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n-torsion Groups

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Abstract—A group is called an n-torsion group if it has a system of defining relations of the form $r^n = 1$ for some elements r, and for any of its finite order element a the defining relation $a^n = 1$ holds. It is assumed that the group can contain elements of infinite order. In this paper, we show that for every odd $n \geq 665$ for each n-torsion group can be constructed a theory similar to that of constructed in S. I. Adian's well-known monograph on the free Burnside groups. This allows us to explore the n-torsion groups by methods developed in that work. We prove that every n-torsion group can be specified by some independent system of defining relations; the center of any non-cyclic n-torsion group is trivial; the n-periodic product of an arbitrary family of n-torsion groups is an n-torsion group; in any recursively presented n-torsion group the word and conjugacy problems are solvable.

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1. INTRODUCTION

If in a group G the identity $x^n = 1$ is satisfied, then G is said to be a periodic group of exponent n, or an n-periodic group. All similar groups form a variety, called a Burnside variety of exponent n. The free groups of rank m of a variety of exponent n is denoted by $B(m, n)$. One of the most well-known problems in algebra and group theory, posed by W. Burnside in 1902 has a simple statement: *is any finitely generated group $B(m, n)$ finite*? Currently, the free groups $B(m, n)$ are also called free Burnside groups in honor of W. Burnside.

A negative answer to the Burnside problem was first obtained in a series of classical works by S. I. Adian and P. S. Novikov. A few years later, in the monograph [1], S. I. Adian has modified and strengthened the constructed theory and proved his celebrated theorem stating that: *for all odd $n \geq 665$ and $m > 1$ the Burnside groups $B(m, n)$ are infinite*. In addition to the study of groups $B(m, n)$, in the monograph [1] were constructed and studied a number of other groups possessing new unusual properties. The structures and ideas of construction of these groups became a starting point for solution of a series of well-known old and difficult problems of the group theory. Here we mention two important classes of groups from [1].

The first important class of the groups, constructed in [1] by means of generating and defining relations, is denoted by $B(m, n, \alpha)$, where m is the number of the generators of the group, $n \geq 665$ is an arbitrary odd number, and α is a natural parameter. In [1] it was proved that the free Burnside group $B(m, n)$ is the direct limit in α of the group $B(m, n, \alpha)$.

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The next important class of groups is connected with the known finite bases problem of group theory, which was posed by B. Neumann in 1937. In [1] it was proved that for any odd $n \geq 1003$ the following family of identities in two variables $\{(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1\}$, where the parameter p runs over the set of all prime numbers, is irreducible, that is none of these identities follows from the others. From this it follows that for any odd $n \geq 1003$ there exists continuum of distinct varieties $\mathcal{A}_n(\Pi)$ corresponding to distinct sets of prime numbers Π. Besides, for any fixed $m > 1$ there exists *continuum* of nonisomorphic groups $\Gamma(m, n, \Pi)$, where $\Gamma(m, n, \Pi)$ is a relatively free group of rank m of the variety $\mathcal{A}_n(\Pi)$.

Observe that for fixed m and n, there is a natural homomorphism from each of the groups $\Gamma(m, n, \Pi)$ and $B(m, n, \alpha)$ onto $B(m, n)$. Moreover, the groups $\Gamma(m, n, \Pi)$ and $B(m, n, \alpha)$ possess the following two properties:
1. each of these (m-generated) groups has a system of defining relations of the form $A^n = 1$ for some elements A;
2. in each of these groups, for every element Y of finite order, the defining relation $Y^n = 1$ is satisfied.

Let X be an arbitrary group alphabet, $\mathcal{R}$ be some set of words, written in this alphabet, $n > 1$ be a fixed natural number, and

$$G = \langle X | R^n = 1, R \in \mathcal{R} \rangle \quad (1.1)$$

be the presentation of some group G.

Definition 1.1. *The group* (2.2) *is said to be an n-torsion group if for every element $Y \in G$, either $Y^n = 1$, or Y is of infinite order.*

