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On Continuous Selections of Set-valued Mappings with Almost Convex Values

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Abstract—In this paper, it is proved that through each point of the graph of a continuous set-valued mapping with almost convex and star-like values can be passed a continuous selection of that mapping.

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1. INTRODUCTION

One of the important problem in the theory of set-valued mappings is the question of existence of single-valued approximations and selections with specified properties. The question of existence of selections possessing certain topological properties is of great importance and has various applications in many fields of mathematics. The problem of existence of *continuous* selections of set-valued mappings goes back to the classical theorem by E. Michael (see [15]). Later on this problem was widely developed and was applied in the theory of differential embeddings, in the control systems and in the general topology (see [1, 4]. The above quoted Michael's theorem state that every lower semicontinuous mapping with *convex* values admits a continuous selection.

In [1, 16], were given examples, illustrating the importance of convexity condition of the set-valued mapping. In [9], it was constructed an example of a continuous mapping with *star-like* values that does not admit any continuous selector (see [9], Example 1(A)). Nevertheless, the existence of continuous selections can also be proved for some classes of mappings with nonconvex values. For instance, in the paper [9] it was considered a subclass (mappings with star-like or *right-convex* values) of continuous set-valued mappings with star-like values, admitting continuous selections (see Theorem 1 of [9]). In the general nonconvex case, in the paper [10] to each closed set M is associated some function $h : R_+ \to R_+$ of non-convexity of the set M. In Theorem 5.1 of [10], it was proved that if a is a lower semicontinuous mapping such that the values of the convexity function $h_{a(x)}$ is strictly less than some monotone nondecreasing function $\alpha : (0, \infty) \to [0, 1)$, then a has a continuous single-valued selection. It should be noted, however, that the definition of function h is of descriptive nature, and it is rather difficult to construct such function for each closed set M.

Also, notice that in the papers [11, 12], by using the method of tangent cones, were extracted differentiable or directional differentiable *local* selections from set-valued mappings both with convex and nonconvex values.

In the present paper, we consider the question of existence of continuous selections for a new class of set-valued mappings with nonconvex values, more precisely, for a class of mappings with *almost convex* values. Notice that the notion of almost convexity was introduced in the papers [7, 8]. Also, the need to study such sets arose in the theory of differential games (see [5]).

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2. SOME NOTATION AND DEFINITIONS

Let X be a metric space and Y be a Banach space. In what follows, by $B_r(a)$ we will denote a ball with center at a and of radius r. For a closed set $M \subseteq Y$, by $diam(M)$ we denote the diameter of M, and by $conv\{M\}$ the convex hull of M. We set

$$Pr_M(x) \equiv \{y \in M / \|x - y\| = \inf_{z \in M} \|x - z\| \equiv d(x, M)\}.$$

Next, we recall the definitions of a set-valued mapping and a selector. Let 2^Y be the collection of all nonempty subsets of Y, and let E be a subset of the space X.

A mapping $a : E \longrightarrow 2^Y$ is called a set-valued mapping. A continuous single-valued mapping $y : E \longrightarrow Y$ is called a continuous selection (or a continuous selector) of the mapping a if $y(x) \in a(x)$, $x \in E$.

A mapping $a : E \longrightarrow 2^Y$ is said to be lower semicontinuous at $x_0 \in E$ if for any $\varepsilon > 0$ there exists $\delta > 0$, such that $a(x_0) \subseteq a(x) + B_\epsilon(0)$ for any $x \in E \cap B_\delta(x_0)$.

A mapping $a : E \longrightarrow 2^Y$ is said to be upper semicontinuous at $x_0 \in E$ if for any $\varepsilon > 0$ there exists $\delta > 0$, such that $a(x) \subseteq a(x_0) + B_\epsilon(0)$ for any $x \in E \cap B_\delta(x_0)$.

If a mapping is lower and upper semicontinuous at x_0, then it is called continuous at x_0 (see [1], the definition 1.2.43 of Hausdorff continuity). The set

$$graph(a) = \{(x, y) \in E \times R^m,\ y \in a(x)\}$$

is called a graph of the mapping a.

Definition 2.1 (see [3]). *Let $M \subseteq Y$. Define*

$$M^0 \equiv \{x \in M :\ \lambda x + (1 - \lambda)y \in M,\ y \in M,\ \lambda \in [0, 1]\}.$$

The subset $M^0 \subseteq M$ is said to be the star-kernel of the set M. If $M^0 \neq \emptyset$, then the set M is said to be a star-like set.

It can easily be shown that M^0 is a convex set. Also, it is clear that if M is a convex set, then $M = M^0$.

Definition 2.2 (see [7]). *We say that a set $M \subseteq Y$ satisfies the almost convexity condition with a constant $\theta \geq 0$ if for any $x_j \in M$, $\lambda_j \geq 0$, $j \in J$, where J is a finite set of indices such that $\sum_{j \in J} \lambda_j = 1$, we have*

$$\sum_{j \in J} \lambda_j x_j \in M + \theta r^2 B_1(0),$$

where $r \equiv \max_{i,j \in J} \|x_i - x_j\|$.

