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Convergence of a Subsequence of Triangular Partial Sums of Double Walsh-Fourier Series

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Abstract—In 1987 Harris proved-among others that for each $1 \le p < 2$ there exists a two-dimensional function $f \in L_p$ such that its triangular Walsh-Fourier series does not converge almost everywhere. In this paper we prove that the set of the functions from the space $L_p(\mathbb{I}^2)$ $(1 \le p < 2)$ with subsequence of triangular partial means $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$. We also prove that for each function $f \in L_2(\mathbb{I}^2)$ a.e. convergence $S_{a(n)}^{\triangle}(f) \to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

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1. INTRODUCTION

We denote the set of all nonnegative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} = [0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$. Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} n_j 2^j, \quad n_j = 0, 1, \quad \text{and} \quad x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, \quad x_j = 0, 1.$$

In the case of $x \in \mathbb{Q}$ chose the expension which terminates in zeros. n_i, x_i are the i-th coordinates of n, x, respectively. Define the dyadic addition $\dotplus$ as

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Denote by $\oplus$ the dyadic (or logical) addition. That is, $k \oplus n = \sum_{i=0}^{\infty} |k_i - n_i| \, 2^i$, where k_i, n_i are the ith coordinate of natural numbers k, n with respect to number system based 2.

The sets $I_n(x) = \{y \in \mathbb{I} : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}$, $I_n = I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) = \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ denote by $|n| = \max\{j \in \mathbb{N} : n_j \ne 0\}$, that is, $2^{|n|} \le n < 2^{|n|+1}$. Set $e_j = 1/2^{j+1}$, the i-th coordinate of e_i is 1, the rest are zeros $(i \in \mathbb{N})$.

The Rademacher system is defined by $r_n(x) = (-1)^{x_n}$, $x \in \mathbb{I}$, $n \in \mathbb{N}$.

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The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}, \quad (x \in \mathbb{I},\ n \in \mathbb{N}).$$

The Walsh-Dirichlet kernel is defined by $D_n(x) = \sum_{k=0}^{n-1} w_k(x)$. Recall that (see [13])

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x \in [0, 2^{-n}) \\ 0, \ \text{ if } x \in [2^{-n}, 1) \end{cases}, \tag{1.1}$$

We consider the double system $\left\{w_n(x^1) \times w_m(x^2) : \ n, m \in \mathbb{N}\right\}$ on the unit square $\mathbb{I}^2 = [0,1) \times [0,1)$.

We denote by $L_0(\mathbb{I}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{I}^2$. $\mu(A)$ is the Lebesgue measure of $A \subset \mathbb{I}^d$. We denote by $L_p\left(\mathbb{I}^2\right)$ the class of all measurable functions f that are 1-periodic with respect to all variable and satisfy

$$\|f\|_p := \left(\int_{\mathbb{I}^2} |f\left(y^1, y^2\right)|^p dy^1 dy^2\right)^{1/p} < \infty.$$

If $f \in L_1\left(\mathbb{I}^2\right)$, then

$$\hat{f}\left(n^1, n^2\right) = \int_{\mathbb{I}^2} f\left(y^1, y^2\right) w_{n^1}(y^1) w_{n^2}(y^2) dy^1 dy^2$$

is the $\left(n^1, n^2\right)$-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{N^1,N^2}\left(x^1, x^2; f\right) = \sum_{n^1=0}^{N^1-1} \sum_{n^2=0}^{N^2-1} \hat{f}\left(n^1, n^2\right) w_{n^1}(x^1) w_{n^2}(x^2).$$

The triangular partial sums defined as

$$S_k^{\triangle}\left(x^1, x^2; f\right) = \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i,j)\, w_i(x^1) w_j(x^2).$$

Let $a = (a(n))$ be a lacunary sequence of positive integers with quotient q. That is, $a(n+1)/a(n) \geq q > 1$ for any $n \in \mathbb{N}$. Now, set the maximal function $S_{a,*}^{\triangle} f = \sup_n \left|S_{a(n)}^{\triangle}(f)\right|$. In 1971 Fefferman proved [2] the following result with respect to the trigonometric system.

