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REAL AND COMPLEX ANALYSIS

On Convergence of Partial Sums of Franklin Series to $+\infty$

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Abstract—In this paper, we prove that if $\{n_k\}$ is an arbitrary increasing sequence of natural numbers such that the ratio n_{k+1}/n_k is bounded, then the n_k-th partial sum of a series by Franklin system cannot converge to $+\infty$ on a set of positive measure. Also, we prove that if the ratio n_{k+1}/n_k is unbounded, then there exists a series by Franklin system, the n_k-th partial sum of which converges to $+\infty$ almost everywhere on $[0, 1]$.

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1. Introduction

In 1915, N. N. Lusin [1] has posed the following problem: can a trigonometric series converge to $+\infty$ on a set of positive measure? Since then many mathematicians have investigated the question of convergence or summability to $+\infty$ of orthogonal series on a set of positive measure.

Yu. B. Germeier [2] proved that a trigonometric series cannot be summed by the Riemann method to $+\infty$ on a set of positive measure. N. N. Lusin and I. I. Privalov [3] have constructed an example of trigonometric series, which is almost everywhere Abel summable to $+\infty$. D. E. Men'shov [4] proved that for any function f, not necessarily finite almost everywhere, there exists a trigonometric series that converges to f in measure. In particular, there exists a trigonometric series that converges to $+\infty$ in measure on $[-\pi, \pi]$. A.A. Talalyan [5] proved that for any measurable on $[-\pi, \pi]$ function f there exists a trigonometric series that converges to f in measure and almost everywhere on the set where f in finite. Finally, in 1988 S. V. Konyagin [6] solved the above posed Lusin's problem by proving the following theorem.

Theorem 1.1. *Let $\underline{S}(x)$ and $\overline{S}(x)$ be the lower and upper limits of the partial sums of a trigonometric series, respectively. Then*

$$\mu\left(\left\{x \in [-\pi, \pi] : -\infty < \underline{S}(x) \le \overline{S}(x) = +\infty\right\}\right) = 0.$$

In particular, a trigonometric series cannot converge to $+\infty$ on a set of positive measure.

For series by Haar and Walsh systems we mention the following results. A. A. Talalyan and F. G. Arutyunyan [7] proved that the series by Haar and Walsh systems cannot converge to $+\infty$ on a set of positive measure. In papers [8] and [9] can be found more simple proofs of this result. However, there exist uniformly bounded orthonormal systems of functions, the series by which can converge to $+\infty$ on a set of positive measure for any permutation of the terms of the series (see [10]). N. B. Pogosyan [11] proved that for each complete orthonormal system there exists a series, which after an appropriate permutation converges to $+\infty$ almost everywhere. In [12], G. G. Gevorkyan proved that a series by Franklin system cannot converge to $+\infty$ on a set of positive measure.

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Theorem 1.2 (see [12]). *Let $S_n(x)$ be the partial sum of a series by Franklin system. Then*

$$\mu\left(\left\{x \in [0,1] : \lim_{k\to\infty} S_{2^k}(x) = +\infty\right\}\right) = 0.$$

In the present paper we study the possibility of convergence to $+\infty$ of partial sums $S_{n_k}(x)$ of series by Franklin system on a set of positive measure.

2. Necessary Definitions and Statement of Main Results

Let $n = 2^\mu + \nu$, where $\mu = 0, 1, 2, \ldots$ and $1 \le \nu \le 2^\mu$. Define

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, & 0 \le i \le 2\nu, \\ \dfrac{i-\nu}{2^\mu}, & 2\nu < i \le n. \end{cases}$$

Also, we set $s_{n,-1} = s_{n,0} = 0$ and $s_{n,n+1} = s_{n,n} = 1$.