A cyclic group of order n and an infinite cyclic group are simple examples of n-torsion groups. Also, it is clear that free groups of arbitrary rank n are n-torsion groups for any natural n. In [2], A. Karrass, W. Magnus and D. Solitar proved that in any group $G = \langle X | A^n = 1 \rangle$, where A is a simple word (that is, a word which is not a proper power of another word), the element A has order n in G, and each element of finite order in G conjugate to some power of the element A. This means that any group with a single defining relation of the form $A^n = 1$ is an n-torsion group. From the result of work [3] by S. I. Adian (see also [4]) it follows that in any finitely presented n-torsion group the algorithmic word and conjugacy problems are solvable for all odd $n \geq 665$. Besides the absolute free groups and the above mentioned groups $B(m, n)$, $B(m, n, \alpha)$, $\Gamma(m, n, \Pi)$, $m \geq 1$, studied in [1], [3]-[5], in the work [6], S. I. Adian has investigated free groups of a variety, satisfying the identity $[x, y]^n = 1$. Based on the result of [6], it is not difficult to deduce that these free groups also are n-torsion groups. Some other groups that essentially are n-torsion groups have been constructed and studied in the works by A .Yu. Olshanskii, S. V. Ivanov, I. G. Lysenok, V. S. Atabekyan and others (see [7]–[12]).
Denote by $B(X, n)$ the free Burnside group of period n with the same as in G system of generators X:

$$B(X, n) = \langle X \mid A^n = 1 \text{ for all words } A \text{ in the group alphabet } X \rangle.$$

Observe that the identical on the set of generators X mapping from G into $B(X, n)$ can be extended to surjective homomorphism, because by the definition of n-torsion group (2.2) all its defining relations have the form $R^n = 1$, where $R \in \mathcal{R}$. Hence, the following proposition holds.

Proposition 1.1. *Every n-torsion group $G = \langle X | R^n = 1, R \in \mathcal{R} \rangle$ is homomorphically mapped onto $B(X, n)$. In particular, every noncyclic n-torsion group is infinite, provided that n has an odd divisor $k \geq 665$ or a divisor of the form $k = 16m \geq 8000$.*

The second part of Proposition 1.1 follows from the above stated theorem by S. I. Adian and a theorem by I. G. Lysenok from [9] (see also [8]).

We show that for odd $n \geq 665$, for each n-torsion group can be constructed a theory similar to that of constructed in monograph [1], which will allow us to investigate n-torsion groups by methods developed in [1], and to study their key properties. Some of these properties we will prove in the present paper. For exponents of the form $n = 16m \geq 8000$ a similar theory can be constructed basing on the results of [8] and [9].

We first introduce one more broad class of n-torsion groups, which is obtained by means of the following theorem.

Theorem 1.1. *The n-periodic product of any family of n-torsion groups is also an n-torsion group for any odd $n \geq 665$.*

Recall that the notion of n-periodic product of groups was introduced by S. I. Adian [13] in 1976. It was proved that periodic products are associative, exact, hereditary by subgroups and also possess important properties such as the Hopf property, the C^*-simplicity, the uniform non-amenability, the SQ-universality, etc. (see [13]-[18]).

In what follows, we will assume that $n \geq 665$ is an arbitrary fixed odd number.

Theorem 1.2. *Every n-torsion group can be presented by some independent system of defining relations $\{A^n = 1 : A \in \mathcal{A}\}$, where $\mathcal{A}$ is some set of words in the alphabet X. Besides, each element of a finite order from G will be conjugate to some power of an element $A \in \mathcal{A}$.*

It is well-known that the centralizer of any nontrivial element of an absolutely free group is cyclic. According to Adian's theorem, the centers of the groups $B(m, n)$ and $B(m, n, \alpha)$ are trivial for all odd $n \geq 665$. Moreover, the centralizer of any nontrivial element of these groups is also cyclic (see [1], Chapter 6, Theorems 3.1 and 3.2). In 2015, the authors have proved that the centralizers of any nontrivial elements of a relatively free groups $\Gamma(m, n, \Pi)$ also are cyclic for all odd $n \geq 1003$ (see [19]). Based on the independent system of defining relations in Theorem 1.2, we show that these properties possess any n-torsion group.

Theorem 1.3. *The centralizer of any nontrivial element of every n-torsion group is a cyclic group.*

Corollary 1.1. *The center of every noncyclic n-torsion group is trivial.*

Corollary 1.2. *Every Abelian subgroup of each n-torsion group is a cyclic group.*

The next result is true for all groups, which have only cyclic centralizers for nontrivial elements (see [20], Lemma 2.3).

Corollary 1.3. *Each nontrivial normal subgroup of every noncyclic n-torsion group is infinite.*

The next result concerns algorithmic problems.

Theorem 1.4. *If the presentation* (2.2) *of an n-torsion group G is recursive, then in G the word and conjugacy problems are solvable.*

Corollary 1.4. *In every finitely defined n-torsion group the word and conjugacy problems are solvable.*

In [21] were considered relatively free groups K, having only cyclic centralizers for nontrivial elements, and it was proved that every automorphism of the semigroup of endomorphisms $End(K)$ of the group K from the mentioned class is uniquely determined by its action on the subgroup of inner automorphisms $Inn(K)$. Putting together this result and Theorem 1.3, we obtain the following result.