Let P be an open polygonal region in $\mathbb{R}^2$, containing the origin. Set $\lambda P = \left\{\left(\lambda x^1, \lambda x^2\right) : \left(x^1, x^2\right) \in P\right\}$ for $\lambda > 0$. Then for every $p > 1$, $f \in L_p\left([-\pi,\pi]^2\right)$ it holds the relation

$$\sum_{(n^1,n^2) \in \lambda P} \widehat{f}\left(n^1, n^2\right) \exp\left(i\left(n^1 y^1 + n^2 y^2\right)\right) \to f\left(y^1, y^2\right) \text{ as } \lambda \to \infty$$

for a. e. $\left(y^1, y^2\right) \in [-\pi, \pi]^2$. That is, $S_{\lambda P} f \to f$ a. e. Sjulin gave [14] a better result in the case when P is a rectangle. He proved a. e. convergence for the class $f \in L\left(\log^+ L\right)^3 \log\log L$ and for functions $f \in L\left(\log^+ L\right)^2 \log\log L$ when P is a square. This result for squares is improved by Antonov [1]. There is a sharp constraint between the trigonometric and the Walsh case. In 1987 Harris proved [8] for the Walsh system that if S is a region in $[0,\infty) \times [0,\infty)$ with piecewise C^1 boundary not always paralled to the axes and $1 \leq p < 2$, then there exists an $f \in L_p\left(\mathbb{I}^2\right)$ such that $S_{\lambda P} f$ does not converges a. e. and in L_p norms as $\lambda \to \infty$. In particular, from theorem of Harris follows that for any $1 \leq p < 2$ there exists an $f \in L_p\left(\mathbb{I}^2\right)$ such that $S_{2^A}^{\triangle} f$ does not converges a. e. as $A \to \infty$.

In this paper we improve this result of Harris for triangular partial sums ($P = \triangle$), In particular, let $1 \leq p < 2$, then we prove that the set of the functions from the space $L_p(\mathbb{I}^2)$ with subsequence of triangular partial means $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$. We also prove that for each function $f \in L_2(\mathbb{I}^2)$ a.e. convergence $S_{a(n)}^{\triangle}(f) \to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

For results with respect to convergence of rectangular and triangular partial sums of Walsh-Fourier series see [6, 7, 9—12, 15].

2. MAIN RESULTS

Theorem 2.1. *Let $1 \leq p < 2$. The set of the functions from the space $L_p(\mathbb{I}^2)$ with subsequence of triangular partial sums $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$.*

Theorem 2.2. *The operator $S_{a,*}^{\triangle}$ is of strong type (L_2, L_2). More precisely, $\|S_{a,*}^{\triangle} f\|_2 \leq C_q \|f\|_2$.*

By Theorem 2.2 and by the usual density argument we have

Corollary 2.1. *As $n \to \infty$ we have $S_{a(n)}^{\triangle}(f) \to f$ a.e. for every $f \in L_2(\mathbb{I}^2)$, where $a(n)$ is a lacunary sequence of positive integers.*

The following theorem is proved in [4, 5].

Theorem GGT. *Let $\{T_m\}_{m=1}^{\infty}$ be a sequence of linear continues operators, acting from space $L_p(\mathbb{I}^2)$ in to the space $L_0(\mathbb{I}^2)$. Suppose that there exists the sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from unit bull $S_p(0,1)$ of space $L_p(\mathbb{I}^2)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0 = \inf_k \mu\{\left(x^1, x^2\right) \in \mathbb{I}^2 : |T_{m_k} \xi_k \left(x^1, x^2\right)| > \lambda_k\} > 0.$$

Then the set of functions f from space $L_p(\mathbb{I}^2)$), for which the sequence $\{T_m f\}$ converges in measure to an a. e. finite function is of first Baire category in space $L_p(\mathbb{I}^2)$.