By S_n we denote the space of functions that are continuous and piecewise linear on $[0,1]$ with knots $\{s_{n,i}\}_{i=0}^n$, that is, $f \in S_n$ if $f \in C[0,1]$ and it is linear on each segment $[s_{n,i-1}, s_{n,i}]$, $i = 1, 2, \ldots, n$. It is clear that $dim S_n = n+1$ and the set $\{s_{n,i}\}_{i=0}^n$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Hence there exists a unique (up to the sign) function $f_n \in S_n$, which is orthogonal to S_{n-1} and $\|f_n\|_2 = 1$. Setting $f_0(x) = 1$ and $f_1(x) = \sqrt{3}\,(2x-1)$ for $x \in [0,1]$, we obtain an orthonormal system $\{f_n(x)\}_{n=0}^\infty$, which in equivalent manner was defined by Franklin in [13].

Let $\{n_k\}$ be an arbitrary increasing sequence of natural numbers. By $\sigma_k(x)$, $k = 1, 2, \ldots$, we denote the sums of the first $n_k + 1$ terms of the series:

$$\sum_{n=0}^{\infty} a_n f_n(x), \tag{2.1}$$

that is, $\sigma_k(x) = \sum\limits_{n=0}^{n_k} a_n f_n(x)$. The main results of this paper are the following theorems.

Theorem 2.1. *If* $\sup\limits_{k\in\mathbb{N}} \frac{n_{k+1}}{n_k} < +\infty$, *then*

$$\mu\left(\left\{x \in [0,1] : \lim_{k\to\infty} \sigma_k(x) = +\infty\right\}\right) = 0.$$

Theorem 2.2. *If* $\sup\limits_{k\in\mathbb{N}} \frac{n_{k+1}}{n_k} = +\infty$, *then there exists a series by Franklin system such that* $\lim\limits_{k\to\infty} \sigma_k(x) = +\infty$ *almost everywhere on* $[0,1]$.

Theorem 2.3. *For an increasing sequence* $\{n_k\}$ *the condition*

$$\mu\left(\left\{x \in [0,1] : \lim_{k\to\infty} \sigma_k(x) = +\infty\right\}\right) = 0$$

is satisfied for all series of the form (2.1) *if and only if* $\sup\limits_{k\in\mathbb{N}} \frac{n_{k+1}}{n_k} < +\infty$.

Note that Theorem 2.3 follows from Theorems 2.1 and 2.2.

In the proof of Theorem 1.1 a key role will play the notion of scalar product of the series (2.1) and a function from space S_n, defined in [14], and then successfully applied in the study of uniqueness problems of series by the Franklin system (see also [15]).

By S we formally denote the series (2.1). From the the definition of Franklin system it follows that if $g \in S_m$ and $n > m$, then

$$\int_0^1 f_n(x)g(x)dx = 0.$$

Hence the scalar product of the series S and a function $g \in S_m$ can be defined by formula:

$$(S, g) = \sum_{n=0}^{\infty} a_n \int_0^1 f_n(x)g(x)dx = \sum_{n=0}^{m} a_n \int_0^1 f_n(x)g(x)dx.$$

It is clear that if $g_1 \in S_{m_1}$ and $g_2 \in S_{m_2}$, then for any α, β, we have

$$(S, \alpha g_1 + \beta g_2) = \alpha (S, g_1) + \beta (S, g_2).$$

Let δ_{ij} be the Kronecker symbol, that is, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. For $n \geq 2$ we define

$$N_{n,i}(s_{n,j}) = \delta_{ij}, j = 0, \ldots, n \quad \text{and} \quad N_{n,i}(t) \text{ is linear on } [s_{n,j-1}, s_{n,j}], \ j = 1, \ldots, n, \quad i = 0, 1, ..., n.$$

Observe that the functions $\{N_{n,i}(t)\}_{i=0}^{n}$ are normed in the space $C[0,1]$, and from $N_{n,i}(s_{n,j}) = \delta_{ij}$, it follows that the system $\{N_{n,i}(t)\}_{i=0}^{n}$ forms a basis in S_n. Denoting

$$M_{n,i}(t) = \frac{2}{s_{n,i+1} - s_{n,i-1}} N_{n,i}(t),$$

we obtain another basis in S_n, which is normed in $L[0,1]$.

