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Generalized Composition Operators from the Lipschitz Space into the Zygmund Space

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Abstract—In this paper, at first we study boundedness and compactness criterions for generalized composition operator from the Lipschitz space into the Zygmund space. Then we estimate the essential norm of this operator.

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1. INTRODUCTION

Let X and Y be Banach spaces. The essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$||T||_{e,X\to Y} = \inf\{||T-K||_{X\to Y} :\ \text{K is compact}\}.$$

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ and H^∞ be the space of bounded analytic functions on $\mathbb{D}$ with norm $||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the set of self-maps of $\mathbb{D}$. The weighted composition operator with symbols u and φ, denoted by uC_φ, is defined as follows

$$uC_\varphi f = M_u C_\varphi f = u(f\circ\varphi), \quad f \in H(\mathbb{D}),$$

where M_u is the multiplication operator with symbol u and C_φ is the composition operator. We refer the interested reader to [4] and [12] for the theory of the composition operators and to [2, 3, 6, 10, 14, 16, 20, 21, 22] for (weighted) composition on various spaces of analytic functions.

Let $\mathcal{Z}$ denote the set of all functions $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$||f|| = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By Theorem 5.3 of [12] and the Closed Graph Theorem, we see that an analytic function f on $\mathbb{D}$ belongs to $\mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty$. Furthermore,

$$||f|| \approx \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|.$$

The preceding quantity is seminorm for the space $\mathcal{Z}$. The norm defined by

$$||f||_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|$$

yields a Banach space structure on $\mathcal{Z}$, which is called the Zygmund space.

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Let $0 < \alpha < \infty$. A function $f \in H(\mathbb{D})$ is said to belong to the Bloch type space $\mathcal{B}^\alpha$ if

$$\beta_f = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Under the seminorm $f \to \beta_f$, $\mathcal{B}^\alpha$ is conformally invariant, and the norm defined by $||f||_{\mathcal{B}^\alpha} = |f(0)| + \beta_f$ yields a Banach space structure on $\mathcal{B}^\alpha$. It is well known that for $0 < \alpha < 1$ $\mathcal{B}^\alpha$ is a subspace of H^∞. When $\alpha = 1$, we get the classical Bloch space $\mathcal{B}$.

The Lipschitz space Lip_α (with $0 < \alpha < 1$) is the space of functions $f \in H(\mathbb{D})$ satisfying the Lipschitz condition of order α, i.e, there exists a constant $C > 0$ such that

$$|f(z) - f(w)| \le C|z - w|^\alpha, \quad z, w \in \mathbb{D}.$$

Such functions f extend continuously to the closure of the disc. The quantity

$$||f||_{\mathrm{Lip}_\alpha} = |f(0)| + \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^\alpha},\ z, w \in \mathbb{D}, z \ne w\right\}$$

defines a norm on Lip_α. Let $f \in \mathrm{Lip}_\alpha$ and set

$$C = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^\alpha},\ z, w \in \mathbb{D}, z \ne w\right\}.$$

Then, for $z \in \mathbb{D}$, we have $|f(z)| \le |f(0)| + C|z|^\alpha \le C|z - w|^\alpha \le ||f||_{\mathrm{Lip}_\alpha}$.

Thus, taking the supremum over $\mathbb{D}$, we obtain $||f||_\infty \le ||f||_{\mathrm{Lip}_\alpha}$. By a theorem of Hardy and Littlewood [5], the elements of Lip_α are characterized by the following Bloch-type condition: A function $f \in H(\mathbb{D})$ belongs to Lip_α if and only if

$$\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\alpha}|f'(z)| < \infty.$$

Moreover,

$$||f||_{\mathrm{Lip}_\alpha} \approx |f(0)| + \alpha(f). \tag{1.1}$$

Composition operators, weighted composition operators, and related operators between the Zygmund space and some various spaces of analytic functions have been studied in [7, 8, 9, 13, 19]. In [8], Li and Stević defined the generalization composition operator C_φ^g as follows

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi. \tag{1.2}$$

Li and Stević studied the boundedness and compactness of the generalized composition operator on the Zygmund space and the Bloch type space and the little Bloch type space in [8]. In this paper, we study boundedness and compactness of the generalization composition operator C_φ^g from Lip_α to $\mathcal{Z}$. Also we give some estimates for the essential norm of this operator. Weighted composition operators uC_φ between Lip_α and $\mathcal{Z}$ spaces were studied by Colonna and Li in [2]. Some characterizations of the boundedness and compactness of the composition operator, as well as Volterra type operator, on the Bloch type space and the Zygmund space can be found in [1, 5, 17].