Proof of Theorem 2.1. First we prove that there exists a function h_A for which

$$\|h_A\|_p \leq 1 \tag{2.1}$$

and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; h_A\right)\right| > \frac{2^{A/p}}{\sqrt{A}}\right\} \geq \frac{A}{2^{A+3}}. \tag{2.2}$$

Let

$$f_A\left(x^1, x^2\right) = \sum_{k=0}^{A-1} \sum_{l=0}^{2^A-1} w_{2^k \oplus l}\left(x^1\right) w_l\left(x^2\right), \quad h_A\left(x^1, x^2\right) = \frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}} f_A\left(x^1, x^2\right).$$

We can write

$$\begin{aligned}
\|f_A\|_p &= \left(\int_{\mathbb{I}^2} \left|\sum_{k=0}^{A-1} w_{2^k}\left(x^1\right) D_{2^A}\left(x^1 \dotplus x^2\right)\right|^p dx^1 dx^2\right)^{1/p} \\
&= \left(\int_{\mathbb{I}} \left|\sum_{k=0}^{A-1} w_{2^k}\left(x^1\right)\right|^p \left(\int_{\mathbb{I}} D_{2^A}^p\left(x^1 \dotplus x^2\right) dx^2\right) dx^1\right)^{1/p}
\end{aligned}$$

$$= \left(\int_{\mathbb{I}}\left|\sum_{k=0}^{A-1} w_{2^k}\left(x^1\right)\right|^p dx^1 \left(\int_{\mathbb{I}} D_{2^A}^p\left(x^2\right) dx^2\right)\right)^{1/p}$$

$$\leq \left(\int_{\mathbb{I}}\left(\sum_{k=0}^{A-1} w_{2^k}\left(x^1\right)\right)^2 dx^1\right)^{1/2} 2^{A(1-1/p)} = \sqrt{A}2^{A(1-1/p)}.$$

Hence (2.1) is proved. From simple calculation we obtain that

$$\begin{aligned}\widehat{h}_A(i,j) &= \int_{\mathbb{I}^2} h_A\left(y^1,y^2\right) w_i\left(y^1\right) w_j\left(y^2\right) dy^1 dy^2\\ &= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\int_{\mathbb{I}^2} f_A\left(y^1,y^2\right) w_{2^A-1}\left(y^1\right) w_i\left(y^1\right) w_j\left(y^2\right) dy^1 dy^2\\ &= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\int_{\mathbb{I}^2} f_A\left(y^1,y^2\right) w_{2^A-1-i}\left(y^1\right) w_j\left(y^2\right) dy^1 dy^2\\ &= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\widehat{f}_A\left(2^A-1-i,j\right).\end{aligned}$$

Hence

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right) = \sum_{i+j<2^A}\widehat{h}_A(i,j)\, w_i\left(x^1\right) w_j\left(x^2\right)$$
$$= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i+j<2^A}\widehat{f}_A\left(2^A-1-i,j\right) w_i\left(x^1\right) w_j\left(x^2\right)$$
$$= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i=0}^{2^A-1}\sum_{j=0}^{2^A-i-1}\widehat{f}_A\left(2^A-1-i,j\right) w_i\left(x^1\right) w_j\left(x^2\right)$$
$$= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i=0}^{2^A-1}\sum_{j=0}^{i}\widehat{f}_A(i,j)\, w_{2^A-1-i}\left(x^1\right) w_j\left(x^2\right).$$

Consequently,

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right) = \frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}\sum_{k=0}^{A-1}\sum_{l\leq 2^k\oplus l} w_{2^k\oplus l}\left(x^1\right) w_l\left(x^2\right).$$

We see that $l \leq 2^k \oplus l$ holds if and only if $l_k = 0$. Hence, we have

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right) = \frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}\sum_{k=0}^{A-1} w_{2^k}\left(x^1\right)\sum_{l\in\{l=0,1,\ldots,2^A-1:l_k=0\}} w_l\left(x^1 \dotplus x^2\right).$$

Let

$$\left(x^1,x^2\right) \in G_{A,s} = I_A\left(t_0,\ldots,t_{s-1},1,t_{s+1},\ldots,t_{A-1}\right) \times I_A\left(t_0,\ldots,t_{s-1},0,t_{s+1},\ldots,t_{A-1}\right).$$

