ISSN 1068-3623, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2018, Vol. 53, No. 4, pp. 187–200. © Allerton Press, Inc., 2018.
Original Russian Text © G. A. Karapetyan, H. A. Petrosyan, 2018, published in Izvestiya Natsional'noi Akademii Nauk Armenii, Matematika, 2018, No. 4, pp. 46-65.

FUNCTIONAL ANALYSIS

# On Solvability of Regular Hypoelliptic Equations in $\mathbb{R}^n$

**G. A. Karapetyan[1*] and H. A. Petrosyan[1**]**

[1] *Russian-Armenian (Slavonic) University, Yerevan, Armenia*[2]

Received 20 March, 2017

**Abstract**—In this paper the unique solvability of regular hypoelliptic equations in multianisotropic weighted functional spaces is proved by means of special integral representation of functions through a regular operator. The existence of the solutions is proved by constructing approximate solutions using multianisotropic integral operators.

**MSC2010 numbers :** 32Q40

**DOI:** 10.3103/S1068362318040015

**Keywords:** *Multianisotropic space; hypoelliptic equation; integral representation; fundamental family of operators.*

## 1. INTRODUCTION

In this paper we study the solvability of a class of regular hypoelliptic equations in $\mathbb{R}^n$. The obtained results generalize the results by G.V. Demidenko [1]-[3], where using a special integral representation (obtained by S.V. Uspenski [4]) were constructed approximate solutions for quasielliptic equations in the whole space. The study of regular hypoelliptic equations is more challenging problem. The issue is that the principal parts of elliptic and quasielliptic operators are homogeneous and generalized homogeneous, respectively, while the principal part of a regular hypoelliptic operator is multi-nonhomogeneous. Notice that the regular operators were introduced and studied by S.M. Nikol'skii [5] and V.P. Mikhailov [6]) (see also [7]). In derivation of our results, we essentially use a special integral representation of functions through a multianisotropic kernel and estimates of such kernels obtained in [8]-[11]. Notice that this approach goes back to the classical work by S.L. Sobolev [12], where has been obtained integral representations of functions through the function itself and its derivatives. Later on, these results were extended for functions belonging to generalized homogeneous spaces (see [13]-[15]).

In the present paper, we prove the unique solvability of regular equations in the special weighted functional spaces. Similar spaces in the case $\sigma = 1$, for elliptic operators have been studied in [16]-[17], and for quasielliptic operators in [18]. For an arbitrary $\sigma \in (0, 1)$, such spaces were introduced and studied by G.V. Demidenko (see [1]).

## 2. APPROXIMATE SOLUTIONS FOR REGULAR EQUATIONS AND THEIR PROPERTIES

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\mathbb{Z}_+^n$ be the set of multiindices from $\mathbb{R}^n$. For $\xi, \eta \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}_+^n$ and $t > 0$ we introduce the following notation: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$, $t^\eta = (t^{\eta_1}, \dots, t^{\eta_n})$, $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$ $(k = 1, \dots, n)$, and let $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ denote the generalized Sobolev derivative of order $\alpha$.

For a given collection of multiindices, by $\mathfrak{N}$ we denote the minimal convex polyhedron containing all points of that collection. The polyhedron $\mathfrak{N}$ is said to be completely regular if $\mathfrak{N}$ has a vertex at the

[2] The research was supported by GKN MON RA under project # SCS 15T-1A 197, and by the thematic financing of Russian-Armenian University from the fund of MON RF.

[*] E-mail: garnik_karapetyan@yahoo.com
[**] E-mail: heghin.petrosyan@gmail.com

origin and vertices on all coordinate axes, and the exterior normals to all $(n-1)$-dimensional non-coordinate sides of $\mathfrak{N}$ have positive coordinates. We denote by $\mathfrak{N}_i^{n-1}$ $(i=1,\dots,I_{n-1},\ I_{n-1}\geq n)$ the $(n-1)$-dimensional non-coordinate sides of the polyhedron $\mathfrak{N}$, by $\partial'\mathfrak{N}$ the set of all multiindices that belong to at least one $(n-1)$-dimensional non-coordinate side of the polyhedron $\mathfrak{N}$, $\mathfrak{N}^{(0)}=\mathfrak{N}\setminus\partial'\mathfrak{N}$, and let $\{\alpha^1,\alpha^2,\dots,\alpha^M\}$ denote the set of all different from zero vertices of the polyhedron $\mathfrak{N}$.

Let $\mu^i$ $(i=1,\dots,I_{n-1})$ be that exterior normal to the side $\mathfrak{N}_i^{n-1}$, for which the equation of the hyperplane containing that side is given by formula $(\alpha,\mu^i)=1$ $(i=1,\dots,I_{n-1})$. In what follows, we will assume that the polyhedron $\mathfrak{N}$ has $(n-1)$-dimensional sides, containing the points $\{\alpha^1,\dots,\alpha^n\}\setminus\{\alpha^i\}$ $(i=1,\dots,n)$, where $\alpha^i=(0,\dots,0,l_i,0,\dots,0)$. The exterior normal to the $i$th side we denote by $\mu^i$ $(i=1,\dots,n)$, and let $\lambda_i=\frac{1}{l_i}$ $(i=1,\dots,n)$ and $\lambda=(\lambda_1,\dots,\lambda_n)$.

Let $\gamma=(\gamma_1,\dots,\gamma_n)$ be the intersection point of hyperplanes, containing the $n$-dimensional sides with exterior normals $\mu^1,\dots,\mu^n$, and, for simplicity, we assume that $\gamma_1<\gamma_2<\dots<\gamma_{n-r}\leq\gamma_{n-r+1}\leq\dots\leq\gamma_n$, where $r=0,1,\dots,n-1$. Consider the differential operator

$$P(D)=\sum_{\alpha\in\partial'\mathfrak{N}}a_\alpha D^\alpha \tag{2.1}$$

with real coefficients $a_\alpha$. Assume that the operator $P(D)$ is regular, that is, there exists a constant $\chi>0$ such that for any $\xi\in\mathbb{R}^n$ the following inequality holds:

$$|P(\xi)|=\left|\sum_{\alpha\in\partial'\mathfrak{N}}a_\alpha\xi^\alpha\right|\geq\chi\sum_{\alpha\in\partial'\mathfrak{N}}|\xi^\alpha|\,. \tag{2.2}$$

For a positive parameter $\nu$ and a natural number $k$, define the functions $G_0(\xi,\nu)=e^{-(\nu P(\xi))^{2k}}$ and $G_1(\xi,\nu)=2ke^{-(\nu P(\xi))^{2k}}(\nu P(\xi))^{2k-1}$, and by $\hat{G}_0(t,\nu)$ and $\hat{G}_1(t,\nu)$ denote the Fourier transforms of $G_0(\xi,\nu)$ and $G_1(\xi,\nu)$, respectively. For functions $\hat{G}_l(t,\nu)$ $(l=0,1)$, we have the following estimates (see [10]).