If no necessity to specify the constant θ, then we will say that the set M is almost convex. Notice that if $\theta = 0$, then M is a convex set. The class of almost convex sets is sufficiently broad.

3. EXAMPLES

Example 3.1. The set $M = \{a, b\}$ consisting of two points is almost convex. Indeed, we have

$$conv\{a, b\} \subseteq M + \frac{1}{2\|a - b\|} \|a - b\|^2 B_1(0),$$

that is, in this case as a constant θ of almost convexity can be taken $1/(2\|a - b\|)$.

Example 3.2. An arc of a circle is an almost convex set. This immediately follows from the sufficiency condition of almost convexity (see [8], Theorem 2). To determine the constant θ of almost convexity, we first assume that the arc M is smaller than the semicircle, and the arc contains the set $Q = \{x_1, x_2, ..., x_k\}$. Let $A = x_1, B = x_k, d = diam(Q) = AB$. Then the set $conv\{Q\}$ is in an α-neighborhood of the set M, where $\alpha = CD$ (see Fig. 1). Hence, we have

$$DC = R - \sqrt{R^2 - \frac{d^2}{4}}.$$

Now the constant θ can be determined from the inequality $DC \leq \theta d^2$, that is,

$$\frac{1}{R+\sqrt{R^2-\frac{d^2}{4}}} \leq \theta.$$

It is clear that the numbers $\theta \geq 1/4R$ satisfy the last inequality. If the arc M is larger than the semicircle, then the almost convexity of M with some constant ϑ follows from Theorem 3 of [8]. If $Q = \{a, b\}$, then the set $conv\{Q\}$ is in a β-neighborhood of the set M, where $\beta = \|a-b\|/2$ (see Fig. 1). Thus, we have $\theta \geq \frac{1}{2\|a-b\|}$, and hence $\theta \to \infty$ as $\|a-b\| \to 0$.

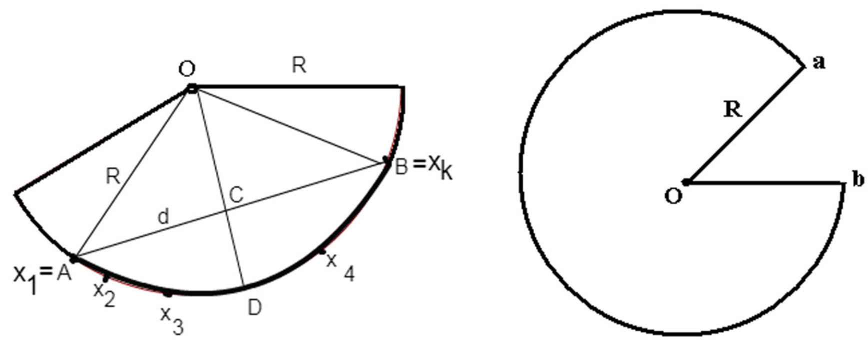

Fig. 1.

Example 3.3. A circle M of radius R is an almost convex set with constant $\theta \geq 1/(\sqrt{3}R)$. Indeed, let $Q \equiv \{x_1, x_2, ..., x_k\} \subset M$. Consider two cases. First, let $0 \notin conv\{Q\}$. This means that the set belongs to some semicircle. Hence, by Example 3.2, we have

$$conv\{Q\} \subseteq M + \frac{1}{4R}(diam(Q))^2 B_1(0). \tag{3.1}$$

Now let $0 \in intQ$. Then in $convQ$ there exists an acute-angled triangle, for which the center 0 of the circle is an interior point, implying that the circle is circumscribed this triangle. Therefore, some of the sides of this triangle is of length at least $R\sqrt{3}$, implying that $diam(Q) \geq \sqrt{3}R$.

It is clear that the set Q lies in a R-neighborhood of M. Now we choose the number θ to satisfy

$$R \leq \theta(diam(Q))^2. \tag{3.2}$$

Observe that this inequality holds if $\theta \geq 1/(\sqrt{3}R)$. If the point O is on the boundary of the set $conv\{Q\}$, then $diam(Q) = 2R$. Hence, the inequality (3.2) is satisfied if $\theta \geq 1/4R$. In the general case, taking into account the inclusion (3.1), we obtain

$$conv\{Q\} \subseteq M + \frac{1}{\sqrt{3}R}(diam(Q))^2 B_1(0),$$

implying that the set M is almost convex with constant $1/(\sqrt{3}R)$.

Now we give an example of a set that is almost convex and star-like, but is not convex.

Example 3.4. In Fig. 2, the shaded domain M with closed boundary $ACBDA$ is a star-like set. We show that this set is almost convex. To this end, we choose $\theta > 0$ to satisfy

$$DE = R - \sqrt{R^2 - \frac{d^2}{4}} \leq \theta d^2, \quad \text{where } d \equiv AB.$$

Observe that this inequality will be satisfied if we take $\theta = \frac{1}{4R}$. Also, it is easy to see that the domain M is an almost convex set with a constant θ. Notice that if $DO = R \to \infty$, then $\theta \to 0$, and the domain $ACDBA$ becomes into the triangle ACB.