Since $x^1 \dotplus x^2 = I_A(e_s)$, we can write

$$\sum_{l\in\{l=0,1,\ldots,2^A-1:l_k=0\}} w_l\left(x^1 \dotplus x^2\right) = \sum_{l_0=0}^{1}\cdots\sum_{l_{k-1}=0}^{1}\sum_{l_{k+1}=0}^{1}\cdots\sum_{l_{A-1}=0}^{1}(-1)^{l_s} = \begin{cases} 2^{A-1}, & \text{if } k=s\\ 0, & k\neq s\end{cases}.$$

Hence,

$$\left|S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)\right| \geq \frac{2^{A-1}}{2^{A(1-1/p)}\sqrt{A}}\sum_{s=0}^{A-1}\mathbb{I}_{G_{A,s}}\left(x^1,x^2\right) = \frac{2^{A/p}}{2\sqrt{A}}\sum_{s=0}^{A-1}\mathbb{I}_{G_{A,s}}\left(x^1,x^2\right). \tag{2.3}$$

Set

$$\Omega_A := \bigcup_{s=0}^{A-1}\bigcup_{t_0=0}^{1}\cdots\bigcup_{t_{s-1}=0}^{1}\bigcup_{t_{s+1}=0}^{1}\cdots\bigcup_{t_{A-1}=0}^{1} G_{A,s}$$

From estimation (2.3) we get

$$\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|S^{\triangle}_{2^A}\left(x^1,x^2;h_A\right)\right|>\frac{2^{A/p}}{2\sqrt{A}}\right\}\geq\mu\left(\Omega_A\right)=\frac{1}{2^{2A}}\sum_{s=0}^{A-1}\sum_{x_0=0}^{1}\cdots\sum_{x_{s-1}=0}^{1}\sum_{x_{s+1}=0}^{1}\cdots\sum_{x_{A-1}=0}^{1}=\frac{A}{2^{A+1}}.$$

Now, we prove that there exists $\left(x^1_1,x^2_1\right),...,\left(x^1_{p(A)},x^2_{p(A)}\right)\in\mathbb{I}^2$, $p\left(A\right):=\left[2^{A+3}/A\right]+1$, such that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x^1_j,x^2_j\right)\right)\right)\geq\frac{1}{2}. \tag{2.4}$$

Indeed,

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x^1_j,x^2_j\right)\right)\right)=1-\mu\left(\bigcap_{j=1}^{p(A)}\left(\overline{\Omega_A\dotplus\left(x^1_j,x^2_j\right)}\right)\right) \tag{2.5}$$

$$=1-\int_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x^1_1,t^2\dotplus x^2_1\right)\cdots\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x^1_{p(A)},t^2\dotplus x^2_{p(A)}\right)dt^1dt^2.$$

Interpreting $\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x^1_1,t^2\dotplus x^2_1\right)\cdots\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x^1_{p(A)},t^2\dotplus x^2_{p(A)}\right)$ as a function of the $2p\left(A\right)+2$ variables t^1,t^2, $\left(x^1_1,x^2_1\right),...,\left(x^1_{p(A)},x^2_{p(A)}\right)$ and integrating over all variables, each over $\mathbb{I}^2$, we note that

$$\int_{\mathbb{I}^2}\cdots\int_{\mathbb{I}^2}\int_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x^1_1,t^2\dotplus x^2_1\right)\cdots\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x^1_{p(A)},t^2\dotplus x^2_{p(1)}\right)dt^1dt^2dx^1_1dx^2_1\cdots dx^1_{p(A)}dx^2_{p(A)}$$

$$=\int_{\mathbb{I}^2}\left(\int_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x^1_1,t^2\dotplus x^2_1\right)dx^1_1dx^2_1\right)...\left(\int_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x^1_{p(A)},t^2\dotplus x^2_{p(A)}\right)dx^1_{p(A)}dx^2_{p(A)}\right)dt^1dt^2$$

$$=\left(\mu\left(\overline{\Omega}_A\right)\right)^{p(A)}=\left(1-\mu\left(\Omega_A\right)\right)^{p(A)}\leq\left(1-\frac{1}{p\left(A\right)}\right)^{p(A)}\leq\frac{1}{2}.$$