Corollary 1.5. *If an automorphism of the semigroup $End(K)$ of a relatively free n-torsion group K acts identically on the subgroup of inner automorphisms $Inn(K)$, then it acts identically on the whole semigroup $End(K)$.*

Corollary 1.6. *For any relatively free n-torsion group K, the group of automorphisms $Aut(End(K))$ is canonically embedded into the group $Aut(Aut(K))$*

Corollaries 1.5 and 1.6 generalize Theorem 1.2 and Corollary 4 from [19], respectively, proved for the groups $\Gamma(m, n, \Pi)$ for odd $n \geq 1003$.

Theorem 1.1 we prove in Section 2. In Section 3, for each n-torsion group, we construct a special independent system of defining relations, which will be used to prove Theorems 1.2 and 1.3 in Sections 4 and 5, respectively.

In what follows, we will frequently cite the monograph [1]. To cite the results from [1] we will use standard notation, for instance, the notation II.5.3 [1] means item 3, section 5, chapter II of [1].

2. Proof of Theorem 1.1

We prove that the *n-periodic product of every family of n-torsion groups is an n-torsion group.*

Recall that the periodic product of exponent n (or the n-periodic product) of a family of groups $\{G_i\}_{i\in I}$ is defined to be the group that is obtained by adding to the relations of free product $F = \prod_{i\in I}{}^* G_i$ all the defining relations of the form $A^n = 1$, where A is an elementary period of some rank α for a specified classification of periodic words of the free product F (see the definition in item 8 of [13]).

By Theorem 2 of [13] the equality $X^n = 1$ in the group $\prod_{i\in I}{}^{\mathrm{n}} G_i$ is satisfied for any word X from the free product F, which in the group $G = \prod_{i\in I}{}^{\mathrm{n}} G_i$ is not conjugate to any element of subgroups G_i, $i \in I$. Besides, by Theorem 3 of [13], each factor G_i, $i \in I$ is identically embedded into the n-periodic product G (see also [16] and [18]).

Assume that all the factors G_i of a given n-periodic product $G = \prod_{i\in I}{}^{\mathrm{n}} G_i$ are n-torsion groups. Then each of the groups G_i will have a presentation of the form $G_i = \langle a_{ik}, k \in S_i |\, g_{il}^n, l \in J_i\rangle$ for some sets of indices S_i, J_i, $i \in I$, and hence, the free product F will have the presentation:

$$\prod_{i\in I}{}^* G_i = \langle a_{ik}, k \in S_i, i \in I |\, g_{il}^n, l \in J_i, i \in I\rangle\,.$$

As it was mentioned above, the n-periodic product of a family of groups $\{G_i\}_{i\in I}$ is obtained by adding to the relations of free product $\prod_{i\in I}{}^* G_i$ all the defining relations of the form $A^n = 1$, where each element A is an elementary period of some rank α for a specified classification of periodic words in the group alphabet $\{a_{ik}\}_{k\in S_i, i\in I}$ of the free product $\prod_{i\in I}{}^* G_i$. Thus, the n-periodic product G has a presentation, in which all the defining relations have the form $r^n = 1$ (where $r = g_{il}, l \in J_i, i \in I$ or $r = A$, and A is an elementary period).

Further, by Theorem 2 of [13], the equality $X^n = 1$ in the group $\prod_{i\in I}{}^{\mathrm{n}} G_i$ is satisfied for each word X from the free product $\prod_{i\in I}{}^* G_i$, which in the group $\prod_{i\in I}{}^{\mathrm{n}} G_i$ is not conjugate to any element of subgroups G_i, $i \in I$. And, if X in the group $\prod_{i\in I}{}^{\mathrm{n}} G_i$ is conjugate to some element of one of the subgroups G_i, $i \in I$, then either X has infinite order, or $X^n = 1$, because, by assumption, each group G_i, $i \in I$, is an n-torsion group, and in addition, by Theorem 3 of [13], each factor G_i, $i \in I$ is identically embedded into the n-periodic product G. Theorem 1.1 is proved.