The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. We say that $a \approx b$ if both $a \preceq b$ and $b \preceq a$ hold.

2. BOUNDEDNESS OF THE OPERATOR $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$

In this section, we give necessary and sufficient conditions for the boundedness of the operator $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$.

Theorem 2.1. *Let $0 < \alpha < 1$, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the operator $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$ is bounded if only if the following quantities are finite*:

$$M_1 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|g'(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}}$$

and

$$M_2 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|g(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}}.$$

Proof. For any $f \in \mathrm{Lip}_\alpha$,

$$(1-|z|^2)|(C^g_\varphi f)''(z)| = (1-|z|^2)|(f'(\varphi(z))g(z))'| \le (1-|z|^2)|f'(\varphi(z)||g'(z))|$$

$$+ (1-|z|^2)|(\varphi'(z))||g(z)||(f''(\varphi(z))| \le C||f||_{\mathrm{Lip}_\alpha}\left(\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}} + \frac{(1-|z|^2)|g'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}}\right),$$

where in the last inequality we have used (1.1) and the following well known characterization of Bloch type functions (see [18]):

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\alpha}|f'(z)| \approx |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{2-\alpha}|f''(z)|.$$

Conversely, assume that $C^g_\varphi : \mathrm{Lip}_\alpha \to \mathcal{Z}$ is bounded. For a fixed $a \in \mathbb{D}$ and for $z \in \mathbb{D}$ and $1 \le j \le 3$, set

$$f_{a,j}(z) = \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j-\alpha}}.$$

A direct calculation shows that

$$f_{a,j}(a) = (1-|a|^2)^\alpha, \quad f'_{a,j}(a) = \frac{(j-\alpha)\overline{a}}{(1-|a|^2)^{1-\alpha}}, \quad f''_{a,j}(a) = \frac{(j-\alpha)(j+1-\alpha)\overline{a}^2}{(1-|a|^2)^{2-\alpha}}.$$

Then, for $w \in \mathbb{D}$, we get,

$$(C^g_\varphi f_{\varphi(w),1})''(w) = \frac{(1-\alpha)g(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(1-\alpha)(2-\alpha)g(w)\varphi'(w)\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}, \tag{2.1}$$

$$(C^g_\varphi f_{\varphi(w),2})''(w) = \frac{(2-\alpha)g(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(2-\alpha)(3-\alpha)g(w)\varphi'(w)\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}} \tag{2.2}$$

and

$$(C^g_\varphi f_{\varphi(w),3})''(w) = \frac{(3-\alpha)g(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(3-\alpha)(4-\alpha)g(w)\varphi'(w)\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}. \tag{2.3}$$

Subtracting (2.1) from (2.2), we get

$$(C^g_\varphi f_{\varphi(w),2})''(w) - (C^g_\varphi f_{\varphi(w),1})''(w) = \frac{g(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(4-2\alpha)g(w)\varphi(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}. \tag{2.4}$$

On the other hand, subtracting (2.1) from (2.3), we obtain

$$(C^g_\varphi f_{\varphi(w),3})''(w) - (C^g_\varphi f_{\varphi(w),1})''(w) = \frac{2g(w)\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{1-\alpha}} + \frac{(10-4\alpha)g(w)\varphi(w)^2\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}}.$$

Subtracting (2.3) from (2.4), we get

$$\frac{2g(w)\varphi'(w)\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{2-\alpha}} = (C^g_\varphi f_{\varphi(w),1})''(w) - 2(C^g_\varphi f_{\varphi(w),2})''(w) + (C^g_\varphi f_{\varphi(w),3})''(w),$$

which implies that

$$\frac{(1-|w|^2)|g(w)\varphi'(w)||\varphi(w)|^2}{(1-|\varphi(w)|^2)^{2-\alpha}}$$

$$= \frac{1}{2}|(C^g_\varphi f_{\varphi(w),1})''(w)| + |(C^g_\varphi f_{\varphi(w),2})''(w)| + \frac{1}{2}|(C^g_\varphi f_{\varphi(w),3})''(w)|$$

$$\leq \frac{1}{2}||(C_\varphi^g f_{\varphi(w),1})(w)||_{\mathcal{Z}} + ||(C_\varphi^g f_{\varphi(w),2})(w)||_{\mathcal{Z}} + \frac{1}{2}||(C_\varphi^g f_{\varphi(w),3})(w)||_{\mathcal{Z}} \leq C. \tag{2.5}$$