Taking into account that below we will work only with functions $M_{n,i}$ for $n = n_k$, for simplicity of notation, instead of $M_{n_k,i}$ we will write M_i^k

Also, we denote $\tau_i^k = s_{n_k,i}$ and $\Delta_i^k = \operatorname{supp} M_i^k = \left[\tau_{i-1}^k, \tau_{i+1}^k\right]$. The lemmas that follow can be found in papers [12], [14], [16].

Lemma 2.1. *Let φ be a function that is linear on the segments $\left[\tau_{i-1}^k, \tau_i^k\right]$ and $\left[\tau_i^k, \tau_{i+1}^k\right]$. Then*

$$\left(\varphi, M_i^k\right) = \int_0^1 \varphi(t) M_i^k(t) = \frac{1}{6}\varphi\left(\tau_{i-1}^k\right) + \frac{2}{3}\varphi\left(\tau_i^k\right) + \frac{1}{6}\varphi\left(\tau_{i+1}^k\right),$$

if $\tau_{i+1}^k - \tau_i^k = \tau_i^k - \tau_{i-1}^k$, and

$$\left(\varphi, M_i^k\right) = \int_0^1 \varphi(t) M_i^k(t) = \frac{1}{9}\varphi\left(\tau_{i-1}^k\right) + \frac{2}{3}\varphi\left(\tau_i^k\right) + \frac{2}{9}\varphi\left(\tau_{i+1}^k\right),$$

if $\tau_{i+1}^k - \tau_i^k = 2\left(\tau_i^k - \tau_{i-1}^k\right)$.

Lemma 2.2. *For any $M_{j_0}^{\nu_0}$ and $\nu > \nu_0$ there exist numbers α_j such that $M_{j_0}^{\nu_0} = \sum_j \alpha_j M_j^{\nu}$, and*

$$\sum_j \alpha_j = 1, \ \alpha_j \geq 0 \text{ and } \ \alpha_j = 0 \text{ if } \ \Delta_j^{\nu} \not\subset \Delta_{j_0}^{\nu_0}.$$

Lemma 2.3. *If $(S, M_{n,i}) =: A < 0$, then*

$$\mu\left(\left\{x \in \Delta_{n,i} : \sum_{i=0}^{n} a_i f_i(x) < \frac{A}{2}\right\}\right) > \frac{\mu\left(\Delta_{n,i}\right)}{9}.$$

3. Proof of Theorem 2.1

Observe first that from the condition $\sup\limits_{k\in\mathbb{N}} \frac{n_{k+1}}{n_k} < +\infty$ it follows that there is an absolute constant $C \in (0,1)$ such that for any $l \in \mathbb{N}$, $i = 0, 1, \ldots, n_l$, and $j = 0, 1, \ldots, n_{l-1}$, we have

$$\mu\left(\Delta_i^l\right) > C\mu\left(\Delta_j^{l-1}\right). \tag{3.1}$$

Denote $E = \left\{x \in [0,1] : \lim\limits_{k\to\infty} \sigma_k(x) = +\infty\right\}$, and assume that $\mu(E) > 0$. Then a segment $\Delta_{i_0}^{k_0}$ can be found such that

$$\mu\left(\Delta_{i_0}^{k_0} \cap E\right) > (1 - 0.0001C)\,\mu\left(\Delta_{i_0}^{k_0}\right). \tag{3.2}$$

Therefore, there is a number $L < 0$ such that

$$\mu(E_L) < 0.0001C\mu\left(\Delta_{i_0}^{k_0}\right), \text{ where } E_L = \left\{x \in \Delta_{i_0}^{k_0} : \inf_k \sigma_k(x) < L\right\}. \tag{3.3}$$

For an integrable function g we denote

$$\mathcal{M}_2(g,x) = \sup_{k,i:\Delta_i^k \ni x} \frac{1}{\mu\left(\Delta_i^k\right)} \int_{\Delta_i^k} |g(t)|\,dt.$$

It is clear that $\mathcal{M}_2(g,x)$ does not exceed the Hardy-Littlewood maximal function of g, and hence, we have

$$\mu(D) < 0.01\mu\left(\Delta_{i_0}^{k_0}\right), \text{ where } D = \left\{x \in \Delta_{i_0}^{k_0} : \mathcal{M}_2\left(\chi_{E_l}, x\right) > \frac{C}{15}\right\}. \tag{3.4}$$