3. Independent systems of defining relations for n-torsion groups

Let the group G with presentation (2.2) be an arbitrary n-torsion group. For each such group, by induction on natural parameter α, we construct some presentation by means of generators of X and a new system of defining relations $\{A^n = 1; A \in \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha\}$. We use the notation and the system of references adopted in [1]. The presented in sections I.4 and VII.2 of the monograph [1] system of definitions by complicated simultaneous induction on natural parameter, called rank, was based on the notion of (signed) elementary words of the form A^n, where n is a fixed odd number ≥ 1003. To prove Theorem 1.2, we construct a similar system of notions, using a given set of words $\mathcal{R}$.

We first can assume that all the words $R \in \mathcal{R}$ in the presentation (2.2) are cyclically uncancellable and simple, that is, any word $R \in \mathcal{R}$ is not a proper power of some other word. Indeed, if, for instance, $R_1^k = R \in \mathcal{R}$, then R_1 in G will have a finite order, and since G is an n-torsion group, then $R_1^n = 1$ in G. Then, the relation $R^n = 1$ can be replaced by the relation $R_1^n = 1$.

For rank 0, all the notions remain the same. In particular, all the uncancellable words are called reduced at the rank 0, and any cyclically uncancellable word is a period of rank 1.

All the words $R \in \mathcal{R}$ are *minimal* periods of rank 1, by their cyclically uncancellability and simplicity (see Definition I.4.9 [1]). Among all reduced at the rank 0 words (the set of which is denoted by $\mathcal{R}_0$) we can extract all *elementary periods* of rank 1, according to Definition I.4.10 from [1]. An elementary period E of rank 1 is called *signed* (at the rank 0), if some cyclic shift of the word E or of its inverse belongs to the set $\mathcal{R}$. Otherwise, an elementary period E of rank 1 is called *unsigned*. Next, we introduce reversals of rank 1 for all periodic words, the periods of which are signed at rank 0 elementary periods of rank 1. These reversals have usual form:

$$PA^tA_1Q \to P(A^{-1})^{n-t-1}A_2^{-1}Q, \tag{3.1}$$

where either A or A^{-1} is a signed elementary period of rank 1 or some its cyclic shift, $A \equiv A_1A_2$, and the words A^tA_1 and $(A^{-1})^{n-t-1}A_2^{-1}$ contain at least $p = 9$ sections, that is, are p-powers.

In a natural manner we define the *real reversals* of rank 1. Based on real reversals we define the notion of a *kernel* of rank 1 for the words $W \in \mathcal{N}_1$, where the set of words $\mathcal{N}_1$ is defined according to I.4.21 from [1]. Further, according to I.4.26 from [1] we define the sets $\mathcal{R}_1, \mathcal{K}_1, \mathcal{L}_1, \mathcal{M}_1$, the equivalence relation of rank 1, denoted by $\overset{1}{\sim}$, as well as all other notions of rank 1. The proofs of all necessary properties introduced in I.4 of [1] of notions of rank 1 remain the same. New is only the restriction of the class of elementary words by signed elementary words of period 1. Finally, for any words $B, C \in \mathcal{R}_1$ we define the binary *operation* $[B, C]_1$ of *coupling* of rank 1 similar to Definition I.4.36 [1]:

$$[B, C]_1 = PQ \leftrightarrow \exists T(B \overset{1}{\sim} PT \ \& \ C \overset{1}{\sim} T^{-1}Q \ \& \ PQ \in \mathcal{R}_1).$$

Further, using simultaneous induction on rank α, all the introduced notions can be defined for all natural ranks. Let the *signed* periods of rank α and the analogs of all notions, that were defined in I.4 of [1], be already defined for all ranks $\leq \alpha$. Then we define them at rank $\alpha + 1$.

If $W \in \mathcal{R}_0$ and $W \equiv x_{i_1}x_{i_2}\cdots x_{i_k}$, where $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ belong to the set of generators X (the symbol $\equiv$ stands for graphic equality), then by $[W]_\alpha$ we denote the result of the following sequence of couplings of rank α:

$$[[\cdots[[x_{i_1}, x_{i_2}]_\alpha, \cdots]_\alpha, x_{i_k}]_\alpha.$$

Thus, we have

$$[W]_\alpha \equiv [[\cdots[[x_{i_1}, x_{i_2}]_\alpha, \cdots]_\alpha, x_{i_k}]_\alpha \in \mathcal{R}_\alpha. \tag{3.2}$$

An elementary period A of rank $\alpha + 1$ is called a *signed elementary period* (at the rank α), if words B and $R \in \mathcal{R}$ can be found to satisfy

$$[R]_\alpha \overset{\alpha}{\sim} [BA^jB^{-1}]_\alpha \tag{3.3}$$

for some integer j. Otherwise, an elementary period of rank $\alpha + 1$ is called a *unsigned elementary period* of rank $\alpha + 1$. It is easy to check that for $\alpha = 1$ the definition of signed elementary period of rank α coincides with the above given definition of signed elementary period of rank 1, because if $BE^jB^{-1} \overset{0}{\sim} R \in \mathcal{R}$, then $BE^jB^{-1} = R$ in the free group. Then we have $|j| = 1$, since the word R is simple. Therefore, one of the words E or E^{-1} is a cyclic shift of the word R, by cyclically uncancellability of elements from $\mathcal{R}$ and elementary period E.