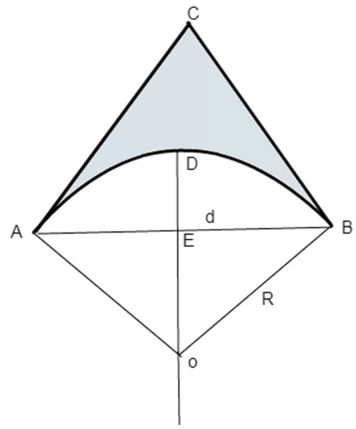

Fig. 2. Almost convex and star-like set

4. PROPERTIES OF ALMOST CONVEX SETS

Proposition 4.1 ([8], Theorem 3). *Let $M \subseteq R^n$ be a closed set satisfying almost convexity condition with constant $\theta > 0$. If $\varepsilon \le 1/(16\theta)$, then the mapping $x \longrightarrow Pr_M(x)$ is single-valued on the set $M + B_\varepsilon(0)$, and*

$$\|Pr_M(x_1) - Pr_M(x_2)\| \le 2\|x_1 - x_2\|.$$

It should be noted that if the set M is convex and closed, then any point from R^n has a unique projection onto M and the projection operator Pr_M satisfies the Lipschitz condition with constant 1.

Remark 4.1. Clarke et al. [14], have defined the notion of a *proximal smooth set* to be the set such that the distance of a space point to this set is a continuously differentiable function in some neighborhood of that set with the exception of the set itself. In the same paper (see Theorem 4.11 of [14]), it was proved that in the Hilbert spaces the condition of proximal smoothness of a set M is equivalent to the fact that the metric projection of any point from sufficiently small neighborhood of M onto M exists, is unique and depends on the projected point continuously. Then, in [13] a similar result was proved in some uniformly convex and smooth Banach spaces. From Proposition 4.1 it follows that if $M \subseteq R^n$ and M is almost convex, then it is also proximal smooth. Thus, in the space R^n, the almost convex sets constitute some subclass in the family of proximal smooth sets.

Proposition 4.2. *If $M \subseteq R^n$ is a closed, star-like and almost convex set, then for small enough $\varepsilon > 0$, the set $M + B_\varepsilon(0)$ is also star-like and almost convex.*

Proof. Let the set M be closed and almost convex with a constant θ. It is known (see [8], Theorem 3 and Corollary 3) that if $\varepsilon \le 1/(16\theta)$, then the set $M + B_\varepsilon(0)$ is almost convex with constant 4θ. Also, it is easy to show that $(M^0 + B_\epsilon(0)) \subseteq (M + B_\varepsilon(0))^0$. Hence M is a star-like set.

Theorem 4.1. *Let $a : [a, b] \longrightarrow 2^{R^n}$ be a set-valued mapping with almost convex valued and a constant θ. Then through each point of the graph of a can be passed a continuous selection of the mapping a.*

Proof. Since the mapping a is Hausdorff continuous on the segment $[a, b]$, then it is also uniformly continuous on $[a, b]$. This means that for any $\varepsilon > 0$, a number $\delta > 0$ can be found so that for a partition of the segment into partial segments $[x_{i-1}, x_i]$ with lengths less than δ, the oscillation of the mapping a on each such partial segment will be less than ε. Choosing $\varepsilon < 1/(16\theta)$, we have

$$a(x_{i-1}) \in a(x) + B_\varepsilon(0), \quad x \in [x_{i-1}, x_i].$$

Let $\overline{y_0} \in a(x_0)$. We set $y_0(x) = Pr_{a(x)}\overline{y_0}\,(x \in [x_0, x_1])$. Since, according to Proposition 4.1, the projection of the point $\overline{y_0}$ onto the set $a(x)$ is unique and the mapping a is continuous, then the mapping y_0 is also continuous (see [2], Section 3.5, Lemma 3, p. 344). We choose a point $y_0(x_1)$ and project it onto the set $a(x)$ $(x \in [x_1, x_2])$. We set $y_1(x) = Pr_{a(x)} y_0(x_1)$, and observe that, according to above arguments, y_1 is a continuous mapping. Continuing this process, we can construct a continuous mapping $y(x)$, defined on the whole segment $[a, b]$ such that

$$y(x) = y_i(x), \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, ..., n.$$

Theorem 4.1 is proved.

Remark 4.2. The mapping y, constructed in Theorem 4.1, also depends on the initial point $\overline{y_0}$. Using the result of Proposition 1, we easily see that the mapping y satisfies Lipschitz condition with respect to the variable $\overline{y_0}$ uniformly on x. Therefore, the mapping y is continuous by the pair $(x, \overline{y_0})$.

Now we give an example of a continuous set-valued mapping $a : R^2 \longrightarrow R^2$ with almost convex and compact values, which does not admit any continuous selector.