Fix $r \in (0,1)$. If $|\varphi(w)| > r$, then by (2.5), we have

$$\frac{(1-|w|^2)|g'(w)|\varphi(w)|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leq \frac{C}{r}.$$

Taking the functions $f(z) = z$ and $f(z) = z^2$ respectively, we obtain

$$N_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)| < \infty \tag{2.6}$$

and $N_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)\varphi(z) + g(z)\varphi'(z)| < \infty$.

If $|\varphi(w)| < r$, then by (2.6) we get

$$M_1 = \frac{(1-|w|^2)|g'(w)|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leq \frac{N_1}{(1-r^2)^{1-\alpha}}, \tag{2.7}$$

which, combined with (2.7), implies that $M_1 < \infty$. Arguing similarly, we get

$$\frac{(1-|w|^2)|g'(w)|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leq C$$

and

$$M_2 = \frac{(1-|w|^2)|g(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{2-\alpha}} \leq \frac{N_2}{(1-r^2)^{2-\alpha}} < \infty.$$

This means $M_2 < \infty$. □

3. COMPACTNESS OF THE OPERATOR $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$

In this section, we study the compactness of the operator $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$. We begin with the following lemma.

Lemma 3.1 ([11], Lemma 3.7). *Let $0 < \alpha < 1$ and T be a bounded linear operator from Lip_α into a normed linear space Y. Then T is compact if and only if $||Tf_n||_Y \to 0$ whenever $\{f_n\}$ is a norm-bounded sequence in Lip_α that converges to 0 uniformly on $\bar{\mathbb{D}}$.*

Theorem 3.2. *Let $0 < \alpha < 1$, $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$ is compact if and only if bounded,*

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)|g'(z_k)|}{(1-|\varphi(z_k)|^2)^{1-\alpha}} = 0 \tag{3.1}$$

and

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)|\varphi'(z_k)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{2-\alpha}} = 0.$$

Proof. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Let $f_{k,j} = \frac{(1-|\varphi(z_k)|^2)^j}{(1-\overline{\varphi(z_k)}z)^{j-\alpha}}$, $k \in \mathbb{N}$. Then $f_{k,j} \in \mathrm{Lip}_\alpha$, $\sup_{k\in\mathbb{N}} ||f_{k,j}||_{\mathrm{Lip}_\alpha} < \infty$ and $f_{k,j} \to 0$ uniformly on $\overline{\mathbb{D}}$ as $k \to \infty$. Let $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$ be compact. By Lemma 3.1 it gives $\lim_{k\to\infty} ||C_\varphi^g f_{k,j}||_{\mathcal{Z}} = 0$. Note that

$$f'_{k,j}(\varphi(z_k)) = \frac{(j-\alpha)\overline{\varphi(z_k)}}{(1-|\varphi(z_k|^2)^{1-\alpha}}, \quad f''_{a,j}(\varphi(z_k)) = \frac{(j-\alpha)(j+1-\alpha)\overline{\varphi^2(z_k)}}{(1-|\varphi(z_k|^2)^{2-\alpha}}.$$

We have

$$||C_\varphi^g f_{k,j}||_{\mathcal{Z}} \geq \frac{(j-\alpha)(1-|z_k|^2)|g'(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{1-\alpha}}\Bigg|$$

$$-\left.\frac{(j-\alpha)(j+1-\alpha)(1-|z_k|^2)|\varphi'(z_k)|g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{2-\alpha}}\right|.$$