By Lemma 2.3 we have

$$\left(\sigma_k, M_i^k\right) > 2L, \text{ if } \Delta_i^k \subset \Delta_{i_0}^{k_0} \text{ and } \Delta_i^k \not\subset D. \tag{3.5}$$

Now we show that by using induction on k, for any $k > k_0$ one can find the following representation:

$$M_{i_0}^{k_0} = \sum_{l=k_0}^{k} \sum_{j\in\Lambda_l} \alpha_j^{(l)} M_j^l + \sum_{j\in B_k} \beta_j^{(k)} M_j^k, \tag{3.6}$$

where

$$\alpha_j^{(l)} \ge 0, \quad \beta_j^{(k)} \ge 0, \quad \sum_{l=k_0}^{k} \sum_{j\in\Lambda_l} \alpha_j^{(l)} + \sum_{j\in B_k} \beta_j^{(k)} = 1, \tag{3.7}$$

and the sets Λ_l and B_k will be specified below.

Indeed, note first that (3.4) implies that $\Delta_{i_0}^{k_0} \not\subset D$. In the case $k = k_0$, denoting $\Lambda_{k_0} = \emptyset$, $B_{k_0} = \{i_0\}$, we get $M_{i_0}^{k_0} = \sum\limits_{j\in B_k} M_j^k$. Now assuming that (3.6) is true for k we prove it for $k+1$. To this end, observe first that by Lemma 2.2, each function $M_i^k, i \in B_k$, can be represented in the form of a linear combination of functions M_i^{k+1} with positive coefficients. Substituting these representations into the second sum in (3.7), we obtain

$$\sum_{j\in B_k} \beta_j^{(k)} M_j^k = \sum_{j\in\Lambda_{k+1}} \alpha_j^{(k+1)} M_j^{k+1} + \sum_{j\in B_{k+1}} \beta_j^{(k+1)} M_j^{k+1}, \tag{3.8}$$

where

$$\Lambda_{k+1} = \left\{j : \Delta_j^{k+1} \subset D \text{ ë } \alpha_j^{k+1} \neq 0\right\}, \tag{3.9}$$

$$B_{k+1}=\left\{j:\Delta_j^{k+1}\not\subset D \text{ ë } \beta_j^{k+1}\neq 0\right\}. \tag{3.10}$$

Instead of the second sum in (3.6), substituting the sums on the right-hand side of (3.8), we obtain (3.6) for $k+1$. The inequalities $\alpha_j^{(l)}\geq 0$ and $\beta_j^{(k)}\geq 0$ in (3.7) follow from the fact that the coefficients in Lemma 2.2 are nonnegative. The equality in (3.7) follows from the fact that the integrals of all functions in (3.6) are equal to 1. Now we prove that

$$\left(S,M_i^k\right)>2L \text{ for } i\in\Lambda_k. \tag{3.11}$$

Assume the opposite, that is, $\left(S,M_i^k\right)\leq 2L$. Then by Lemma 2.3 we have

$$\mu\left(\left\{x\in\Delta_i^k:\sigma_k(x)<L\right\}\right)>\frac{\mu\left(\Delta_i^k\right)}{9}. \tag{3.12}$$

From the definition of Λ_l in representation (3.6) we have that if $i\in\Lambda_k$, then there exists $j\in B_{k-1}$ such that $\Delta_i^k\subset\Delta_j^{k-1}$. By (3.1) and (3.12) we have

$$\mu\left(\left\{x\in\Delta_j^{k-1}:\sigma_k(x)<L\right\}\right)>\frac{\mu\left(\Delta_i^k\right)}{9}>\frac{C\mu\left(\Delta_j^{k-1}\right)}{9}.$$