Example 4.1. Let

$$a(x) = S_1 \setminus B_{\|x\|}(\frac{x}{\|x\|}), \ x \neq 0, \ a(0) = S_1.$$

In [1], Example 1.4.6, p.58 it was proved that the mapping a is continuous and does not admit any continuous selector. Also, observe that the mapping a is of almost convex values. Indeed, since the set $a(x)$ is an arc on the circle S_1, then by Example 3.4 it is almost convex. Besides, if the arc $a(x)$ is smaller than the semicircle, then it is almost convex with constant $\theta = 1/4$, while if the arc $a(x)$ is larger than the semicircle, then it is almost convex with some constant θ. And, the unit circle S_1 is almost convex with constant $1/\sqrt{3}$.

5. MAIN RESULTS

Let $a : E \to 2^{R^m}$ be a set-valued mapping. Define a set-valued mapping $a_0 : E \to 2^{R^m}$ as follows: $a_0(x) \equiv (a(x))^0 \ \forall x \in E$. It is clear that such defined mapping a_0 has convex values.

Theorem 5.1. *Let E be a compact subset of a metric space X and $a : E \to 2^{R^m}$ be a continuous set-valued mapping with compact, star-like and almost convex values. Assume that the constants $\theta(x)$ of almost convexity of the sets $a(x)$, $(x \in E)$ satisfy the condition:*

$$\sup_{x \in E} \theta(x) = \eta < \infty.$$

Let $(x_0, y_0) \in graph(a)$. Then there exists a continuous selection y for the mapping a, passing through the point (x_0, y_0), that is,

$$y(x_0) = y_0, \quad y(x) \in a(x), \quad x \in E.$$

The proof is based on a number of lemmas that follow.

Lemma 5.1. *Let X be a metric space and Y be a Banach space, and let $a : X \to 2^Y$ and $b : X \to 2^Y$ be set-valued mappings with compact and star-like values. Let the mappings a, a_0 and b, b_0 be continuous at a point x_0 and*

$$0 \subseteq int\ (a_0(x_0) - b_0(x_0)). \quad (5.1)$$

Then the mapping $c(x) \equiv a(x) \bigcap b(x)$ is continuous at x_0.

Proof. We first prove the lower semicontinuity of the mapping c at x_0. Since the lower semicontinuous mapping $\Gamma \equiv a_0 - b_0$ has convex closed values, and the inclusion (5.1) is satisfied, then there exist a number $\tau > 0$ and a neighborhood U of x_0, such that

$$B_\tau(0) \subseteq \Gamma(x) = (a_0(x) - b_0(x)), \quad x \in U. \quad (5.2)$$

Indeed, since the mapping Γ is semicontinuous at x_0, there exist a number $\tau > 0$ and a neighborhood U of x_0, such that

$$B_{2\tau}(0) \subseteq \Gamma(x) + B_\tau(0) :$$

Hence, for any continuous liner functional y^* with $\| y^* \|= 1$, we have

$$\max_{u\in B_{2\tau}(0)} < y^*, u > \leq \max_{u\in \Gamma(x)} < y^*, u > + \max_{u\in B_{\tau}(0)} < y^*, u >,$$

implying that

$$2\tau \leq \max_{u\in\Gamma(x))} < y^*, u > +\tau,$$

that is, $\tau \leq \max_{u\in\Gamma(x)} < y^*, u >$. Therefore, taking into account that $\Gamma(x)$ is a convex closed set in the Banach space Y, we get $B_\tau(0) \subseteq \Gamma(x)$, $x \in U$. Next, since the set-valued mapping b is upper semicontinuous in a neighborhood U, then it is bounded on U, that is, there exists a bounded set G, such that $b(x) \subseteq G$, $x \in U$. Let $diam(G) = D$, and let $\varepsilon > 0$ be such that $\varepsilon < 2D$. We set $\alpha = \tau\varepsilon/(2D - \varepsilon)$ and choose $\tau > 0$ small enough to satisfy $\alpha < \varepsilon/2$. Since a and b are lower semicontinuous mappings at x_0, a neighborhood $\overline{U} \subseteq U$ of x_0 can be found to satisfy

$$a(x_0) \subseteq a(x) + B_{\alpha/2}(0),\ b(x_0) \subseteq b(x) + B_{\alpha/2}(0), \quad x \in \overline{U}.$$

Let $x \in \overline{U}$. Then for any $y \in c(x_0)$ there exists a vector $\overline{y}_x \in b(x)$ such that

$$\overline{y}_x \in a(x) + B_\alpha(0) \text{ and } \|y - \overline{y}_x\| \leq \alpha. \tag{5.3}$$

We set $\theta = \tau/(\alpha + \tau) < 1$. Multiplying the inclusion (5.3) by θ, and observing that $\theta\alpha = (1 - \theta)\tau$, we obtain

$$\theta\overline{y}_x \in \theta a(x) + \theta\alpha B_1(0) = \theta a(x) + (1 - \theta)\tau B_1(0). \tag{5.4}$$

Now multiplying the inclusion (5.2) by $(1 - \theta)$, we get

$$(1 - \theta)\tau B_1(0) \subseteq (1 - \theta)a_0(x) - (1 - \theta)b_0(x).$$

Hence, in view of (5.4), there exists a vector $y' \in b_0(x)$ such that

$$\theta\overline{y}_x + (1 - \theta)y' \in a(x) \tag{5.5}$$

On the other hand, since $\overline{y}_x \in b(x)$ and $y' \in b_0(x)$, we have

$$\overline{y} \equiv \theta\overline{y}_x + (1 - \theta)y' \in b(x). \tag{5.6}$$

From (5.5) and (5.6) it follows that $\overline{y} \in c(x)$.