Using the definition of signed elementary period, by analogy with VII.2 of [1], we make some changes in the definitions of some notions given in I.4 of [1]. Namely, in all places where the phrase "normalized occurrence of elementary words of rank α" appears, it should be understood as "marked elementary periods of rank α". All the remaining definitions formally remain without any change. All the assertions of the chapters II-V of [1], as well as all the assertions from items 2.3, 2.4 and 2.7-2.10 of chapter VII [1], and Lemma 2.6 of [22] (which are true not only for odd $n \geq 1003$, but also for odd $n \geq 665$) and their proofs formally remain the same and are true within the above given correction on the notion of marked elementary period. Notice, in particular, that according to Lemma V.1.8 [1] the binary operation of coupling of rank α is an associative operation for any $\alpha \geq 0$.

Based on the above introduced notions we construct a new presentation for the group G. Let $\Gamma_G(X, 0)$ be a free group with generators X. We first construct auxiliary groups $\Gamma_G(X, \alpha)$, using induction on the rank α (by analogy with Definition VI.2.2 of groups $B(m, n, \alpha)$ from [1]).

Assume that $\alpha > 0$ and that all the groups $\Gamma_G(X, \gamma)$ are already constructed for all $\gamma \leq \alpha - 1$. By $\mathcal{E}_\alpha$ we denote the set consisting of those marked elementary periods A of rank α, for which the following conditions are satisfied:

(a) for every marked elementary period E of rank α there is one and only one word $A \in \mathcal{E}_\alpha$ such that in the group $\Gamma_G(X, \alpha - 1)$ the period E is conjugate with the period A, or with the period A^{-1}.

(b) if $A \in \mathcal{E}_\alpha$, then for some words P and Q the inclusion $PA^nQ \in \overline{\mathcal{M}}_{\alpha-1}$ holds.

Remark. *The existence of elementary periods A with the above properties* (a) *and* (b) *is proved in Lemma* 3.5 *below.*

Denote by $\Gamma_G(X, \alpha)$ the group with the same generators X and the system of defining relations $A^n = 1$, where $A \in \bigcup_{\alpha=1}^{n} \mathcal{E}_\alpha$:

$$\Gamma_G(X, \alpha) = \langle X \mid A^n = 1, \; A \in \bigcup_{\alpha=1}^{n} \mathcal{E}_\alpha \rangle.$$

Also, denote

$$\mathcal{E} = \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha. \tag{3.4}$$

By the definition, the group $\Gamma_G(X)$ is generated by generators X and has the system of defining relations $A^n = 1$, where $A \in \mathcal{E}$:

$$\Gamma_G(X) = \langle X \mid A^n = 1, \; A \in \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha \rangle. \tag{3.5}$$

Proposition 3.1. *The groups G and $\Gamma_G(X)$ coincide:*

$$G = \Gamma_G(X) = \langle X \mid A^n = 1, \; A \in \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha \rangle. \tag{3.6}$$

To prove Proposition 3.1 we need some lemmas. The proof of the next lemma is similar to that of Lemma VI.2.8 of [1], with the only difference that the group $B(m, n, \alpha)$ should be replaced by the group $\Gamma_G(X, \alpha)$.