**Lemma 2.1.** *Let $\gamma_1<\gamma_2<\dots<\gamma_{n-r}\leq\gamma_{n-r+1}\leq\dots\leq\gamma_n$ $(r=0,1,\dots,n-1)$. Then for any multiindex $m=(m_1,m_2,\dots,m_n)$ and for any even number $N$ $(N>N_0)$ there exist constants $C_i$ $(i=0,1,\dots,n-1)$, such that for every $\nu:0<\nu<1$ the following inequalities are satisfied:*

$$\left|D^m\hat{G}_l\left(t,\nu\right)\right|\leq\frac{\nu^{-\max\limits_{i=1,\dots,I_{n-1}}\left(|\mu^i|+(m,\mu^i)\right)}(C_{n-1}|\ln\nu|^{n-1}+\dots+C_1|\ln\nu|+C_0)}{(1+\nu^{-N}(t^{N\gamma}+t^{N\beta}+\dots+t^{N\sigma}))\dots(1+\nu^{-N}(t^{N\gamma}+t^{N\delta}+\dots+t^{N\tau}))}, \tag{2.3}$$

*where $(\{\gamma,\beta,\dots,\sigma\},\dots,\{\gamma,\delta,\dots,\tau\})$ is some set of $n$ vectors and $l=0,1$.*

**Lemma 2.2.** *Let the vector $\gamma$ be as in Lemma 2.1. Then there exist constants $C_i$ $(i=0,1,\dots,l)$ and a natural number $N_0$, such that for any number $N:N>N_0$ and any $\nu:0<\nu<1$ the following inequality holds:*

$$\int_0^\infty\dots\int_0^\infty\frac{dt_1dt_2\dots dt_n}{(1+\nu^{-N}(t^{N\gamma}+t^{N\beta}+\dots+t^{N\sigma}))\dots(1+\nu^{-N}(t^{N\gamma}+t^{N\delta}+\dots+t^{N\tau}))}$$

$$\leq\nu^{\min\limits_{i=1,\dots,r+1}|\mu^i|}\left(C_l|\ln\nu|^l+\dots+C_1|\ln\nu|+C_0\right), \tag{2.4}$$

*here $l$ is the number of equalities between the coordinates of the vector $\gamma=(\gamma_1,\dots,\gamma_n)$.*

**Lemma 2.3.** *There exists a constant $C>0$ such that for any $\nu>1$ the following inequality holds:*

$$\left|D^m\hat{G}_l(t,\nu)\right|\leq\frac{C\nu^{-(|\lambda|+(m,\lambda))}}{1+\nu^{-N}|t|_\lambda^N}, \tag{2.5}$$

*where $|t|_\lambda=\left(t_1^{2l_1}+\dots+t_n^{2l_n}\right)^{1/2}$, and $l=0,1$.*

For a function $f \in L_p(\mathbb{R}^n)$, we denote (see [8])

$$U_h(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_h^{h^{-1}} d\nu \int_{\mathbb{R}^n} f(t) \int_{\mathbb{R}^n} e^{-i(t-x,\xi)} G_1(\xi,\nu) d\xi dt. \tag{2.6}$$

By means of the vertices $\alpha^i : \alpha^i \neq 0$ $(i = 1, \dots, M)$ of the polyhedron $\mathfrak{N}$ we introduce the multianisotropic distance: $\rho_{\mathfrak{N}}(x) = \left(\sum_{i=1}^M x^{2\alpha^i}\right)^{1/2}$ and the weighted spaces $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$, which are the completions of the space $C_0^\infty(\mathbb{R}^n)$ by the norm

$$\|U\|_{W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)} = \sum_{\alpha \in \mathfrak{N}} \left\| (1 + \rho_{\mathfrak{N}}(x))^{-\sigma(1 - \max_i (\mu^i, \alpha))} D_x^\alpha U(x) \right\|_{L_p(\mathbb{R}^n)}, \tag{2.7}$$

where $0 < \sigma < 1$. Let $L_{p,\gamma}(\mathbb{R}^n)$ be the space of summable functions, having finite norm

$$\|U\|_{L_{p,\gamma}(\mathbb{R}^n)} = \left\| (1 + \rho_{\mathfrak{N}}(x))^{-\gamma} U(x) \right\|_{L_p(\mathbb{R}^n)}.$$

Denote by $\mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n$ the subspace of functions $f \in L_p(\mathbb{R}^n) \bigcap L_{1,\gamma}(\mathbb{R}^n)$, $\gamma = -(\sigma + N|\lambda|)$, such that $\int_{\mathbb{R}^n} x^\beta f(x) dx = 0$, $|\beta| = 0, 1, \dots, N-1$.

**Lemma 2.4.** *Let $\beta \in \partial' \mathfrak{N}$. Then there exists a constant $C > 0$ such that for every $f \in L_p(\mathbb{R}^n) \bigcap L_1(\mathbb{R}^n)$*

$$\left\| D^\beta U_h \right\|_{L_p(\mathbb{R}^n)} \le C \|f\|_{L_p(\mathbb{R}^n)} \quad 0 < h < 1, \tag{2.8}$$

$$\left\| D^\beta U_{h_1} - D^\beta U_{h_2} \right\|_{L_p(\mathbb{R}^n)} \le \varepsilon(h_1, h_2) \|f\|_{L_p(\mathbb{R}^n)} \quad \text{for} \quad 0 < h_1 < h_2 < 1, \tag{2.9}$$

*where $\varepsilon(h_1, h_2) \to 0$ as $h_1, h_2 \to 0$.*

**Proof.** Let $\beta \in \partial' \mathfrak{N}$, that is, there exists $\mu^{i_0}$ such that $(\beta, \mu^{i_0}) = 1$. From the representation of $U_h(x)$ we have

$$D_x^\beta U_h(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_h^{h^{-1}} d\nu \int_{\mathbb{R}^n} f(t) \int_{\mathbb{R}^n} e^{-i(t-x,\xi)} \xi^\beta G_1(\xi,\nu) d\xi dt.$$

Hence, applying Fubini's theorem, we get

$$D_x^\beta U_h(x) = \int_h^{h^{-1}} d\nu \int_{\mathbb{R}^n} \hat{f}(\xi) \xi^\beta e^{i(x,\xi)} G_1(\xi,\nu) d\xi.$$

Again applying Fubini's theorem, we obtain

$$D_x^\beta U_h(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) \xi^\beta e^{i(x,\xi)} (2k) \int_h^{h^{-1}} (\nu P(\xi))^{2k-1} e^{-(\nu P(\xi))^{2k}} d\nu d\xi$$

$$= \int_{\mathbb{R}^n} (2k) \frac{\hat{f}(\xi) \xi^\beta}{P(\xi)} e^{i(x,\xi)} \int_{hP(\xi)}^{h^{-1}P(\xi)} t^{2k-1} e^{-t^{2k}} dt d\xi = (2\pi)^{\frac{n}{2}} \widetilde{F_h(\xi) \hat{f}(\xi)}, \tag{2.10}$$

where $\tilde{f}$ is the inverse Fourier transform, and

$$F_h(\xi) = (2k) \frac{\xi^\beta}{P(\xi)} \int_{hP(\xi)}^{h^{-1}P(\xi)} t^{2k-1} e^{-t^{2k}} dt.$$