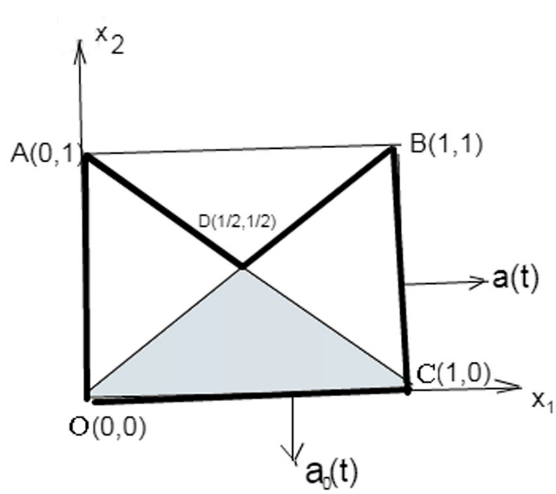

Fig. 3. Intersection of continuous mappings with star-like values

Next, we show that $\|y - \overline{y}\| \leq \varepsilon$. Indeed, we have

$$\|y - \overline{y}\| \leq \|y - (\theta\overline{y}_x + (1 - \theta)y')\| = \|\theta y + (1 - \theta)y - \theta\overline{y}_x - (1 - \theta)y'\|$$

$$\leq \theta\|y-\overline{y}_x\| + (1-\theta)\|y-y'\| \leq \alpha + \frac{\alpha}{\alpha+\tau}D \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $c(x_0) \subseteq c(x) + B_\varepsilon(0)\ \forall x \in \overline{U}$, showing that the mapping c is lower semicontinuous at x_0. Similarly it can be proved upper semicontinuity of the mapping c. Lemma 5.1 is proved.

Example 5.1. Let the domain on Fig. 3 with closed boundary $OADBCO$ represents the set $a(t)$, $t \in [0, 1/2]$. Then $a_0(t)$ is the triangle ODC. We set $b(t) \equiv \{(x_1, x_2) \in [0,1] \times [0,1]/x_2 = tx_1\}$, $t \in R$. It is easy to see that the mappings a and b with star-like values are continuous, but their intersection $a \cap b$ is discontinuous at point $t = 1/2$. This is because here the condition (5.1) is violated at point $t = 1/2$.

Now we give an example of a continuous set-valued mapping a for which the mapping a_0 is not continuous.

Example 5.2. Let the domain on Fig. 4 with boundary $OAFDHO$ represents the set $a(t)$, $t \in [1/2, 1]$. Then $a(t)$ is a star-like set, and its kernel $a_0(t)$ is the set with boundary $OFEHO$. For $t = 1$ the set of values of mapping a_0 is the square $OAEH$. It is clear that the set-valued mapping $a : [1/2, 1] \longrightarrow 2^{R^2}$ is continuous at all points of the segment $[1/2, 1]$, but the mapping a_0 has discontinuity at point 1. Also, notice that the values $a(t)$, $t \in [1/2, 1)$ of the mapping a are not almost convex, because any point on the bisectrix of the angle $\angle AFD$ has two projections on the set $a(t)$, which contradicts Proposition 4.1.

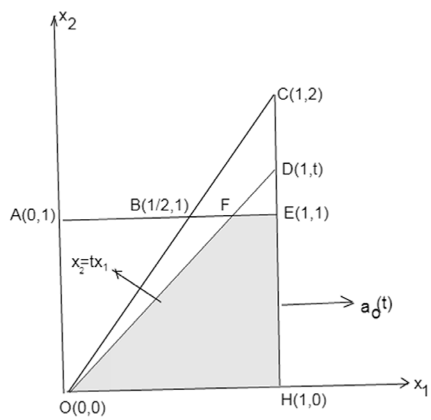

Fig. 4. The mapping a is continuous, while the mapping a_0 is not continuous

In the general case, we have the following result on continuity of the mapping a_0.

Proposition 5.1. *Let $E \subseteq X$ be a compact subset of a metric space X and Y be a Banach space, and let $a : E \to 2^Y$ be a continuous mapping with compact star-like values. Then the interior of points, where a_0 is not continuous, is empty.*

Proof. We first prove that the mapping $a_0 : E \to 2^{R^m}$ is upper semicontinuous. Let $x_n \to x_0$, $y_n \in a_0(x_n)$, and $y_n \to y_0$. We show that $y_0 \in a_0(x)$. Let $z_0 \in a(x_0)$. Since a is a lower semicontinuous mapping, there exists a sequence $z_n \in a(x_n)$ such that $z_n \to z_0$.