Lemma 3.1. *For any two words $C, D \in \mathcal{R}_\alpha$ $(\alpha \geq 0)$ the following relation holds:*

$$C \overset{\alpha}{\sim} D \quad \Leftrightarrow C = D \text{ in } \; \Gamma_G(X, \alpha).$$

Lemma 3.2. *For any rank α and any word C in $\Gamma_G(X, \alpha)$ we have $C = [C]_\alpha$.*

Proof. We use induction on the length $\partial(C)$ of the word C. For $\partial(C) = 0$ the assertion is obvious. Let $C \equiv C_1 x_k$, where $x_k \in X$. By the definition we have $[C]_\alpha = [[C_1]_\alpha, x_k]_\alpha$. By induction assumption we have $[C_1]_\alpha = C_1$ in $\Gamma_G(X, \alpha)$. Next, by Lemma V.1.4 of [1], a word $C_2 \in \mathcal{K}_\alpha$ can be found such that $[C_1]_\alpha \overset{\alpha}{\sim} C_2$ and $[[C_1]_\alpha, x_k]_\alpha = [C_2, x_k]_0$. Then by Lemma 2.10 we have $[C_1]_\alpha = C_2$ in $\Gamma_G(X, \alpha)$, and hence in $\Gamma_G(X, \alpha)$ we have

$$C = C_1 x_k = [C_1]_\alpha x_k = C_2 x_k = [[C_1]_\alpha, x_k]_\alpha = [C]_\alpha.$$

Lemma 3.3. *For any rank α and any word C, a word $D \in \mathcal{K}_\alpha$ can be found to satisfy $C = D$ in $\Gamma_G(X, \alpha)$. If $\alpha \geq \partial(C)$, then such D can be found in $\mathcal{A}_{\alpha+1}$.*

Proof. The proof is similar to that of Lemma VI.2.4 of [1], by using induction on the length $\partial(C)$ of the word C.

Lemma 3.4. *A period E is a marked elementary period of rank α if and only if words $R \in \mathcal{R}$ and B can be found, such that $R = BA^jB^{-1}$ in the group $\Gamma_G(X, \alpha - 1)$ for some integer j.*

Proof. The result follows from the definition of marked elementary period, the equivalence (3.3), and Lemmas 2.10 and 3.2.

Lemma 3.5. *Each marked elementary period E of rank $\alpha \geq 1$ is conjugate in the group $\Gamma_G(X, \alpha - 1)$ to some elementary period A of rank α such that for some words P and Q the inclusion $PA^nQ \in \overline{\mathcal{M}}_{\alpha-1}$ holds. In addition, if E is a marked (unmarked) elementary period of rank α, then A is a marked (unmarked) elementary period of rank α as well.*

Proof. Let E be an arbitrary elementary period of rank $\alpha \geq 1$ and $P * F * Q$ be a normal generating occurrence into some word $Y \in$ Integ $(X, \alpha, E))$. By Lemma IV.3.12 of [1] one can find a word $Z \in \overline{\mathcal{M}}_{\alpha-1}$ such that $Z \overset{\alpha-1}{\sim} Y$. According to II.2.16 [1], to within a cyclic shift of the period E, we can assume that the occurrence $P * F * Q$ has the form $P_1 * E^k E_1 * Q_1$, where E_1 is the beginning of E. By II.4.1 [1], the occurrence $f_{\alpha-1}(P * F * Q; Y, Z)$ has the form $P_2 * A^k A_1 * Q_2$, where A is the image of the period E in $f_{\alpha-1}(P * F * Q; Y, Z)$. Then, by II.4.3 [1] the word A is a period of rank α, and a generating occurrence $P' * F * Q'$ into some word $Y' \in$ Per (α, E) can be found such that $\rho_{\alpha,E}(X) = \rho_{\alpha,A}(Y')$ and MutNorm $_{\alpha-1}(P * F * Q, P' * F * Q')$.

By Lemma 2.6 of [22] (which is a strengthening of Lemma II.7.15 [1]), the periods E and A are conjugate in $\Gamma_G(X, \alpha - 1)$. In view of $Z \in \overline{\mathcal{M}}_{\alpha-1}$ and II.7.10 [1] it follows that the period A^n occurs in some word from $\overline{\mathcal{M}}_{\alpha-1}$. This proves the first assertion of the lemma.

Let $E = TAT^{-1}$ in $\Gamma_G(X, \alpha - 1)$ and E be a marked elementary period of rank α. By Lemma 3.4, can be found words $R \in \mathcal{R}$ and B, such that $R = BE^jB^{-1}$ in the group $\Gamma_G(X, \alpha - 1)$ for some integer j. Then $R = (BT)A^j(BT)^{-1}$ in the group $\Gamma_G(X, \alpha - 1)$, and A is marked by Lemma 3.4.

Lemma 3.6. *If E is a marked elementary period of some rank $\gamma \geq 1$ (or if $E \in \mathcal{E}_\gamma$), then E has order n in the group $\Gamma_G(X, \gamma)$ (and in the group $\Gamma_G(X)$).*

Proof. By definition, every marked elementary period E of some rank $\gamma \geq 1$ is conjugate in the group $\Gamma_G(X, \gamma - 1)$ to some elementary period $A \in \mathcal{E}_\gamma$ or to its inverse. Since $A^n = 1$ is one of the defining relations of the group $\Gamma_G(X, \gamma)$, then $E^n = 1$ in $\Gamma_G(X, \gamma)$ as well.