Consequently, there exists $\left(x^1_1,x^2_1\right),...,\left(x^1_{p(A)},x^2_{p(A)}\right)\in\mathbb{I}^2$ such that

$$\int_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x^1_1,t^2\dotplus x^2_1\right)\cdots\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x^1_{p(A)},t^2\dotplus x^2_{p(A)}\right)dt^1dt^2\leq\frac{1}{2}. \tag{2.6}$$

Combining (2.5) and (2.6) we conclude that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x^1_j,x^2_j\right)\right)\right)\geq1-\frac{1}{2}=\frac{1}{2}.$$

Hence (2.4) is proved. Let $\left(t:=t^1\dotplus t^2\in\mathbb{I}\right)$

$$F_A\left(x^1,x^2,t\right)=\frac{1}{\left(4p\left(A\right)\right)^{1/p}}\sum_{j=1}^{p(A)}r_j\left(t^1\dotplus t^2\right)h_A\left(x^1\dotplus x^1_j,x^2\dotplus x^2_j\right)$$

$$=\frac{1}{\left(4p\left(A\right)\right)^{1/p}}\sum_{j=1}^{p(A)}r_j\left(t\right)h_A\left(x^1\dotplus x^1_j,x^2\dotplus x^2_j\right).$$

Then it is proved in ([3], pp. 7-12) that there exists $t_0\in\mathbb{I}$, such that

$$\int_{\mathbb{I}}\left|F_A\left(x^1,x^2,t_0\right)\right|^p dx^1dx^2\leq1,\quad\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|S^{\triangle}_{2^A}\left(x^1,x^2;F_A\right)\right|>\frac{2^{A/p}/\left(2\sqrt{A}\right)}{\left(p\left(A\right)\right)^{1/p}}\right\}\geq\frac{1}{8}. \tag{2.7}$$

Set $\xi_A\left(x^1,x^2\right):=F_A\left(x^1,x^2,t_0\right)$. Then from (2.7) we have $\|\xi_A\|_p\le 1$ and

$$\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|S_{2^A}^{\triangle}\left(x^1,x^2;\xi_A\right)\right|>2^{1-3/p}A^{1/p-1/2}\right\}\ge\frac{1}{8}$$

and using Theorem GGT we complete the proof of Theorem 2.1.

Proof of Theorem 2.2. First, we suppose that $q\ge 2$. Let $S_n^{\square}(f)$ be n-th square partial sums of the two-dimensional Walsh-Fourier series. It is easy to see that the spectrums of the polynomials $S_{a(n)}^{\square}(f)-S_{a(n)}^{\triangle}(f)$, $n=1,2,\ldots$ are pairwise disjoint that implies

$$\left\|\sup_n\left|S_{a(n)}^{\triangle}(f)\right|\right\|_2^2\le 2\left\|\sup_n\left|S_{a(n)}^{\square}(f)\right|\right\|_2^2+2\left\|\sup_n\left|S_{a(n)}^{\triangle}(f)-S_{a(n)}^{\square}(f)\right|\right\|_2^2$$
$$\le 2\left\|\sup_n\left|S_{a(n)}^{\square}(f)\right|\right\|_2^2+2\sum_n\left\|S_{a(n)}^{\triangle}(f)-S_{a(n)}^{\square}(f)\right\|_2^2\le 2\left\|\sup_n\left|S_{a(n)}^{\square}(f)\right|\right\|_2^2+2\left\|f\right\|_2^2\le c\left\|f\right\|_2^2,$$

where the last inequality is obtained from the L_2 boundedness of the square partial sums majorant operator (see [13]). This completes the proof of Theorem 2.2 in the case of $q\ge 2$. If $2>q>1$, then let Q the least natural number for which $q^Q\ge 2$. For any fixed $j=0,\ldots,Q-1$ we have that the quotient of lacunary sequence n integers $(a(Qn+j))$ is at least 2 since $a(Q(n+1)+j)\ge q^Q a(Qn+j)$. From the above written we have

$$\left\|\sup_n\left|S_{a(Qn+j)}^{\triangle}f\right|\right\|_2^2\le C\left\|f\right\|_2^2$$

and consequently we also have $\left\|S_{a,*}^{\triangle}f\right\|_2^2\le C_q\left\|f\right\|_2^2$.

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