Consequently,

$$\begin{aligned}&\lim_{|\varphi(z_k)|\to 1}\frac{(j-\alpha)(1-|z_k|^2)|g'(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{1-\alpha}}\\ &=\lim_{|\varphi(z_k)|\to 1}\frac{(j-\alpha)(j+1-\alpha)(1-|z_k|^2)|\varphi'(z_k)|g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{2-\alpha}}\end{aligned} \tag{3.2}$$

if one of these two limits exists. Next, set

$$h_{k,j}=\frac{(1-|\varphi(z_k)|^2)^j}{(1-\overline{\varphi(z_k)}z)^{j-\alpha}}-\frac{j-\alpha}{j+1-\alpha}\frac{(1-|\varphi(z_k)|^2)^{j+1}}{(1-\overline{\varphi(z_k)}z)^{j+1-\alpha}}.$$

Then $h'_{k,j}(\varphi(z_k))=0$, $\sup_{k\in\mathbb{N}}||h_{k,j}||_{\mathcal{Z}}<\infty$ and $h_{k,j}$ converges to 0 uniformly on $\overline{\mathbb{D}}$ as $k\to\infty$. Since $C_\varphi^g:\mathrm{Lip}_\alpha\to\mathcal{Z}$ is compact, we have $\lim_{k\to\infty}||C_\varphi^g h_{k,j}||_{\mathcal{Z}}=0$. On the other hand,

$$||C_\varphi^g h_{k,j}||_{\mathcal{Z}}\geq\frac{(j-\alpha)(j+1-\alpha)(1-|z_k|^2)|\varphi'(z_k)|g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{2-\alpha}}.$$

Hence,

$$\lim_{k\to\infty}\frac{(j-\alpha)(j+1-\alpha)(1-|z_k|^2)|\varphi'(z_k)|g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{2-\alpha}}=0.$$

Therefore,

$$\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)|\varphi'(z_k)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{2-\alpha}}=\lim_{k\to\infty}\frac{(j-\alpha)(j+1-\alpha)(1-|z_k|^2)|\varphi'(z_k)|g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{2-\alpha}}=0.$$

This together with (3.2) imply that

$$\lim_{|\varphi(z_k)|\to 1}\frac{(j-\alpha)(1-|z_k|^2)|g'(z_k)|}{(1-|\varphi(z_k)|^2)^{1-\alpha}}=0.$$

Conversely, assume that $C_\varphi^g:\mathrm{Lip}_\alpha\to\mathcal{Z}$ is bounded and (3.1) holds. Since $C_\varphi^g:\mathrm{Lip}_\alpha\to\mathcal{Z}$ is bounded, we have $||C_\varphi^g f||_{\mathcal{Z}}\leq C||f||_{\mathrm{Lip}_\alpha}$ for all $f\in\mathrm{Lip}_\alpha$. Taking the functions $f(z)=z$ and $f(z)=z^2$ respectively, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|<\infty \tag{3.3}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)\varphi(z)+g(z)\varphi'(z)|<\infty. \tag{3.4}$$

Using these facts and the boundedness of the function $\varphi(z)$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|<\infty.$$

Then,

$$C_1=\sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)||\varphi'(z)|<\infty \tag{3.5}$$

and

$$C_2=\sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)||\varphi'(z)|<\infty.$$

On the other hand, from (3.1), for every $\epsilon>0$, there is a $\delta\in(0,1)$ such that

$$\frac{(1-|z_k|^2)|g'(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{1-\alpha}}<\epsilon\quad\text{and}\quad\frac{(1-|z_k|^2)|\varphi'(z_k)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{2-\alpha}}<\epsilon, \tag{3.6}$$

whenever $\delta < |\varphi(z)| < 1$. Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in Lip_α such that $\sup_{k\in\mathbb{N}} ||f_k||_{\mathrm{Lip}_\alpha} < \infty$ and (f_k) converges to 0 uniformly on $\overline{\mathbb{D}}$ as $k \to \infty$. Let $U = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$. Then by (3.5) and (3.6), it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(C_\varphi^g f_k)''(z)| \leq \sup_{z\in\mathbb{D}}(1-|z|^2)|f_k'(\varphi(z))||g'(z)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(\varphi'(z))||g(z)||f_k''(\varphi(z))|$$
$$+ \sup_{z\in\mathbb{D}\setminus U}(1-|z|^2)|f_k'(\varphi(z))||g'(z)| + \sup_{z\in\mathbb{D}\setminus U}(1-|z|^2)|(\varphi'(z))||g(z)||f_k''(\varphi(z))|$$
$$\leq C_1 \sup_{z\in\mathbb{D}}|f_k'(\varphi(z))| + \sup_{z\in\mathbb{D}\setminus U}\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|^2)^{1-\alpha}}||f||_{\mathrm{Lip}_\alpha} + C_2\sup_{z\in\mathbb{D}}|f_k''(\varphi(z))|$$
$$+ \sup_{z\in\mathbb{D}\setminus U}\frac{(1-|z|^2)|g'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{2-\alpha}}||f||_{\mathrm{Lip}_\alpha} \leq C_1\sup_{|\lambda|\leq\delta}|f_k'(\lambda)| + C_2\sup_{|\lambda|\leq\delta}|f_k''(\lambda)| + 2C_\epsilon||f||_{\mathrm{Lip}_\alpha}.$$