The fact that the order of the elementary period $A \in \mathcal{E}_\gamma$ is not less than n in the group $\Gamma_G(X)$, follows in standard manner from Lemma IV.2.16 [1].

Lemma 3.7. *If E is an unmarked elementary period of some rank γ, then E has an infinite order in the group $\Gamma_G(X)$.*

Proof. By Lemma 3.5, an elementary period E of rank $\gamma \geq 1$ is conjugate in the group $\Gamma_G(X, \alpha - 1)$ to some elementary period A of rank γ, such that for some words P and Q the inclusion $PA^nQ \in \overline{\mathcal{M}}_{\gamma-1}$ holds. By Lemma IV.2.1 of [1] we have the inclusion $A^q \in \mathcal{K}_{\gamma-1}$. If E is an unmarked elementary period of rank γ, then by Lemma VII.2.9 of [1] for every $i > 10$ and $\alpha \geq \gamma$ we have $A^i \in \mathcal{R}_\alpha$. Then, by Lemma IV.2.16 of [1] and Lemma 2.10 we obtain that the period A has an infinite order in the group $\Gamma_G(X, \alpha)$ for every $\alpha \geq \gamma$.

Lemma 3.8. *For every word C, which is not equal to 1 in the group $\Gamma_G(X)$, can be found words T and E, such that $C = TE^rT^{-1}$ in $\Gamma_G(X)$ for some integer r, where either $E \in \mathcal{E}$, or E is an unmarked elementary period of some rank γ, and the word E^q occurs in some word from the class $\overline{\mathcal{M}}_{\gamma-1}$.*

Proof. The proof is similar to that of Lemma VI.2.5 of [1], the only difference is that instead of the group $B(m, n)$ should be considered the group $\Gamma_G(X)$, and instead of Lemmas 1.2, 2.3 and 2.4 of chapter VI of [1] should be applied Lemma VII.2.7 of [1] and Lemmas 2.10 and 3.3, respectively.

Lemma 3.9. *The group $\Gamma_G(X)$ is an n-torsion group.*

Proof. The result is an immediate consequence of Lemmas 3.8, 3.6 and 3.7.

Now we are in position to prove Proposition 1.1. We prove that in the groups G and $\Gamma_G(X)$ are satisfied the same defining relations.

Let $R \in \mathcal{R}$. According to Lemma 3.8, if the word R is not equal to 1 in the group $\Gamma_G(X)$, then some words T and E can be found to satisfy $C = TE^rT^{-1}$ in $\Gamma_G(X)$ for some integer r, where either $E \in \mathcal{E}$, or E is a unmarked elementary period of some rank γ, and the word E^q occurs in some word from the class $\overline{\mathcal{M}}_{\gamma-1}$. The equality $R = TE^rT^{-1}$ in $\Gamma_G(X)$ implies the equality $T^{-1}RT = E^r$ in $\Gamma_G(X, \alpha)$ for some $\alpha \geq \gamma$. According to Lemma 3.3, we can assume that $T^{-1}RT, E^r \in \mathcal{A} = \bigcup_{\beta=1}^{\infty} \mathcal{A}_\beta = \bigcap_{\beta=1}^{\infty} \mathcal{R}_\beta$. Then, by Lemma 2.10, we get $T^{-1}RT \overset{\alpha}{\sim} E^r$.

The period E is an elementary period of rank $\gamma \leq \alpha$. Therefore in E^r do not occur active kernels of ranks $\geq \gamma$. Hence, by Lemma IV.2.13 of [1] and the equivalence $T^{-1}RT \overset{\alpha}{\sim} E^r$ it follows that $T^{-1}RT \overset{\gamma-1}{\sim} E^r$. By Lemma 2.10 we obtain $T^{-1}RT = E^r$ in $\Gamma_G(X, \gamma - 1)$. Therefore, by Lemma 3.4, the period E is a marked elementary period of rank γ. By Lemma 3.6 in the group $\Gamma_G(X)$ the relation $E^n = 1$ holds, and hence, by the equality $R = TE^rT^{-1}$, in the group $\Gamma_G(X)$ the relation $R^n = 1$ is satisfied as well.