This implies that (see also (3.3) and (3.4)), $\Delta_j^{k-1}\subset D$ for some $j\in B_{k-1}$, which contradicts (3.10), and so, (3.11) is proved. Thus, for any $k\geq k_0$ we have the representation (3.6), and the relations (3.7) and (3.11). Then for any k we have

$$d=\left(S,M_{i_0}^{k_0}\right)=\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)}\left(S,M_j^l\right)+\sum_{j\in B_k}\beta_j^{(k)}\left(S,M_j^k\right) \tag{3.13}$$

$$=\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)}\left(\sigma_l,M_j^l\right)+\sum_{j\in B_k}\beta_j^{(k)}\left(\sigma_k,M_j^k\right)=:I_1\left(k\right)+I_2\left(k\right).$$

For $I_1\left(k\right)$, in view of (3.11), we have

$$I_1\left(k\right)\geq 2L\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)}. \tag{3.14}$$

For an arbitrary positive $L_0>-100L$ we denote $\Omega_k=\left\{i\in B_k:\sigma_k\left(\tau_i^k\right)>L_0\right\}$. Then for $i\in\Omega_k$ we have $\Delta_i^k\not\subset D$, and hence

$$\mu\left(\left[\tau_{i-1}^k,\tau_i^k\right]\cap E_L\right)<\frac{C}{5}\left(\tau_i^k-\tau_{i-1}^k\right), \tag{3.15}$$

$$\mu\left(\left[\tau_i^k,\tau_{i+1}^k\right]\cap E_L\right)<\frac{C}{5}\left(\tau_{i+1}^k-\tau_i^k\right). \tag{3.16}$$

Denote $\omega_1=\sigma_k\left(\tau_{i-1}^k\right)$, $\omega_2=\sigma_k\left(\tau_i^k\right)$, and $\omega_3=\sigma_k\left(\tau_{i+1}^k\right)$. From (3.15) and linearity of functions σ_k on $\left[\tau_{i-1}^k,\tau_i^k\right]$, it follows that if $\omega_1<L$, then $\frac{\omega_2-\omega_1}{L-\omega_1}>\frac{5}{C}$. Therefore, in any case, we have $\omega_1>\frac{5L}{5-C}-\frac{C}{5-C}\omega_2$. Similarly, from (3.16), we get $\omega_3>\frac{5L}{5-C}-\frac{C}{5-C}\omega_2$. Hence, by Lemma 2.1, we obtain

$$\left(\sigma_k,M_i^k\right)\geq\min\left\{\frac{1}{6}\omega_1+\frac{2}{3}\omega_2+\frac{1}{6}\omega_3,\frac{1}{9}\omega_1+\frac{2}{3}\omega_2+\frac{2}{9}\omega_3\right\}$$

$$\geq\frac{2}{3}\omega_2+\frac{1}{3}\left(\frac{5L}{5-C}-\frac{C}{5-C}\omega_2\right)\geq\frac{7}{12}\omega_2-\frac{0.05L_0}{12}>\frac{L_0}{2},\ \text{for } i\in\Omega_k. \tag{3.17}$$

From (3.13), (3.14), (3.5), (3.10) and (3.17), it follows that

$$d \geq 2L\left(\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)}+\sum_{j\in B_k\setminus\Omega_k}\beta_j^{(k)}\right)+0.5L_0\sum_{j\in\Omega_k}\beta_j^{(k)}. \tag{3.18}$$

Taking into account that the integrals of functions $M_j^l, l \geq k_0, l \in \Lambda_k$ are equal to 1, and (3.6) is satisfied, we obtain

$$\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)}=\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)}\int_{\Delta_j^l}M_j^l(x)dx \leq \int_{F_1}M_{i_0}^{k_0}(x)dx,$$

where $F_1=\bigcup_{l=k_0}^{k}\bigcup_{j\in\Lambda_k}\Delta_j^l$. By the definition of Λ_k we have $F_1\subset D$. Therefore, $\mu(F_1)\leq 0.01\mu\left(\Delta_{i_0}^{k_0}\right)$, and hence

$$\sum_{l=k_0}^{k}\sum_{j\in\Lambda_l}\alpha_j^{(l)} \leq \mu(F_1)\left\|M_{i_0}^{k_0}\right\|_\infty \leq 0.02. \tag{3.19}$$