So,

$$||C_\varphi^g f_k||_{\mathcal{Z}} = |f_k'(\varphi(0))||g(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(C_\varphi^g f_k)''(z)|$$
$$\leq C_1\sup_{|\lambda|\leq\delta}|f_k'(\lambda)| + C_2\sup_{|\lambda|\leq\delta}|f_k''(\lambda)| + 2C_\epsilon||f||_{\mathrm{Lip}_\alpha} + |f_k'(\varphi(0))||g(0)|.$$

The proof is complete. □

4. ESSENTIAL NORM OF $C_\varphi^g f : \mathrm{Lip}_\alpha \to \mathcal{Z}$

In this section, we give some estimates for the essential norm of operator $C_\varphi^g f : \mathrm{Lip}_\alpha \to \mathcal{Z}$.

Theorem 4.1. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$ such that $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$ is bounded. Then*

$$||C_\varphi^g f||_{e,\mathrm{Lip}_\alpha\to\mathcal{Z}} \approx \max\{A_1, A_2\},$$

where

$$A_j := \limsup_{|a|\to1}||C_\varphi^g\left(\frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j-\alpha}}\right)||_{\mathcal{Z}}, \quad j = 1, 2.$$

Proof. First we prove that $\max\{A_1, A_2\} \leq ||C_\varphi^g||_{e,\mathrm{Lip}_\alpha\to\mathcal{Z}}$. Let $a \in \mathbb{D}$. Define

$$f_{a,j}(z) = \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j-\alpha}}.$$

It is easy to check that $f_{a,j} \in \mathrm{Lip}_\alpha$ for all $a \in \mathbb{D}$ and $f_{a,j}$ converges uniformly to 0 on compact subset of Lip_α as $|a| \to 1$ Thus, for any compact operator $T : \mathrm{Lip}_\alpha \to \mathcal{Z}$, we have $\lim_{|a|\to1}||Tf_{a,j}||_{\mathcal{Z}} = 0, \quad j = 1, 2$. Hence,

$$||C_\varphi^g - T||_{\mathrm{Lip}_\alpha\to\mathcal{Z}} \gtrsim \limsup_{|a|\to1}||C_\varphi^g - Tf_{a,j}||_{\mathcal{Z}} \gtrsim \limsup_{|a|\to1}||C_\varphi^g f_{a,j}||_{\mathcal{Z}} - \limsup_{|a|\to1}||Tf_{a,j}||_{\mathcal{Z}} = A_j.$$

Therefore, based on the definition of the essential norm, we obtain

$$||C_\varphi^g||_{e,\mathrm{Lip}_\alpha\to\mathcal{Z}} = \inf_k ||C_\varphi^g - T||_{\mathrm{Lip}_\alpha\to\mathcal{Z}} \gtrsim A_j, \quad j = 1, 2.$$

Now, we prove that $||C_\varphi^g f||_{e,\mathrm{Lip}_\alpha\to\mathcal{Z}} \lesssim \max\{A_1, A_2\}$. For $r \in [0, 1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(K_r f)(z) = f_r(z) = f(rz)$. It is obvious that $f_r - f \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on $\mathcal{B}$ and $||K_r||_{\mathcal{B}\to\mathcal{B}} \leq 1$(see [10]). By a similar argument it can be proved that the operator K_r is compact on Lip_α and $||K_r||_{\mathrm{Lip}_\alpha\to\mathrm{Lip}_\alpha} \leq 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integer j, the operator $C_\varphi^g K_{r_j} : \mathrm{Lip}_\alpha \to \mathcal{Z}$ is compact. By the definition of the essential norm, we get

$$||C_\varphi^g||_{e,\mathrm{Lip}_\alpha\to\mathcal{Z}} \leq \limsup_{j\to\infty}||C_\varphi^g - C_\varphi^g K_{r_j}||_{\mathrm{Lip}_\alpha\to\mathcal{Z}}.$$