Now we prove the inverse assertion. We use induction on rank $\alpha \geq 1$ to show that if $A^n = 1$ is an arbitrary defining relation of the group $\Gamma_G(X)$, then the relation $A^n = 1$ is also satisfied in the group G. Since $A \in \mathcal{E}_\alpha$, then A is a marked elementary period of rank α. According to Lemma 3.4, a word $R \in \mathcal{R}$ can be found such that $R = BA^jB^{-1}$ in the group $\Gamma_G(X, \alpha - 1)$ for some word B and an integer j.

For $\alpha = 1$ we have $R = BA^{\pm 1}B^{-1}$ in the free group $\Gamma_G(X, 0)$ due to the simplicity of the word R. Since $R^n = 1$ in G, we get $A^n = 1$ in G as well.

Let $\alpha \geq 2$ and the assertion be true for all defining relations of the form $A_1^n = 1$ with elementary periods $A_1 \in \mathcal{E}$ of rank $\leq \alpha - 1$. Since in $\Gamma_G(X, \alpha - 1)$ the equality $R = BA^jB^{-1}$ holds, then by the induction assumption it holds also in the group G. By the relation $R^n = 1$ in G we have $A^{jn} = 1$. Therefore the period A has a finite order in G. Since G is an n-torsion group, then $A^n = 1$ in G. Proposition 3.1 is proved.

Proposition 3.2. *The system of defining relations* $\{A^n = 1,\ A \in \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha\}$ *of the group* G *in* (3.6) *is an independent system of relations, that is, any of these relations does not follow from the others.*

Proof. To prove that the system of defining relations of the n-torsion group G from the presentation (3.6) is an independent system, we use the proof of independence constructed in VI.2.2. of [1] of the system of defining relations of the group $B(m, n)$. This proof is given in [23]. So, to prove the proposition, we can repeat the arguments of the proof from [23] with the only difference that the group $B(m, n)$ should be replaced by the group $\Gamma_G(X)$, and the set $\mathcal{E} = \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha$ in VI.2.1 [1] should be replaced by the set $\mathcal{E}$ defined by the equality (3.4).

4. Proof of Theorems

Proof of Theorem 1.2. We prove that every n-torsion group can be specified by means of some independent system of defining relations $\{A^n = 1 : A \in \mathcal{A}\}$, such that each element of a finite order from G will be conjugate to some power of certain element $A \in \mathcal{A}$.

By Proposition 3.1, every n-torsion group has a presentation of the form (3.6). According to Proposition 3.2, the system of defining relations $\{A^n = 1,\ A \in \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha\}$ of the group G from (3.6) is an independent system of defining relations. By Lemma 3.8, for each nontrivial element C from G, some words T and E can be found such that $C = TE^rT^{-1}$ in G for some integer r, where either $E \in \mathcal{E}$, or E is an unmarked elementary period of some rank γ, and the word E^q occurs in some word from the class $\overline{\mathcal{M}}_{\gamma-1}$. In addition, by Lemmas 3.6 and 3.7, if C has a finite order, then $E \in \mathcal{E} = \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha$. Choosing the set $\mathcal{E}$ to be the mentioned in the theorem set of words $\mathcal{A}$, we complete the proof of Theorem 1.2.

Proof of Theorem 1.3. To prove Theorem 1.3, it is enough to repeat the arguments of the proof of Theorem 1 of [19], in which it is stated that the centralizer of every non-identity element of a relatively free group $\Gamma = \Gamma(m, n, \Pi)$ is a cyclic group. It should only be applied the corresponding lemmas from §3 of this paper instead of lemmas from §2 of [19], and the groups $\Gamma(m, n, \Pi)$ and $\Gamma(m, n, \Pi, \alpha)$ should be replaced by the groups $\Gamma_G(X)$ and $\Gamma_G(X, \alpha)$, respectively. Theorem 1.3 is proved.

Proof of Theorem 1.4 Observe first that the solvability of the problem of recognition of equality of words in all the constructed intermediate groups $\Gamma_G(X, \alpha)$ and in $G = \Gamma_G(X)$ naturally follows from the recursivity of the sets of relations in the presentation (3.6) and the algorithmic efficiency (see the efficiency principle in I.5.4 [1]) of all the definitions similar to that of as it was obtained in [1] for the groups $B(X, n, \alpha)$.

The solvability of the conjugacy problem for the group G, we obtain basing on the obtained presentation (3.6) of the group G exactly in the same way as in VI.3.5 from [1], where was proved the solvability of the conjugacy problem for the free Burnside group $B(m, n)$. Theorem 1.4 is proved.

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