It follows from (2.10) that, the inequality (2.8) will be satisfied with some constant $C > 0$, if we prove that the function $F_h(\xi)$ is a $(L_p, L_p)$-multiplicator (see [19]), which is uniformly bounded in $h$. We have

$$|F_h(\xi)| \le C \left| \frac{\xi^\beta}{P(\xi)} \right| \cdot \int_0^\infty t^{2k-1} e^{-t^{2k}} dt \le M,$$

because $\xi^\beta\ P(\xi)$ is a multiplicator for $\beta \in \partial'\mathfrak{N}$ (see [20]). Hence, for some constant $M_1 > 0$ and any $\xi \in \mathbb{R}^n$, we have $\left|\xi^\beta/P(\xi)\right| \leq M_1$. Since the product of two multiplicators is again a multiplicator, it is enough to show that $\int_{hP(\xi)}^{h^{-1}P(\xi)} t^{2k-1}e^{-t^{2k}}dt$ is a multiplicator. To this end, we estimate

$$\left|\xi_i D_{\xi_i}\int_{hP(\xi)}^{h^{-1}P(\xi)} t^{2k-1}e^{-t^{2k}}dt\right|$$

$$\leq \left|\xi_i h^{-1}P'_{\xi_i}(\xi)\big(h^{-1}P(\xi)\big)^{2k-1}e^{-\left(h^{-1}P(\xi)\right)^{2k}}\right| + \left|\xi_i hP'_{\xi_i}(\xi)(hP(\xi))^{2k-1}e^{-(hP(\xi))^{2k}}\right|$$

$$= \left|\frac{\xi_i P'_{\xi_i}(\xi)}{P(\xi)}\right|\big(h^{-1}P(\xi)\big)^{2k}e^{-\left(h^{-1}P(\xi)\right)^{2k}} + \left|\frac{\xi_i P'_{\xi_i}(\xi)}{P(\xi)}\right|(hP(\xi))^{2k}e^{-(hP(\xi))^{2k}} \leq C,$$

where $C$ is a constant, independent of $h$, and $i = 1, \ldots, n$. Similarly, it can be shown that

$$\left|\xi_1^{k_1}\ldots\xi_n^{k_n}\frac{\partial^{k_1+\cdots+k_n}}{\partial\xi_1^{k_1}\ldots\partial\xi_n^{k_n}}F_h(\xi)\right| \leq M,$$

where $k_i$ $(i = 1, \ldots, n)$ are equal to 0 or 1. Thus, the conditions of Lizorkin's theorem are satisfied (see [19]), implying that $F_h(\xi)$ is a $(L_p, L_p)$-multiplicator, and hence the inequality (2.8) is satisfied with some constant $C > 0$.

Now we prove the inequality (2.9). Taking into account that

$$\int_{\mathbb{R}^n} D_t^m\hat{G}_1(t,\nu)t^\alpha dt = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} t^\alpha\int_{\mathbb{R}^n} e^{-i(t,\xi)}(-\xi)^m G_1(\xi,\nu)d\xi dt$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} D_\xi^\alpha((-\xi)^m G_1(\xi,\nu))e^{-i(t,\xi)}d\xi\right)dt = (-1)^{|\alpha|}(2\pi)^{\frac{n}{2}}D_\xi^\alpha((-\xi)^m G_1(\xi,\nu))|_{\xi=0},$$

the number $k$ can be chosen to satisfy

$$\int_{\mathbb{R}^n} D_t^m\hat{G}_1(t,\nu)t^\alpha dt = (-1)^{|\alpha|}(2\pi)^{\frac{n}{2}}D_\xi^\alpha\left((-\xi)^m e^{-(\nu P(\xi))^{2k}}(2k)(\nu P(\xi))^{2k-1}\right)|_{\xi=0} = 0 \qquad (2.11)$$

for any $\alpha : |\alpha| \leq l$, where $l$ is a given number.

Let $\varepsilon > 0$ be an arbitrary number. We choose a function $\tilde{f} \in C_0^\infty(\mathbb{R}^n)$ to satisfy $\left\|f - \tilde{f}\right\|_{L_p(\mathbb{R}^n)} < \varepsilon$ and $\left\|D^\alpha\tilde{f}\right\|_{L_p(\mathbb{R}^n)} \leq C_{\alpha,\varepsilon}\|f\|_{L_p(\mathbb{R}^n)}$ for any $\alpha : |\alpha| \leq l$. Assuming that $h_1 < h_2 < 1$, we can write

$$\left\|D^\beta U_{h_1} - D^\beta U_{h_2}\right\|_{L_p(\mathbb{R}^n)} \leq \left\|\int_{h_1}^{h_1^{-1}} d\nu\int_{\mathbb{R}^n} D^\beta\hat{G}_1(t-\cdot,\nu)[f-\tilde{f}]dt\right\|_{L_p(\mathbb{R}^n)}$$

$$+\left\|\int_{h_2}^{h_2^{-1}} d\nu\int_{\mathbb{R}^n} D^\beta\hat{G}_1(t-\cdot,\nu)[f-\tilde{f}]dt\right\|_{L_p(\mathbb{R}^n)} + \left\|\int_{h_2^{-1}}^{h_1^{-1}} d\nu\int_{\mathbb{R}^n} D^\beta\hat{G}_1(t-\cdot,\nu)\tilde{f}dt\right\|_{L_p(\mathbb{R}^n)}$$

$$+\left\|\int_{h_1}^{h_2} d\nu\int_{\mathbb{R}^n} D^\beta\hat{G}_1(t-\cdot,\nu)\tilde{f}dt\right\|_{L_p(\mathbb{R}^n)} =: I_1 + I_2 + I_3 + I_4.$$

Now we estimate the terms $I_i$ $(i = 1, 2, 3, 4)$ separately. Observe first that by the already proved inequality (2.8), for $I_1$ and $I_2$, we have $I_1, I_2 \leq C\left\|f - \tilde{f}\right\|_{L_p(\mathbb{R}^n)}$.

Next, we estimate $I_3$. Since $1 < h_2^{-1} < h_1^{-1}$, we can apply Young's inequality and the estimate (2.5) for $\hat{G}_1(t,\nu)$ with $\nu > 1$ (see Lemma 2.3), to obtain

$$I_3 \le C\int_{h_2^{-1}}^{\infty}\left\|D^\beta \hat{G}_1(\cdot,\nu)\right\|_{L_p(\mathbb{R}^n)} d\nu\cdot\left\|\tilde{f}\right\|_{L_1(\mathbb{R}^n)} \le C\int_{h_2^{-1}}^{\infty}\nu^{-|\lambda|-(\lambda,\beta)+\frac{|\lambda|}{p}}\left\|\frac{1}{1+|t|_\lambda^N}\right\|_{L_p(\mathbb{R}^n)} d\nu\cdot\|f\|_{L_p(\mathbb{R}^n)}$$

$$\le Ch_2^{|\lambda|+(\lambda,\beta)-\frac{|\lambda|}{p}-1}\|f\|_{L_p(\mathbb{R}^n)}\to 0 \quad as \quad h_2\to 0,$$

because $(\lambda,\beta)\ge 1$. For $I_4$ we have

$$I_4 = \left\|\int_{h_1}^{h_2} d\nu\int_{\mathbb{R}^n} D_t^\beta \hat{G}_1(t,\nu)\tilde{f}(x+t)dt\right\|_{L_p(\mathbb{R}^n)}$$

$$= \left\|\int_{h_1}^{h_2} d\nu\int_{\mathbb{R}^n} D_t^\beta \hat{G}_1(t,\nu)\left[\tilde{f}(x+t)-\sum_{|\alpha|\le l}\frac{t^\alpha}{\alpha!}\tilde{f}^{(\alpha)}(x)\right]dt\right\|_{L_p(\mathbb{R}^n)},$$

where, in view of (2.11), all the terms in the square brackets, except $\tilde{f}(x+t)$, vanish after integration. Taking into account that by Taylor formula the expression in the square brackets is equal to $\sum\limits_{|\alpha|=l+1}\frac{t^\alpha}{\alpha!}\tilde{f}^{(\alpha)}(t+\theta_\alpha(t)x)$, from (2.3) and (2.4), for $\nu < 1$ we obtain

$$\left|t^\alpha D^\beta \hat{G}_1(t,\nu)\right| \le \frac{\nu^{\max\limits_i(-|\mu^i|-(\beta,\mu^i)+(\alpha,\mu^i))}(C_{n-1}|\ln\nu|^{n-1}+...+C_1|\ln\nu|+C_0)}{(1+\nu^{-N}(t^{N\gamma}+t^{N\beta}+...+t^{N\sigma}))...(1+\nu^{-N}+(t^{N\gamma}+t^{N\delta}+...+t^{N\tau}))}.$$

