that, taking into account that $w'' = gw^\mu$, we have for an arbitrary $z_0 \in \bar{D}$

$$w'(z) - w'(z_0) = \int\limits_{z_0}^{z} w''(Z)dZ = \int\limits_{z_0}^{z} g(Z)\,(w(Z))^\mu\,dZ.$$

Consequently, we have $|w'(z)| \leq M(g)M^\mu(w)l_D(z, z_0) + |w'(z_0)|$, where $l_D(z, z_0)$ is the length of a curve, say γ, which lies in $\bar{D}$ and connects z and z_0. We always can connect z with a point $z^* \in \partial D$ and z_0 with a point $z_0^* \in \partial D$ by some curves with the lengths $l(D)/2$ and then can connect the points z^* and z_0^* by a part of the boundary ∂D of the length $l(D)/2$. Thus, we always can take γ such that $l_D(z, z_0) \leq 3l(D)/2$. Also, we can take z_0 such that $|w'(z_0)|$ reaches its minimum in $\bar{D}$ (that is $|w'(z_0)| := m(w') := \min_{z\in\bar{D}} |w'(z)|$). With similar notations we obtain

$$\iint\limits_{D} |w'(z)|d\sigma \leq \frac{3}{2}M(g)M^\mu(w)l(D)S(D) + m(w')S(D).$$

Consequently, (4.10) implies

$$\begin{aligned} L(D, \Gamma, w) &\leq K(\Gamma)\frac{3\pi + 3\mu}{|a_\Gamma|}M(g)M^\mu(w)l(D)S(D) \\ &+ K(\Gamma)\frac{2\pi + 2\mu}{|a_\Gamma|}m(w')S(D) + K(\Gamma)\iint\limits_{D}\left|\frac{g'(z)}{g(z)}\right| d\sigma \\ &+ K(\Gamma)\frac{\pi + 2}{2}l(D) = K_{21}M^\mu(w) + K_{22}m(w') + K_{23}, \end{aligned} \tag{4.11}$$

with K_{21}, K_{22}, and K_{23} given after Theorem 3.1. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. This theorem is a particular case of Theorem 3.1, where we deal with a straight line Γ. Due to Comment 3, we see that the constant $K(\Gamma)$ in (4.7) is replaced by $1/2$ for the straight line; respectively we should apply (4.8) (instead of (4.7)) in the above proofs. Applying (4.8), we obtain (4.9), (4.10), and (4.11) with $K(\Gamma)$ replaced by $1/2$. Respectively, we get the proof of Theorem 3.2 with the coefficients K_{31}, K_{32}, and K_{33} given after Theorem 3.2.

Proof of Theorem 2.1. The next inequality giving interrelations between Blaschke characteristic and Gamma-lines was proved in [2, item 1.5], (see also [4, item 7.1])): for any regular function w in D and any smooth Jordan curve Γ connecting a with ∞ we have $\mathcal{N}(D,a,w) \leq L(D,\Gamma,w)$. Since any straight line passing through a contains two parts connecting a with ∞, we have for any straight line Γ

$$\mathcal{N}(D,a,w) \leq \frac{1}{2}L(D,\Gamma,w).$$

Due to Theorem 3.2 we have upper bounds $L(D,\Gamma,w)$ for any straight line Γ, which does not pass through zero. Respectively, Theorem 3.2 and the previous inequality give the following upper bounds for $\mathcal{N}(D,a,w)$:

$$\mathcal{N}(D,a,w) \leq \frac{1}{2}L(D,\Gamma,w) \leq \frac{1}{2}\left[K_{31}M^{\mu}(w) + K_{32}m(w') + K_{33}\right].$$

Denoting $K_{11} = \frac{1}{2}K_{31}$, $K_{12} = \frac{1}{2}K_{32}$, and $K_{13} = \frac{1}{2}K_{33}$, we obtain Theorem 2.1.

FUNDING

The work is supported by the NSF of China (11701111), the NSF of Guangdong Province (2016A030310257).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. A. Alonso, G. Barsegian, and A. Fernandez, "Gamma-lines approach in the study of global behavior of solutions of complex differential equations," Complex Var. Elliptic Equations **56**, 119–135 (2011). https://doi.org/10.1080/17476930903394747
2. G. Barsegian, *Gamma-Lines: On the Geometry of Real and Complex Functions* (Taylor & Francis, London, 2002). https://doi.org/10.1201/9781482264852
3. G. Barsegian, "Some interrelated results in different branches of geometry and analysis," in *Further Progress in Analysis: Proceedings of the 6th International ISAAC Congress, Ankara, Turkey, 2007*, Ed. by H. G. W. Begehr, A. O. Celebi, and R. P. Gilbert (World Scientific, 2009), pp. 3–32. https://doi.org/10.1142/9789812837332_0001
4. G. Barsegian, "Initiating a new trend in complex equations studying solutions in a given domain: problems, results, approaches," Preprint.
5. G. Barsegian and G. Sukiasyan, "Methods for study level sets of enough smooth functions," in *Topics in Analysis and Applications: The NATO Advanced Research Workshop, Yerevan, 2002*, Ed. by G. Barsegian and H. Begehr, NATO Science Publications (Kluwer, 2004).
6. J. Heittokangas, "Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequences as the zeros," Comput. Methods Funct. Theory **5**, 49–63 (2005). https://doi.org/10.1007/BF03321085
7. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Mathematics, vol. 15 (de Gruyter, Berlin, 1993).
8. *Blaschke Products and Their Application*, Ed. by J. Mashreghi and E. Fricain, The Fields Institute Communications, vol. 65 (Springer, New York, 2013). https://doi.org/10.1007/978-1-4614-5341-3
9. R. Nevanlinna, *Eindeutige analytische Funktionen* (Berlin, Springer, 1936).
10. Ch. Pommerenke, "On the mean growth of the solutions of complex linear differential equations in the disk," Complex Var., Theory Appl. **1**, 23–38 (1982). https://doi.org/10.1080/17476938208814004