Now we prove that for large enough k the following inequality holds:

$$\sum_{j\in B_k\setminus\Omega_k}\beta_j^{(k)} \leq 0.9. \tag{3.20}$$

Denote

$$\Gamma_0^k=\left\{i:\Delta_i^k\subset\Delta_{i_0}^{k_0}\right\},\quad m=\operatorname{card}\left(\Gamma_0^k\right)+1, \tag{3.21}$$

and observe that for each $i\in\Gamma_0^k$, we have

$$\frac{1}{2m}\mu\left(\Delta_{i_0}^{k_0}\right)\leq\left|\tau_{i+1}^k-\tau_i^k\right|\leq\frac{2}{m}\mu\left(\Delta_{i_0}^{k_0}\right). \tag{3.22}$$

Denoting $\tilde{\Delta}_j^k=\left[\frac{\tau_{j-1}^k+\tau_j^k}{2},\frac{\tau_j^k+\tau_{j+1}^k}{2}\right]$, we can write

$$\sum_{j\in B_k\setminus\Omega_k}\beta_j^{(k)}=\sum_{j\in B_k\setminus\Omega_k}\beta_j^{(k)}M_j^k\left(\tau_j^k\right)\frac{\tau_{j+1}^k-\tau_{j-1}^k}{2}\leq\sum_{j\in B_k\setminus\Omega_k}M_{i_0}^{k_0}\left(\tau_j^k\right)\frac{\tau_{j+1}^k-\tau_{j-1}^k}{2}$$

$$\leq\sum_{\substack{j\in B_k\setminus\Omega_k\\ j\notin\{i_1,i_2\}}}\int_{\tilde{\Delta}_j^k}M_{i_0}^{k_0}(t)dt+M_{i_0}^{k_0}\left(\tau_{i_0}^{k_0}\right)\frac{\tau_{i_1+1}^k-\tau_{i_1-1}^k}{2}+M_{i_0}^{k_0}\left(\tau_{i_2}^k\right)\frac{\tau_{i_2+1}^k-\tau_{i_2-1}^k}{2}, \tag{3.23}$$

where i_1 and i_2 are chosen to satisfy $\tau_{i_1}^k=\tau_{i_0}^{k_0}$ and $\tau_{i_2+1}^k-\tau_{i_2}^k=2\left(\tau_{i_2}^k-\tau_{i_2-1}^k\right)$ (if there is no such i_2, then instead of the last term in (3.23) should be taken 0). From (3.22) and (3.23) we get

$$\sum_{j\in B_k\setminus\Omega_k}\beta_j^{(k)}\leq\sum_{\substack{j\in B_k\setminus\Omega_k\\ j\notin\{i_1,i_2\}}}\int_{\tilde{\Delta}_j^k}M_{i_0}^{k_0}(t)dt+\frac{4}{m}M_{i_0}^{k_0}\left(\tau_{i_0}^{k_0}\right)\mu\left(\Delta_{i_0}^{k_0}\right). \tag{3.24}$$

It is clear that if $\tau_i^k,\tau_{i+1}^k\in B_k\setminus\Omega_k$, then $\sigma_k(x)<L_0$, for $x\in\left[\tau_i^k,\tau_{i+1}^k\right]$. Hence, taking into account (3.22) and that for large enough k the following estimate holds:

$$\mu\left(\left\{x\in\Delta_{i_0}^{k_0}:\sigma_k(x)>L_0\right\}\right)>0.999\mu\left(\Delta_{i_0}^{k_0}\right),$$

we obtain card $\{\Gamma_1^k\} \le 0.002m$ for large enough k, where $\Gamma_1^k = \{i \in B_k \setminus \Omega_k : i+1 \in B_k \setminus \Omega_k\}$. Therefore, from (3.22), we get

$$\sum_{j\in\Gamma_1^k} \int_{\tilde{\Delta}_j^k} M_{i_0}^{k_0}(t)dt \le \text{card}\left(\Gamma_1^k\right) \cdot \frac{2}{m}\mu\left(\Delta_{i_0}^{k_0}\right) \left\|M_{i_0}^{k_0}\right\|_\infty \le 0.008. \tag{3.25}$$