Therefore, applying Young's inequality, for $I_4$ we have

$$I_4 \le C\sum_{|\alpha|=l+1}\int_0^{h_2}\left\|t^\alpha D_t^\beta \hat{G}_1(t,\nu)\right\|_{L_1(\mathbb{R}^n)} d\nu\cdot\left\|\tilde{f}^{(\alpha)}\right\|_{L_p(\mathbb{R}^n)}$$

$$\le \sum_{|\alpha|=l+1} h_2^{-\max\limits_i(|\mu^i|+(\beta,\mu^i)-(\alpha,\mu^i))+\min\limits_{i=1,\dots,r+1}|\mu^i|+1}(a_{\alpha,n+l-1}|\ln h_2|^{n+l-1}+...+a_{\alpha,1}|\ln h_2|+a_{\alpha,0})\|f\|_{L_p(\mathbb{R}^n)}$$

for some constants $a_{\alpha,0},a_{\alpha,1},...,a_{\alpha,n+l-1}$. Since $l$ is arbitrary, it can be chosen so that the function of $h_2$ in the last formula tends to zero as $h_2\to 0$. Then, we have $I_4\to 0$ as $h_2\to 0$, and the result follows. Lemma 2.4 is proved.

We assume that for polyhedron $\mathfrak{N}$ the following condition is satisfied:

$$\max_{\substack{i=1,\dots,I_{n-1}\\ \beta\in\mathfrak{N}^{(0)}}}(|\mu^i|+(\beta,\mu^i)) - \min_{i=1,\dots,r+1}|\mu^i| < 1.$$

**Lemma 2.5.** *Let $f\in L_p(\mathbb{R}^n)$, $(1+\rho_{\mathfrak{N}}(x))^\sigma f\in L_1(\mathbb{R}^n)$, $|\lambda|>1$, $\frac{|\lambda|}{p}>\sigma>1-\frac{|\lambda|}{p'}$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. Then for any $\beta\in\mathfrak{N}^{(0)}$ the following inequalities hold: for $0<h<1$,*

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_j(\beta,\mu^j))}D_x^\beta U_h(x)\right\|_{L_p(\mathbb{R}^n)}$$

$$\le C\left(\|f\|_{L_p(\mathbb{R}^n)}+\left\|(1+\rho_{\mathfrak{N}}(x))^{\sigma(1-\max\limits_j(\beta,\mu^j))}f(x)\right\|_{L_1(\mathbb{R}^n)}\right), \tag{2.12}$$

*and for* $0 < h_1 < h_2 < 1$,

$$\left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_j(\beta,\mu^j))} (D_x^\beta U_{h_1}(x) - D_x^\beta U_{h_2}(x)) \right\|_{L_p(\mathbb{R}^n)}$$

$$\leq \varepsilon(h_1,h_2) \left( \|f\|_{L_p(\mathbb{R}^n)} + \left\| (1+\rho_{\mathfrak{N}}(x))^{\sigma(1-\max\limits_j(\beta,\mu^j))} f(x) \right\|_{L_1(\mathbb{R}^n)} \right), \tag{2.13}$$

*where* $\varepsilon(h_1,h_2) \to 0$ *as* $h_1, h_2 \to 0$.

**Proof.** By Minkowski inequality we have

$$\left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_j(\beta,\mu^j))} D_x^\beta U_h(x) \right\|_{L_p(\mathbb{R}^n)}$$

$$\leq \left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_j(\beta,\mu^j))} \int_h^1 d\nu \int_{\mathbb{R}^n} f(t) D_x^\beta \hat{G}_1(t-x,\nu)dt \right\|_{L_p(\mathbb{R}^n)}$$

$$+ \left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_j(\beta,\mu^j))} \int_1^{h^{-1}} d\nu \int_{\mathbb{R}^n} f(t) D_x^\beta \hat{G}_1(t-x,\nu)dt \right\|_{L_p(\mathbb{R}^n)} = I_1 + I_2.$$

Now we estimate the terms $I_1$ and $I_2$ separately. Applying (2.3) and (2.4) for function $\hat{G_1}(t,\nu)$ with $0 < \nu < 1$ and Young's inequality, for $I_1$ we obtain

$$I_1 \leq C \int_h^1 d\nu \left\| D_t^\beta \hat{G}_1(t,\nu) \right\|_{L_1(\mathbb{R}^n)} \cdot \|f\|_{L_p(\mathbb{R}^n)} \leq \int_h^1 d\nu\, \nu^{-\max\limits_j(|\mu^j|+(\beta,\mu^j))} (C_{n-1}|\ln\nu|^{n-1} + \ldots$$

$$+ C_1|\ln\nu| + C_0) \int_{\mathbb{R}^n} \frac{dt_1 \ldots dt_n}{(1+\nu^{-N}(t^{N\gamma}+\ldots+t^{N\sigma}))\ldots(1+\nu^{-N}(t^{N\gamma}+\ldots+t^{N\tau}))} \cdot \|f\|_{L_p(\mathbb{R}^n)}$$

$$\leq \int_h^1 \nu^{-\max\limits_j(|\mu^j|+(\beta,\mu^j)) + \min\limits_{j=1,\ldots,r+1}|\mu^j|} (C_{n+l-1}|\ln\nu|^{n+l-1} + \ldots + C_1|\ln\nu| + C_0) d\nu \|f\|_{L_p(\mathbb{R}^n)} \leq C\|f\|_{L_p(\mathbb{R}^n)}.$$

To estimate $I_2$, we use the inequality $\rho_{\mathfrak{N}}(x-y)(1+\rho_{\mathfrak{N}}(x))^{-1} \leq a(1+\rho_{\mathfrak{N}}(y))$ and Young's inequality, to obtain

$$I_2 \leq C \int_1^{h^{-1}} d\nu \left\| \int_{\mathbb{R}^n} \rho_{\mathfrak{N}}(x-t)^{-\sigma(1-\max\limits_j(\beta,\mu^j))} D_x^\beta \hat{G}_1(t-x,\nu)(1+\rho_{\mathfrak{N}}(t))^{\sigma(1-\max\limits_j(\beta,\mu^j))} f(t)dt \right\|_{L_p(\mathbb{R}^n)}$$

$$\leq C \int_1^{h^{-1}} d\nu \left\| \rho_{\mathfrak{N}}(x)^{-\sigma(1-\max\limits_j(\beta,\mu^j))} \int_{\mathbb{R}^n} e^{i(x,\xi)} \xi^\beta G_1(\xi,\nu)d\xi \right\|_{L_p(\mathbb{R}^n)} \left\| (1+\rho_{\mathfrak{N}}(t))^{\sigma(1-\max\limits_j(\beta,\mu^j))} f(t) \right\|_{L_1(\mathbb{R}^n)}. \tag{2.14}$$