On the other hand, since $y_n \in a_0(x_n)$, for any $\lambda \in [0, 1]$ we have

$$\lambda z_n + (1-\lambda)y_n \in a(x_n),$$

implying that $\lambda z_0 + (1-\lambda)y_0 \in a(x_0)$. This means that $y_0 \in a_0(x_0)$. Thus, the mapping a_0 has a closed graph. Now we can apply Theorem 10 from [6] (see Section 1.1, p. 118) to conclude that the interior of the set of points, where a_0 is not continuous, is empty. Proposition 5.1 is proved.

As an illustration of the result stated in Proposition 5.1, can be considered Example 5.2, where the mapping a_0 is defined on the segment $[1/2, 1]$ and it is discontinuous only at point $t = 1$. However, if the values of a continuous mapping a are star-like and almost convex sets, then the mapping a_0 will be continuous. More precisely, we have the following lemma.

Lemma 5.2. *Let $E \subseteq X$ be a compact subset of a metric space X, and let $a : E \to 2^{R^m}$ be a continuous mapping. Further, let the sets $a(x)$ be compact, star-like and satisfy the convexity* *condition with some constant $\theta(x)$. Assume that int $a_0(x) \neq \emptyset$ for each x and $\eta = \sup_{x\in E}\theta(x) < \infty$. Then the mapping a_0 is continuous.*

Proof. Observe first that the upper semicontinuity of the mapping a_0 was proved in Proposition 5.1. Now we show that a_0 is lower semicontinuous. Let $y_0 \in int\ a_0(x_0)$. Assume that there exist a sequence $x_k \to x_0$ and a number $\delta > 0$ such that $d(y_0, a_0(x_k)) \geq \delta$ for sufficiently large k. Then, we can assume that $B_\delta(y_0) \subseteq a_0(x_0)$, but $B_\delta(y_0) \bigcap a_0(x_k) = \emptyset$ for large k. Since the mapping a is pointwise continuous (see [1], Theorem 1.3.8, p. 45), there exists a neighborhood $B_{\delta_0}(y_0) \subseteq B_\delta(y_0)$ such that $B_{\delta_0}(y_0) \subseteq a(x_k)$ for sufficiently large k. Therefore, since $y_0 \notin a_0(x_k)$, there exists a point $y_k \in a(x_k)$, which is not visible from the point y_0, that is, on the segment $[y_0, y_k]$ there exists a point $\overline{y_k} \notin a(x_k)$.

Since $a(x_k)$ is a closed set, there is a ball V_k with center at $\overline{y_k}$ such that $V_k \bigcap a(x_k) = \emptyset$. We will shift this ball from point $\overline{y_k}$ to y_k along the segment $[y_0, y_k]$. By the compactness of $a(x_k)$, among these balls there is a ball V_k' which touches the set $a(x_k)$ only at one point $z_k \in a(x_k)$. It is clear that the tangent to V_k' at point z_k, the hyperplane L_k, strongly separates the point y_0 from the set $a_0(x_k)$. Let H_{z_k} be the half-space containing the point y_0. Note that, by construction, this half-space contains the ball V_k'. Without loss of generality, we can assume that $z_k \to z_0 \in a(x_0)$. Since the mapping a satisfies the convexity condition with a specified constant, there exists a ball $\widetilde{V_k}$ of a fixed radius $r = 1/(8\eta)$, which also touches the set at z_k and which is in the half-space H_{z_k} (see Lemma 2.7 of [7] and Theorem 1 of [8]).

Next, we can assume that the sequence of balls $\widetilde{V_k}$ converges in the Hausdorff metric to some ball V_0 of radius r. The limiting hyperplane L_0 touches the ball V_0 at point z_0. Notice that if $u \in int\ V_0$, then there exists a number $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(u) \subseteq \widetilde{V_k}$ for sufficiently large k. Hence, we have $int\ B_0 \bigcap a(x_0) = \emptyset$. Also, observe that the limiting closed half-space H_{z_0} contains the ball V_0 and the point y_0. Thus, the ball $B_\delta(y_0)$ contains points that are not visible from z_0. But this is impossible, because the ball $B_\delta(y_0)$ entirely is contained in the kernel of the set $a(x_0)$. The obtained contradiction completes the proof of Lemma 5.2.