On the other hand, if $\Gamma_2^k = \{i \in B_k \setminus \Omega_k : i+1 \notin B_k \setminus \Omega_k \text{ and } i \notin \{i_1, i_2\}\}$, then we have

$$1 = \int_{\Delta_{i_0}^{k_0}} M_{i_0}^{k_0}(t)dt \ge \sum_{j\in\Gamma_2^k} \int_{\tilde{\Delta}_j^k} M_{i_0}^{k_0}(t)dt + \sum_{j\in\Gamma_2^k} \int_{\tilde{\Delta}_{j+1}^k} M_{i_0}^{k_0}(t)dt \ge \frac{9}{8} \sum_{j\in\Gamma_2^k} \int_{\tilde{\Delta}_j^k} M_{i_0}^{k_0}(t)dt. \tag{3.26}$$

From (3.24)-(3.26), for large enough k, we obtain $\sum_{j\in B_k\setminus\Omega_k} \beta_j^{(k)} \le 0.9$. Thus, the estimate (3.20) is proved. It follows from (3.20), (3.19) and (3.7) that

$$\sum_{j\in\Omega_k} \beta_j^{(k)} \ge 0.08. \tag{3.27}$$

Finally, from (3.18)-(3.20) and (3.27), for any $L_0 > -100L$ and for large enough k, we obtain $d > 0.01L_0$, that is, the number d is greater than any number. The obtained contradiction completes the proof of the theorem. Theorem 2.1 is proved.

4. Proof of Theorem 2.2

Lemma 4.1. *For any $p, q \in \mathbb{N}$ with $q > p+2$ and any $i = 0, 1, \dots, 2^p - 1$ there exists a function $\varphi_{p,q}^i \in S_{2^q}$ satisfying the following conditions:*

1) supp $\left(\varphi_{p,q}^i\right) = \left[\frac{i}{2^p}, \frac{i+1}{2^p}\right]$,

2) $\mu\left(\left\{x \in [0,1] : \varphi_{p,q}^i(x) = 1\right\}\right) = \frac{1}{2^p} - \frac{1}{2^{q-2}}$,

3) $\int_0^1 f_n(x)\varphi_{p,q}^i(x)dx = 0$, for $n \in \mathbb{N} \setminus \{2^p, 2^p+1, \dots, 2^q\}$.

Proof. Let $p, q \in \mathbb{N}$ and $q > p+2$. Consider the following function defined on $[0,1]$:

$$\varphi_{p,q}^i(x) = \begin{cases} 1, \text{ for } x \in \left[\frac{i}{2^p} + \frac{1}{2^{q-1}}, \frac{i+1}{2^p} - \frac{1}{2^{q-1}}\right], \\ \frac{3}{2} - 2^{q-p-1}, \text{ for } x \in \left\{\frac{i}{2^p} + \frac{1}{2^q}, \frac{i+1}{2^p} - \frac{1}{2^q}\right\}, \\ 0, \text{ for } x \notin \left[\frac{i}{2^p}, \frac{i+1}{2^p}\right], \end{cases}$$

and which is piecewise linear with knots:

$$\left\{0, \frac{i}{2^p}, \frac{i}{2^p} + \frac{1}{2^q}, \frac{i}{2^p} + \frac{1}{2^{q-1}}, \frac{i+1}{2^p} - \frac{1}{2^{q-1}}, \frac{i+1}{2^p} - \frac{1}{2^q}, \frac{i+1}{2^p}, 1\right\}.$$

It is easy to check that for this function the conditions 1) and 2) of the lemma are satisfied. Next, from $\varphi_{p,q}^i \in S_{2^q}$ and the definition of Franklin system it follows that $\int_0^1 f_n(x)\varphi(x)dx = 0$, for $n > 2^q$. If $n < 2^p$, then f_n is linear on the segment $\left[\frac{i}{2^p}, \frac{i+1}{2^p}\right]$. Let

$$f_n(x) = a + b\left(x - \frac{2i+1}{2^{p+1}}\right), \quad x \in \left[\frac{i}{2^p}, \frac{i+1}{2^p}\right].$$