Using the inequality (2.5) for $\nu > 1$, the first factor on the right-hand side of (2.14) can be estimated from above by

$$C \left\| \left( \sqrt{x_1^{2l_1} + \ldots + x_n^{2l_n}} \right)^{-\sigma(1-\max\limits_j(\beta,\mu^j))} \frac{\nu^{-(|\lambda|+(\beta,\lambda))}}{1+\nu^{-N}|x|_\lambda^N} \right\|_{L_p(\mathbb{R}^n)}.$$

Hence, making a change of variable $x = \nu^\lambda \eta$ and taking into account that $N > N_0$, the first factor on the right-hand side of (2.14) can be estimated from above by the integral

$$C \int_1^{h^{-1}} \frac{d\nu}{\nu^{|\lambda|+(\beta,\lambda)+\sigma(1-\max\limits_j(\beta,\mu^j)-\frac{|\lambda|}{p}}}.$$

Next, since $\sigma p < |\lambda|$, $1 - \frac{|\lambda|}{p'} < \sigma$, $|\lambda| > \max_j |\mu^j|$, $\sigma < 1$, then $\frac{|\lambda|}{p'} + \sigma + (\beta, \lambda) - \sigma \max_j (\beta, \mu^j) > 1$, and hence the integral of interest converges, and for $I_2$ we get the estimate

$$I_2 \leq C \left\| (1 + \rho_{\mathfrak{N}}(x))^{\sigma(1 - \max_j (\beta, \mu^j))} f(x) \right\|_{L_1(\mathbb{R}^n)}.$$

Thus, the inequality (2.12) is proved. The inequality (2.13) can be proved similarly, using the arguments applied in the proof of Lemma 2.4. Lemma 2.5 is proved.

**Proposition 2.1.** [7] *Let $\theta(\xi) = \sum_\alpha \gamma_\alpha \xi^\alpha$ be a polynomial with constant coefficients, and let $\mathfrak{N}(\theta) = \{\alpha \in \mathbb{Z}_+^n, \gamma_\alpha \neq 0\}$. A necessary and sufficient condition for existence of a constant $C > 0$ to satisfy the inequality $|\theta(\xi)| \leq C \rho_{\mathfrak{N}}(\xi)$ for every $\xi \in \mathbb{R}^n$ with $\rho_{\mathfrak{N}}(\xi) > 1$ is that $\mathfrak{N}(\theta) \subset \mathfrak{N}$.*

The number $c_0 = \min_{1 \leq l \leq n} \frac{\min_{1 \leq j \leq I_{n-1}} \mu_l^j}{\max_{1 \leq j \leq I_{n-1}} \mu_l^j}$ is called the regularity index of an operator $P(D)$.

**Lemma 2.6.** *A necessary and sufficient condition for fulfillment of the inequality*

$$\sum_{\alpha \in \mathbb{Z}_+^n} \left| \frac{P^{(\alpha)}(\xi)}{P(\xi)} \right| < A |P(\xi)|^{-c \max_{1 \leq j \leq I_{n-1}} (\alpha, \mu^j)} \tag{2.15}$$

*for every $\xi \in \mathbb{R}^n$ with $|P(\xi)| > 1$, and some positive constants $c$ and $A$ is that $c \leq c_0$, where $c_0$ is the regularity index of operator $P(D)$.*

**Proof.** Observe first that if $|P(\xi)| > 1$, then for every $c > 0$ and $\alpha \in \mathbb{Z}_+^n$ with some constant $C > 1$, we have

$$\frac{1}{C} \sum_{j=1}^{M} \left| \xi^{\alpha^j} \right|^{1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l)} \leq |P(\xi)|^{1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l)} \leq C \sum_{j=1}^{M} \left| \xi^{\alpha^j} \right|^{1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l)}.$$

Hence, in view of Proposition 2.1, the estimate (2.15) is equivalent to the following embedding:

$$\mathfrak{N}\left(P^{(\alpha)}\right) \subset \mathfrak{N}\left(1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l)\right) \quad \text{for any} \quad \alpha \in \mathbb{Z}_+^n. \tag{2.16}$$

So, to prove the lemma it is enough to show that the embedding (2.16) holds if and only if $c \leq c_0$. We first show that the fulfilment of (2.16) implies $c \leq c_0$. Let $\alpha = e^j = (0, ..., 0, \underbrace{1}_{j}, 0, ..., 0)$. Then for any $i, j$: $1 \leq i \leq I_{n-1}$, $1 \leq j \leq n$ there is a vertex $\beta$ on $\mathfrak{N}_i^{n-1}$ such that $(\beta, \mu^i) = 1$ and $\beta_j \geq 1$.

Since $\beta - \alpha = \beta - e^j \in \mathfrak{N}\left(P^{(\alpha)}\right)$, by (2.16) we have

$$\mathfrak{N}(P^{(\alpha)}) \subset \mathfrak{N}\left(1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l)\right), \quad (\beta - \alpha, \mu^i) \leq 1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l)$$

for any $i : i \leq 1 \leq I_{n-1}$, that is,

$$1 - \mu_j^i = (\beta, \mu^i) - (\alpha, \mu^i) = (\beta - \alpha, \mu^i) \leq 1 - c \max_{1 \leq l \leq I_{n-1}} (\alpha, \mu^l) \leq 1 - c \max_{1 \leq l \leq I_{n-1}} \mu_j^l.$$

This implies that for any $1 \leq l \leq I_{n-1}$ and $1 \leq j \leq n$, $c \leq \frac{\mu_j^l}{\max_{1 \leq l \leq I_{n-1}} \mu_j^l}$. Therefore, we have

$$c \leq \min_{1 \leq j \leq n} \frac{\min_{1 \leq l \leq I_{n-1}} \mu_j^l}{\max_{1 \leq l \leq I_{n-1}} \mu_j^l} = c_0,$$

showing that the fulfilment of (2.16) implies $c \leq c_0$.

Now we proceed to prove the converse assertion, that is, the condition $c \le c_0$ implies (2.16). Let $\alpha \in \mathbb{Z}^n_+$. Since $\mathfrak{N}(P^{(\alpha)}) \subset \mathfrak{N}\{\beta-\alpha, \beta \in (P^{(\alpha)}), \beta \ge \alpha\}$, for any $l : 1 \le l \le I_{n-1}$ we have

$$(\beta-\alpha, \mu^l) = (\beta, \mu^l) - (\alpha, \mu^l) \le 1 - (\alpha, \mu^l) = 1 - c_0\left(\alpha, \frac{\mu^l}{c_0}\right).$$

In view of definition of the number $c_0$, for any $l, i : 1 \le l, i \le I_{n-1}$, we have $\frac{\mu^l}{c_0} \ge \mu^i$. Therefore, for any $l, i : 1 \le l, i \le I_{n-1}$, we obtain

$$(\beta-\alpha, \mu^l) = 1 - c_0\left(\alpha, \frac{\mu^l}{c_0}\right) \le 1 - c_0(\alpha, \mu^i) \le 1 - c_0 \max_{1 \le i \le I_{n-1}} (\alpha, \mu^i),$$

that is, for any $l : 1 \le l \le I_{n-1}$,

$$(\beta-\alpha, \mu^l) \le 1 - c_0 \max_{1 \le i \le I_{n-1}} (\alpha, \mu^i) \le 1 - c \max_{1 \le i \le I_{n-1}} (\alpha, \mu^i)$$

implying that, for all $\alpha \in \mathbb{Z}^n_+$ and $\beta \in (P)$, $\beta - \alpha \in \mathfrak{N}(1 - c \max\limits_{1 \le i \le I_{n-1}} (\alpha, \mu^i))$. Taking into account that $\mathfrak{N}(1 - c \max\limits_{1 \le i \le I_{n-1}} (\alpha, \mu^i))$ is a convex polyhedron, and $(P^{(\alpha)}) = \{\beta-\alpha, \beta \in (P), \beta \ge \alpha\}$, we conclude that for any $\alpha \in \mathbb{Z}^n_+$, $\mathfrak{N}(P^{(\alpha)}) \subset \mathfrak{N}(1 - c \max\limits_{1 \le i \le I_{n-1}} (\alpha, \mu^i))$, and the result follows. Lemma 2.6 is proved.