For any $f \in \mathrm{Lip}_\alpha$ such that $||f||_{\mathrm{Lip}_\alpha} \le 1$,

$$
\begin{aligned}
||(C_\varphi^g - C_\varphi^g K_{r_j})f||_{\mathcal{Z}} &\le |(C_\varphi^g f(0)| + |(f - f_{r_j})'(\varphi(0))g(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)||(f-f_{r_j})'(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)\varphi'(z)||(f-f_{r_j})''(\varphi(z))| \\
&\le \underbrace{\limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)|g'(z)||(f-f_{r_j})'(\varphi(z))|}_{M_1} \\
&\quad + \underbrace{\limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|g'(z)||(f-f_{r_j})'(\varphi(z))|}_{M_2} \\
&\quad + \underbrace{\limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)|g(z)\varphi'(z)||(f-f_{r_j})''(\varphi(z))|}_{M_3} \\
&\quad + \underbrace{\limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|g(z)\varphi'(z)||(f-f_{r_j})''(\varphi(z))|}_{M_4},
\end{aligned}
$$

where $N \in \mathbb{N}$ is large enough such that $r_j \ge \frac{1}{2}$ for all $j \in \mathbb{N}$. Since $C_\varphi^g : \mathrm{Lip}_\alpha \to \mathcal{Z}$ is bounded, by (3.3) and (3.4), we have

$$
\widetilde{F_1} = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)| < \infty, \quad \widetilde{F_2} = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)\varphi(z) + g(z)\varphi'(z)| < \infty.
$$

Since $r_j f_{r_j} \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, so

$$
M_1 \le \widetilde{F_1} = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)| = 0, \quad M_3 \le \widetilde{F_2} = \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)\varphi(z) + g(z)\varphi'(z)| = 0.
$$

Next we consider M_2. We have $M_2 \le \limsup_{j\to\infty}(Q_1 + Q_2)$, where

$$
Q_1 = \sup_{|\varphi(z)|>r_N}(1-|z|^2)|(f'(\varphi(z))||g(z)\varphi'(z)|, \quad Q_2 = \sup_{|\varphi(z)|>r_N}(1-|z|^2)r_j|(f'(\varphi(z))||g(z)\varphi'(z)|.
$$

Using the fact that $||f||_{\mathrm{Lip}\alpha} \le 1$ and (1.1), we obtain

$$
\begin{aligned}
Q_1 &= \sup_{|\varphi(z)|>r_N}(1-|z|^2)|(f'(\varphi(z))||g(z)\varphi'(z)|\frac{(1-|\varphi(z)|^2)^{1-\alpha}}{(j-\alpha)\overline{\varphi(z)}}\frac{(j-\alpha)\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{1-\alpha}} \\
&\preceq \frac{(j-\alpha)||f||_{\mathrm{Lip}_\alpha}}{r_N}\sup_{|\varphi(z)|>r_N}(1-|z|^2)|g(z)\varphi'(z)|\frac{(j-\alpha)\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{1-\alpha}} \\
&\preceq \sup_{|\varphi(z)|>r_N}(1-|z|^2)|g(z)\varphi'(z)|\frac{(j-\alpha)\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{1-\alpha}} \preceq \sup_{|a|>r_N}||C_\varphi^g(f_{a,j})||, \quad j = 1, 2.
\end{aligned}
$$

Taking the limit as $N \to \infty$, we obtain

$$
\limsup_{j\to\infty} Q_1 \le \limsup_{|a|\to\infty} ||C_\varphi^g(f_{a,j})||_{\mathcal{Z}}.
$$

Similarly,

$$
\limsup_{j\to\infty} Q_2 \le \limsup_{|a|\to\infty} ||C_\varphi^g(f_{a,j})||_{\mathcal{Z}}.
$$

Hence, we get $M_2 \preceq \max\{A_1, A_2\}$. Similarly, it can be shown that $M_4 \preceq \max\{A_1, A_2\}$. This completes the proof of the theorem. □

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