Lemma 5.3. *Let $E \subseteq X$ be a compact subset of a metric space X, and let $a : E \to 2^{R^m}$ be a set-valued mapping with compact and star-like values, such that int $a_0(x) \neq \emptyset$ for any x. Also, assume that the mappings a and a_0 are continuous. Then for any $(x_0, y_0) \in graf(a)$ there exists a continuous mapping $y(x)$ such that $y(x) \in a(x)\ \forall x \in E$ and $y(x_0) = y_0$.*

Proof. Observe first that since the mapping a_0 is lower semicontinuous, there exists a continuous mapping $\tilde{y}(x)$ such that $\tilde{y}(x) \in inta_0(x)$, $x \in R^n$. Indeed, since the mapping a_0 is pointwise continuous, then for any $y \in int\ a_0(x)$ there exist neighborhoods $V(y)$, $U_y(x)$, such that $V(y) \subseteq a_0(x')\ \forall x' \in U_y(x)$. Denote $U_y = \bigcup_{x\in E} U_y(x)$, and observe that the family of open sets $\{U_y\}_{y\in Y}$, $(Y \equiv \bigcup_{x\in E} a_0(x))$ forms an open covering of the compact set E. Let $\{U_{y_j}\}_{j\in J}$ be a finite subcovering from this covering. Consider the partition of unity $\{p_{y_j}\}_{j\in J}$, corresponding to the covering $\{U_{y_j}\}_{j\in J}$, and define a continuous mapping y as follows: $\tilde{y}(x) = \sum_{j\in J} p_{y_j}(x) y_j$. It is easy to check that $\tilde{y}(x) \in int\ a_0(x)$, $x \in E$. Next, consider a mapping b defined as follows:

$$b(x) = \{y : y = \lambda y_0 + (1-\lambda)\tilde{y}(x),\ \lambda \in [0,1]\}.$$

It is clear that b is lower semicontinuous, and for any x we have $0 \in int(a_0(x) - b(x))$. Then by Lemma 5.1, the mapping $c(x) \equiv a(x) \bigcap b(x)$ is lower semicontinuous. Also, it is clear that $c(x)$ has convex closed values. Therefore, according to Michael's theorem, through the point $(x_0, y_0) \in graf(c)$ can be passed a continuous selection y of the mapping $c(x)$. Lemma 5.3 is proved.

Proof of Theorem 5.1. Let $\epsilon < 1/(16\eta)$. Consider the set-valued mapping $a(x) + B_\varepsilon(0)$, and observe that, in view of Proposition 5.1, it satisfies the conditions of Lemma 5.3. Hence, through the point $(x_0, y_0) \in graph(a)$ can be passed a continuous single-valued mapping $\tilde{y}$ such that $\tilde{y}(x) \in a(x) + B_\varepsilon(0)$, $x \in E$. Since $\tilde{y}(x) \in a(x) + B_{1/(16\theta(x))}(0)$, according to Proposition 5.1, the projection $y(x)$ of the point $\tilde{y}(x)$ onto the set $a(x)$ is single-valued. Also, since the mappings a and $\tilde{y}$ are continuous, then as it was pointed out above, the mapping y will also be continuous. It is clear that y is the desired mapping. Theorem 5.1 is proved.

The next theorem contains a sufficient condition for existence of continuous selections of set-valued mappings with almost convex values (without star-like condition).

Theorem 5.2. *Let $E \subseteq X$ be a compact subset of a metric space X, and let $a : E \to 2^{R^m}$ be a continuous mapping, such that for any $x \in E$ the set $a(x)$ is compact and the convexity condition with a constant $\theta(x)$ is satisfied. Also, assume that*

$$\eta = \sup_{x \in E} \theta(x) < \infty, \quad diam(a(x)) \le \frac{1}{4\theta(x)}, \tag{5.7}$$

Then through any point of the graph of the mapping a can be passed a continuous selection y of the mapping a.

Proof. We first assume $int\ a(x) \neq \emptyset$, and show that the mapping a is pointwise continuous. Let $y_0 \in int\ a(x_0)$. Then, according to lower semicontinuity of the mapping a, for any $\varepsilon > 0$ can be chosen a neighborhood U of point x_0 to satisfy $y_0 + B_{2\varepsilon} \subseteq a(x) + B_\varepsilon(0)$ for any $x \in U$. This implies that

$$y_0 + B_\varepsilon(0) \subseteq \bigcap_{s \in B_\varepsilon(0)} (a(x) + B_\varepsilon(0) - s). \tag{5.8}$$

Next, since $a(x)$ satisfies the convexity condition with a constant $\theta(x)$, by Lemma 2.11 of [7] (see also [8], Theorem 5), for $\varepsilon \le 1/16\eta$ the right-hand side of the inclusion (5.8) is equal to $a(x)$. Hence, we have $y_0 + B_\varepsilon(0) \subseteq a(x)\ \forall x \in U$. Now we show that there exists a continuous mapping $\tilde{y}(x)$, such that

$$\tilde{y}(x) \in a(x) + [diam(a(x))]^2 \theta(x) B_1(0).$$

Indeed, let $u_x \in int\ a(x)$. Then, in view of pointwise continuity, there exists a neighborhood $U(x)$, such that $u_x \in int\ a(\overline{x})\ \forall \overline{x} \in U_x \equiv U(x)$.