Let $\chi(s) = \begin{cases} 1 \text{ for } 0 \le s \le 1 \\ 0 \text{ for } s \ge 2 \end{cases}$ and $\chi \in C^\infty(\overline{\mathbb{R}^1_+})$.

**Lemma 2.7.** *Under the conditions of Lemma 2.5, for any $h$ and $\sigma$ $(0 < h < 1, 0 \le \sigma < 2c_0)$, and for any multiindex $\beta \in \mathfrak{N}$ as $\rho \to \infty$*

$$\left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))} \left( D^\beta_x \left( U_h(x) - U_h(x)\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right)\right)\right\|_{L_p(\mathbb{R}^n)} \to 0. \tag{2.17}$$

**Proof.** Let $\beta = 0$. Then by the definition of function $\chi(s)$ we have

$$\left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma} \left( U_h(x) - U_h(x)\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right)\right\|_{L_p(\mathbb{R}^n)} \le \left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma} U_h(x)\right\|_{L_p(\rho_{\mathfrak{N}}(x) > \rho)} \to 0$$

as $\rho \to \infty$, because by Lemma 2.5, $U_h \in L_{p,\sigma}(\mathbb{R}^n)$.

Now let $\beta \ne 0$. Then by Leibnitz formula we have

$$D^\beta_x \left( U_h(x) - U_h(x)\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right) = D^\beta_x U_h(x)\left(1 - \chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right)$$

$$-U_h(x) D^\beta_x \chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right) - \sum_{\substack{s+q=\beta \\ |s|,|q|>0}} C_{s,q} D^s_x U_h(x) D^q_x \chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right) =: \Phi_{1,\rho} + \Phi_{2,\rho} + \Phi_{3,\rho}.$$

Now we estimate the terms $\Phi_{i,\rho}$ $(i = 1, 2, 3)$ separately. Using Lemmas 2.4 and 2.5, and the arguments applied in [1], for $\Phi_{1,\rho}$ we have

$$\left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))} \Phi_{1,\rho}(x)\right\|_{L_p(\mathbb{R}^n)} \le \left\| (1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))} D^\beta_x U_h(x)\right\|_{L_p(\rho_{\mathfrak{N}}(x) > \rho)} \to 0$$

as $\rho \to \infty$, because $D^\beta U_h \in L_{p,\sigma}$ for any $\beta \in \mathfrak{N}$.

To estimate $\Phi_{2,\rho}$ and $\Phi_{3,\rho}$, we use Frankel's formula for derivative of a composite function (see [21]), to obtain

$$D^\beta \chi(\varphi(x_1, ..., x_n)) = \sum_{i=1}^{|\beta|} \chi^{(i)}_s(s)|_\varphi \cdot Q_{\beta,i}(\varphi), \tag{2.18}$$

where $Q_{\beta,i}(\varphi)$ is a homogeneous polynomial of degree $i$ of the form:

$$Q_{\beta,i}(\varphi)=\sum_{r^1+...+r^n=\beta} P^i_{r^1}(\varphi)...P^i_{r^n}(\varphi),\quad i=1,...,|\beta|.$$

Observe that for $\varphi(x_1,...,x_n)=\rho^2_{\mathfrak{N}}(x)/\rho^2$ each $P^l_{r^k}$ $(l-1,...,|\beta|)$ is given by

$$P^l_{r^k}\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)=\sum_{\theta\in R(r^k)}\frac{l!}{\theta!}\left(D^{\alpha^1}\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right)^{\theta_1}...\left(D^{\alpha^l}\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right)^{\theta_l},$$

where $R(r^k)=\{\theta;\sum_{i=1}^{l}\theta_i\alpha^i=r^k;|\theta|=l\}$, and $\alpha^1,...,\alpha^l$ are vectors such that $0<\alpha^i\le r^k$ $(i=1,...,l)$.

By Lemma 2.6, for any $i=1,...,l$ and $\rho_{\mathfrak{N}}(x)>1$, we have

$$\left(D^{\alpha^i}\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right)^{\theta_i}\le C\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)^{\theta_i}\left(\rho^2_{\mathfrak{N}}(x)\right)^{-c_0\theta_i\max\limits_j(\alpha^i,\mu^j)}.$$

Since

$$\sum_i\theta_i\max_j(\alpha^i,\mu^j)\ge\max_j\sum_i(\theta_i\alpha^i,\mu^j)=\max_j(r^k,\mu^j),$$

we have

$$\left|D^\beta_x\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right|\le\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)^{l}\left(\rho^2_{\mathfrak{N}}(x)\right)^{-c_0\max\limits_j(\beta,\mu^j)},$$

where, in view of definition of function $\chi(s)$, the variables $x_1,...,x_n$ vary in the compact $\overline{K_\rho}=\{x\in\mathbb{R}^n;\rho\le\rho_{\mathfrak{N}}(x)\le\sqrt{2}\rho\}$. Therefore, by Lemma 2.6, for any $\beta\in\mathfrak{N}$ there exists a constant $C>0$ such that

$$\left|D^\beta_x\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right|\le C\rho^{-2c_0\max\limits_i(\beta,\mu^i)},\quad\rho\ge1.\tag{2.19}$$

Now we use (2.19), to estimate $\Phi_{2,\rho}$ and $\Phi_{3,\rho}$. Taking into account that the function $\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)$ is different from zero only in the compact $\overline{K_\rho}$, and all the derivatives of function $\chi(s)$ are bounded, for $\Phi_{2,\rho}$ we have

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}\Phi_{2,\rho}(x)\right\|_{L_p(\mathbb{R}^n)}$$

$$\le\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}U_h(x)D^\beta_x\chi\left(\frac{\rho^2_{\mathfrak{N}}(x)}{\rho^2}\right)\right\|_{L_p(K_\rho)}$$

$$\le C\rho^{-2c_0\max\limits_i(\beta,\mu^i)}(1+\sqrt{2}\rho)^{\sigma\max\limits_i(\beta,\mu^i))}\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma}U_h(x)\right\|_{L_p(K_\rho)}.$$

Taking into account that by Lemma 2.5, $U_h\in L_{p,\sigma}(\mathbb{R}^n)$, for $\sigma<2c_0$ we obtain

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}\Phi_{2,\rho}(x)\right\|_{L_p(\mathbb{R}^n)}\to0\quad as\quad\rho\to\infty.$$

Next, using the inequality $\max\limits_i(s+q,\mu^i)\le\max\limits_i(s,\mu^i)+\max\limits_i(q,\mu^i)$, for $\Phi_{3,\rho}(x)$ we can write

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}\Phi_{3,\rho}(x)\right\|_{L_p(\mathbb{R}^n)}$$

$$\leq C\sum_{s+q=\beta}\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}D_x^sU_h(x)D_x^q\chi\left(\frac{\rho_{\mathfrak{N}}^2(x)}{\rho^2}\right)\right\|_{L_p(K_\rho)}$$

$$\leq C\sum_{s+q=\beta}\rho^{-2c_0\max\limits_i(q,\mu^i)}\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(s+q,\mu^i))}D_x^sU_h(x)\right\|_{L_p(K_\rho)}$$

$$\leq C\sum_{s+q=\beta}\rho^{-2c_0\max\limits_i(q,\mu^i)}\rho^{\sigma\max\limits_i(q,\mu^i)}\left(\sqrt{2}\right)^{\sigma\max\limits_i(q,\mu^i)}\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(s,\mu^i))}D_x^sU_h(x)\right\|_{L_p(K_\rho)},$$

Therefore, in view of Lemmas 2.4 and 2.5, for $\sigma<2c_0$ we get

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}\Phi_{3,\rho}(x)\right\|_{L_p(\mathbb{R}^n)}\to 0\quad as\quad \rho\to\infty.$$

Lemma 2.7 is proved.