From the equality

$$\varphi_{p,q}^i\left(x + \frac{2i+1}{2^{p+1}}\right) = \varphi_{p,q}^i\left(x - \frac{2i+1}{2^{p+1}}\right), \quad x \in \left[\frac{i}{2^p}, \frac{i+1}{2^p}\right]$$

it follows that

$$\int_0^1 f_n(x)\varphi_{p,q}^i(x)dx = a\int_0^1 \varphi_{p,q}^i(x)dx + b\int_{\frac{i}{2^p}}^{\frac{i+1}{2^p}} \left(x - \frac{2i+1}{2^{p+1}}\right)\varphi_{p,q}^i(x)dx = 0,$$

showing that the condition 3) of the lemma is satisfied. Lemma 4.1 is proved.

Lemma 4.2. *For any $p, q \in \mathbb{N}$ with $q > p + 2$ there exists a function $\psi_{p,q} \in S_{2^q}$ satisfying the following conditions:*

1) $\mu(\{x \in [0,1] : \psi_{p,q}(x) \neq 1\}) = \frac{1}{2^{q-p-2}}$,

2) $\int_0^1 f_n(x)\psi_{p,q}(x)dx = 0$, for $n \in \mathbb{N} \setminus \{2^p, 2^p + 1, \ldots, 2^q\}$.

Proof. Let $p, q \in \mathbb{N}$ and $q > p + 2$. Denote $\psi_{p,q} = \sum_{i=0}^{2^p-1} \varphi_{p,q}^i$, where $\varphi_{p,q}^i$ is as in Lemma 4.1. Observe that the assertion 2) of the lemma immediately follows from Lemma 4.1. Next, we have

$$\mu(\{x \in [0,1] : \psi_{p,q}(x) \neq 1\}) = 1 - 2^p \left(\frac{1}{2^p} - \frac{1}{2^{q-2}}\right) = \frac{1}{2^{q-p-2}},$$

and assertion 1) of the lemma follows. Lemma 4.2 is proved.

Proof of Theorem 2.2. Let $p_k = [\log_2(n_k)] + 1$ and $q_k = [\log_2(n_{k+1})] - 1$. From the conditions of the theorem it follows that $\limsup(q_k - p_k) = +\infty$. Therefore, for each $j \in \mathbb{N}$ there exists a strictly increasing sequence $k_j \in \mathbb{N}$ such that $\frac{1}{2^{q_{k_i} - p_{k_i} - 2}} \leq \frac{1}{i^2}$. From Lemma 4.2 it follows that for any $j \in \mathbb{N}$ there exists $\psi_j \in S_{2^{q_{k_j}}}$ such that:

1) $\mu(\{x \in [0,1] : \psi_j(x) \neq 1\}) \leq \frac{1}{j^2}$,

2) $\int_0^1 f_n(x)\psi_j(x)dx = 0$, for $n \in \mathbb{N} \setminus \left\{2^{p_{k_j}}, 2^{p_{k_j}} + 1, \ldots, 2^{q_{k_j}}\right\}$.

Let $E_j = \{x \in [0,1] : \psi_j(x) = 1\}$, $j \in \mathbb{N}$. Observe that for any $m \in \mathbb{N}$

$$\mu\left(\bigcap_{j=m}^{\infty} E_j\right) = 1 - \mu\left(\bigcup_{j=m}^{\infty} E_j^c\right) \geq 1 - \sum_{j=m}^{\infty} \mu\left(E_j^c\right) \geq 1 - \sum_{j=m}^{\infty} \frac{1}{j^2},$$

and hence

$$\mu\left(\liminf_{j\to\infty} E_j\right) = 1. \tag{4.1}$$

From 2) it follows that $\psi_j = \sum_{n=2^{p_{k_j}}}^{2^{q_{k_j}}} a_n^{(j)} f_n$, and, from (4.1) we get $\sum_{j=1}^{\infty} \sum_{n=2^{p_{k_j}}}^{2^{q_{k_j}}} a_n^{(j)} f_n = +\infty$ almost everywhere on $[0,1]$, which completes the proof of Theorem 4.2.

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