The family of open neighborhoods $\{U_x\}_{x \in E}$ forms a covering of the compact set E. Let $\{U_{x_j}\}_{j \in J}$ be a finite subcovering from this covering. Consider the partition of unity $\{p_j\}_{j \in J}$, corresponding to this covering, and define a continuous mapping $\tilde{y}$ as follows: $\tilde{y}(x) = \sum_{j \in J} p_j(x) u_j$. Denote $J(x) = \{j \in J : x \in U_{x_j}\}$, and observe that if $x \in U(x_j)$, then $u_j \in a(x)$, and hence we have

$$\tilde{y}(x) = \sum_{j \in J(x)} p_j(x) u_j \in a(x) + \theta(\max_{i,j \in J(x)} \|u_j - u_i\|)^2 B_1(0) \subseteq a(x) + \theta(x)[diam(a(x))]^2 B_1(0).$$

Now if $\theta(x)[diam(a(x))]^2 \le 1/16\theta(x)$, that is, $diam(a(x)) \le 1/4\theta(x)$, then according to Proposition 4.1 there exists a unique projection $y(x)$ of point $\tilde{y}(x)$ on the set $a(x)$. Since the mapping a with compact values is continuous, then the mapping $y(x)$ also is continuous.

Now we consider the general case. We set $b(x) = a(x) + B_\varepsilon(0)$, $\varepsilon < 1/(16\eta)$, and use Proposition 4.2 to conclude that $b(x)$ is almost convex with constant $4\theta(x)$. Also, it is clear that $diam(b(x)) = diama(x) + \epsilon$. Hence, according to above arguments, through any point of the graph of the mapping can be passed a continuous selection of that mapping, provided that

$$diam\ b(x) \le \frac{1}{4(4\theta(x))} = \frac{1}{16\theta(x)}.$$

Therefore, if

$$diam\ a(x) \le \frac{1}{16\theta(x)} - \varepsilon < \frac{1}{4\theta(x)},$$

then there exists a continuous mapping $\overline{y}$ such that $\overline{y}(x_0) = y_0$, $\overline{y}(x) \in b(x)$ for any $x \in E$. It is easy to see that $y(x) = Pr_{a(x)}\overline{y}(x)$ is the desired mapping. Theorem 5.2 is proved.

Remark 5.1. For Example 4.1, the inequality (5.7) in Theorem 5.2 is not satisfied. Indeed, for unit circle S_1 we have $diam(S_1) = 2,\ \theta = 1/\sqrt{3}$, implying that the inequality $diam(S_1) \le 1/4\theta$ is violated. The inequality $sup_{x \in E}\theta(x) < \infty$ is also violated, because Example 3.2 shows that $sup_{x \in E}\theta(x) = \infty$.

REFERENCES

1. Yu.G. Borisovich, B.D. Gel'man, A.D. Myshkis, V.V. Obukhovskii, *Introduction to set-valued mappings and differential embeddings* (KomKniga, Moscow, 2005).
2. F. P. Vasilyev, *Methods of solution of extremal problems* (Nauka, Moscow, 1981).
3. M.A. Krasnosselsky, "Sur un critére pour qu'un domaine soit étoilé", Mat. Sb., **19(61)** (2), 309-310, 1946.
4. E. S. Polovinkin, *Multivalued analysis and differential embeddings* (Fizmatlit, Moscow, 2014).
5. L.S. Pontryagin, "Linear differential games of pursuit", Mat. Sb., **112** (3), 307-330, 1980.
6. J.-P. Aubin and I. Ekeland, *Applied nonlinear analysis* (Mir, Moscow, 1988).
7. V.V. Ostapenko, "Approximate solution of the approach-deviation problem", Preprint-82-16, Inst. Kibernetiki AN USSR, Kiev, 1982.
8. V.V. Ostapenko, "On an almost convexity condition", Ukr. Mat. J., **35** (2), 169-172, 1983.
9. P. V. Semenov, "On paraconvexity of starlike sets", Sib. Mat. Zh., **37** (2), 399-405, 1996.
10. D. Repovs, P.V. Semenov, "Michael's theory of continuous selections. Development and applications", UMN, **49** (6), 151-188, 1994.
11. R.A. Khachatryan, "On directional derivatives of selections of set-valued mappings", Journal of Contemporary Mathem. Analysis (Armenian Academy of Sciences), **51** (3), 65-84, 2016.
12. R.A. Khachatryan, "On existence of continuous and smooth selections of set-valued mappings", Journal of Contemporary Mathem. Analysis (Armenian Academy of Sciences), **37** (2), 65-76, 2002.
13. F. Bernard, L. Thibold, N. Zlateva "Characterizations of prox-regular sets in uniformly convex Banach spaces", Journal Convex Analysis, **13** (3/4), 525-559, 2006.
14. F.H. Clarke, R.J. Stern, P.R. Wolenski, "Proximal Smoothness and the Lower-C^2 Property", Journal of Convex Analysis, **2** (1/2), 117-144, 1995.
15. E. Michael, "Continuous selections 1", Ann. Math., **63**, 361-381, 1956.
16. H. Heremes, "On continuos and measurables selections and the existence of solutions of generalized differential equations", Proc. Amer. Math. Sci., **29** (3), 535–542, 1971.