**Definition 2.1.** *(see [1]). Let $V$ and $W$ be normed spaces. A family of linear operators $P_h$ ($h\in(0,1)$) is said to be fundamental in the pair of spaces $\{V,W\}$ as $h\to 0$, if for any $h\in(0,1)$ the operator $P_h:V\to W$ is bounded, and the following conditions are satisfied:*

$$\sup_h\|P_h\|\leq C<\infty,\quad \|P_{h_1}-P_{h_2}\|\to 0\quad as\quad h_1,h_2\to 0. \tag{2.20}$$

For a function $f\in L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n)$ we denote $U_h=P_hf$. We use Lemmas 2.4, 2.5 and 2.7 to prove the following theorem.

**Theorem 2.1.** *Let $|\lambda|>1$, $\frac{|\lambda|}{p}>\sigma>1-\frac{|\lambda|}{p'}$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. Then the family of operators $P_h$ is fundamental in the pair of spaces $\{L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n),W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)\}$ as $h\to 0$.*

**Proof.** It follows from Lemma 2.7 that for any function $f\in L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n)$ the function $U_h=P_hf$ belongs to the space $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$. Hence, in view of Lemmas 2.4 and 2.5, we obtain

$$\|U_h\|_{W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)}\leq C\left(\|f\|_{L_p(\mathbb{R}^n)}+\|(1+\rho_{\mathfrak{N}}(x))^{\sigma}f(x)\|_{L_1(\mathbb{R}^n)}\right), \tag{2.21}$$

where the constant $C$ does not depend on $f$ and $h$, $h\in(0,1)$. Thus, the condition (2.20) is satisfied. Also, we have

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\alpha,\mu^i))}(D_x^\alpha U_{h_1}(x)-D_x^\alpha U_{h_2}(x))\right\|_{L_p(\mathbb{R}^n)}$$

$$\leq\varepsilon(h_1,h_2)\left(\|f\|_{L_p(\mathbb{R}^n)}+\left\|(1+\rho_{\mathfrak{N}}(x))^{\sigma(1-\max\limits_i(\alpha,\mu^i))}f(x)\right\|_{L_1(\mathbb{R}^n)}\right),$$

where $\varepsilon(h_1,h_2)\to 0$ as $h_1,h_2\to 0$. The last relation implies (2.20), and the result follows. Theorem 2.1 is proved.

For $|\lambda|\leq 1$ we have the following analogs of Lemmas 2.5 and 2.7.

**Lemma 2.8.** *Let $1\geq|\lambda|>1-N\lambda_{\min}$, $\sigma<\min\{c_0,\frac{|\lambda|}{p}\}$, $\sigma>1-|\lambda|+\frac{|\lambda|}{p}-N\lambda_{\min}$. Then for any function $f\in\mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n)$ and for any $\beta\in\mathfrak{N}$ the following inequalities hold: for $0<h<1$,*

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max\limits_i(\beta,\mu^i))}D_x^\beta U_h(x)\right\|_{L_p(\mathbb{R}^n)}$$

$$\leq C\left(\|f\|_{L_p(\mathbb{R}^n)}+\left\|(1+\rho_{\mathfrak{N}}(x))^{\sigma(1-\max_i(\beta,\mu^i))+N|\lambda|}f(x)\right\|_{L_1(\mathbb{R}^n)}\right), \tag{2.22}$$

*where the constant $C>0$ does not depend on $f$ and $h$, and for $0<h_1<h_2<1$,*

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma(1-\max_i(\beta,\mu^i))}(D_x^\beta U_{h_1}(x)-D_x^\beta U_{h_2}(x))\right\|_{L_p(\mathbb{R}^n)}$$

$$\leq \varepsilon(h_1,h_2)\left(\|f\|_{L_p(\mathbb{R}^n)}+\left\|(1+\rho_{\mathfrak{N}}(x))^{\sigma(1-\max_i(\beta,\mu^i))+N|\lambda|}f(x)\right\|_{L_1(\mathbb{R}^n)}\right), \tag{2.23}$$

*where $\varepsilon(h_1,h_2)\to 0$ as $h_1,h_2\to 0$.*

**Proof.** We prove the inequality (2.22). We consider the case $\beta=0$ (the remaining cases can be treated similarly). As in the proof of Lemma 2.5, we have

$$\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma}U_h(x)\right\|_{L_p(\mathbb{R}^n)}\leq\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma}\int_h^1 d\nu\int_{\mathbb{R}^n}f(t)\int_{\mathbb{R}^n}e^{-i(t-x,\xi)}G_1(\xi,\nu)d\xi dt\right\|_{L_p(\mathbb{R}^n)}$$

$$+\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma}\int_1^{h^{-1}} d\nu\int_{\mathbb{R}^n}f(t)\int_{\mathbb{R}^n}e^{-i(t-x,\xi)}G_1(\xi,\nu)d\xi dt\right\|_{L_p(\mathbb{R}^n)}=:I_1+I_2.$$

The term $I_1$ can be estimated as in Lemma 2.5. So, we have to estimate only $I_2$. Since $f\in\mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n)$, that is, $\int_{\mathbb{R}^n}x^\beta f(x)dx=0$ for $|\beta|=0,1,...,N-1$, the Fourier transform of function $f$ can be written in the form (see [1]):

$$\hat f(\xi)=\frac{1}{(2\pi)^{\frac n2}}\int_0^1...\int_0^1\left(\int_{\mathbb{R}^n}e^{-i(\lambda_1...\lambda_N y,\xi)}(-iy,\xi)^N f(y)dy\right)\cdot\lambda_1^{N-1}...\lambda_{N-2}^2\lambda_{N-1}d\lambda_1...d\lambda_N.$$

Therefore, we can write

$$I_2\leq C\int_1^{h^{-1}}d\nu\int_0^1...\int_0^1\lambda_1^{N-1}...\ \lambda_{N-2}^2\lambda_{N-1}$$

$$\cdot\left\|(1+\rho_{\mathfrak{N}}(x))^{-\sigma}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(x-\lambda_1...\lambda_N y,\xi)}G_1(\xi,\nu)(-iy,\xi)^N f(y)d\xi dy\right\|_{L_p(\mathbb{R}^n)}d\lambda_1...d\lambda_N$$

$$\leq\sum_{|\rho|=N}C_\rho\int_1^{h^{-1}}d\nu\left\|(\rho_{\mathfrak{N}}(x))^{-\sigma}\int_{\mathbb{R}^n}e^{i(x,\xi)}G_1(\xi,\nu)\xi^\rho d\xi\right\|_{L_p(\mathbb{R}^n)}\cdot\|y^\rho(1+\rho_{\mathfrak{N}}(y))^\sigma f(y)\|_{L_1(\mathbb{R}^n)}$$

$$\leq\sum_{|\rho|=N}C_\rho\int_1^{h^{-1}}\nu^{-|\lambda|-(\lambda,\rho)}d\nu\ \|y^\rho(1+\rho_{\mathfrak{N}}(y))^\sigma f(y)\|_{L_1(\mathbb{R}^n)}$$

$$\cdot\left\|\left(\sqrt{x_1^{2l_1}+...+x_n^{2l_n}}\right)^{-\sigma}\left[1+\nu^{-K}\left(\sqrt{x_1^{2l_1}+...+x_n^{2l_n}}\right)^K\right]^{-1}\right\|_{L_p(\mathbb{R}^n)}$$

$$\leq\sum_{|\rho|=N}C_\rho\int_1^{h^{-1}}\nu^{-|\lambda|-(\lambda,\rho)+\frac{|\lambda|}{p}-\sigma}d\nu\ \|y^\rho(1+\rho_{\mathfrak{N}}(y))^\sigma f(y)\|_{L_1(\mathbb{R}^n)}$$

$$\cdot\left\|\left(\sqrt{y_1^{2l_1}+...+y_n^{2l_n}}\right)^{-\sigma}\left[1+\left(\sqrt{y_1^{2l_1}+...+y_n^{2l_n}}\right)^K\right]^{-1}\right\|_{L_p(\mathbb{R}^n)}.$$

In the last integral we have used the change of variables $x=\nu^\lambda y$. Since $\frac{|\lambda|}{p}>\sigma$, then for large enough $K$ the integral in $L_p$ converges. The integral in $\nu$ also converges, because by assumption $|\lambda|+(\lambda,\rho)-\frac{|\lambda|}{p}+\sigma>|\lambda|+N\lambda_{\min}-\frac{|\lambda|}{p}+\sigma>1$. As a result, we obtain

$$I_2\le C\left\|(1+\rho_{\mathfrak{N}}(x))^{\sigma+N|\lambda|}f(x)\right\|_{L_1(\mathbb{R}^n)}.$$

Thus, the inequality (2.22) is proved. The inequality (2.23) can be proved similarly. Lemma 2.8 is proved.

**Lemma 2.9.** *Let the conditions of Lemma 2.8 be satisfied, and let the function $\chi(s)$ be defined as above. Then for any $h\in(0,1)$ and $\beta\in\mathfrak{N}$ the relation (2.17) holds.*

The proof is similar to that of Lemma 2.7 (by applying Lemmas 2.4 and 2.8).

**Theorem 2.2.** *Let the conditions of Lemma 2.8 be satisfied. Then the family of operators $P_h$ is fundamental in the pair of spaces $\{\mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n), W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n\}$ as $h\to 0$.*

The proof is similar to that of Theorem 2.1 (by applying Lemmas 2.8 and 2.9).
For non-weighted spaces (when $\sigma=0$) we have the following result.

**Theorem 2.3.** *Let $|\lambda|\left(1-\frac{1}{p}\right)>1$. Then the family of operators $P_h$ is fundamental in the pair of spaces $\{L_p(\mathbb{R}^n)\bigcap L_1(\mathbb{R}^n), W^{\mathfrak{N}}_p(\mathbb{R}^n)\}$ as $h\to 0$. Moreover, if $|\lambda|\left(1-\frac{1}{p}\right)\le 1$ and $|\lambda|\left(1-\frac{1}{p}\right)+N\lambda_{\min}>1\ge|\lambda|\left(1-\frac{1}{p}\right)+(N-1)\lambda_{\min}$, then the family of operators $P_h$ is fundamental in the pair of spaces $\{\mathfrak{L}_{p,0,N}(\mathbb{R}^n), W^{\mathfrak{N}}_p(\mathbb{R}^n)\}$.*

## 3. REGULAR EQUATIONS IN $\mathbb{R}^n$

In this section, we use the results obtained in Section 1, to prove existence and uniqueness of a solution of the following equation:

$$P(D)U=f, \tag{3.1}$$

where the operator $P(D)$ is defined by (2.1), and satisfies the regularity condition (2.2).

**Theorem 3.1.** *Let $|\lambda|>1$, $\frac{|\lambda|}{p}>\sigma>1-|\lambda|+\frac{|\lambda|}{p}$. Then for any function $f\in L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n)$ the equation (3.1) has a unique solution $U$ from the class $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$, which is the limit (as $h\to 0$,) in the class $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ of approximate solutions $U_h$, defined by formula (2.6), and there exists a constant $C>0$ such that for any function $f\in L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n)$ the following inequality holds:*

$$\|U\|_{W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)}\le C\left(\|f\|_{L_p(\mathbb{R}^n)}+\|f\|_{L_{1,-\sigma}(\mathbb{R}^n)}\right). \tag{3.2}$$

**Proof.** Let $f\in L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n)$. We consider a family of operators $P_h$ and construct a sequence of functions $U_k(x)$ by formula

$$U_k(x)=P_{h_k}f(x), \tag{3.3}$$

where $h_k\to 0$ as $k\to\infty$.

If $|\lambda|>1$, then we can apply Theorem 2.1 to conclude that the family of operators $P_h$ is fundamental in the pair of spaces $\{L_p(\mathbb{R}^n)\bigcap L_{1,-\sigma}(\mathbb{R}^n), W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)\}$ as $h\to 0$. If $|\lambda|\le 1$, then by Theorem 2.2, the family of operators $P_h$ is fundamental in the pair of spaces $\{\mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n), W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)\}$ as $h\to 0$. Thus, for any $|\lambda| > 0$, the sequence $\{U_k(x)\}$ is fundamental in the space $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ with respect to the norm (2.7). And by the completeness of the space $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ there exists a function $U \in W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ such that $\|U_k - U\|_{W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Also, for $|\lambda| > 1$ the inequality (2.12) holds, while for $|\lambda| \le 1$ holds (2.22). From formula (2.1) of [8] and properties of averaging $f_h$, for almost all $x$ from $\mathbb{R}^n$ we have

$$f(x) = -\lim_{h\to 0} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_h^{h^{-1}} d\nu \frac{\partial}{\partial\nu} \int_{\mathbb{R}^n} f(t)\hat{G}_0(t-x,\nu)dt. \tag{3.4}$$

On the other hand, applying formulas (2.6) and (3.3), we can write

$$P(D_x)U_k = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{h_k}^{h_k^{-1}} d\nu \int_{\mathbb{R}^n} f(t) \int_{\mathbb{R}^n} P(D_x)e^{-i(t-x,\xi)}G_1(\xi,\nu)d\xi dt$$

$$= -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{h_k}^{h_k^{-1}} d\nu \int_{\mathbb{R}^n} f(t) \int_{\mathbb{R}^n} e^{-i(t-x,\xi)} \frac{\partial}{\partial\nu} G_0(\xi,\nu)d\xi dt = -\int_{h_k}^{h_k^{-1}} d\nu \frac{\partial}{\partial\nu} \int_{\mathbb{R}^n} f(t)\hat{G}_0(t-x,\nu)dt.$$

Passing to the limit as $k \to \infty$ and applying the integral representation (3.4), we can state that $U$ is a solution of equation (3.1). Also, taking into account Lemma 2.4, we conclude that for any $\beta \in \partial'\mathfrak{N}$ the following inequality holds:

$$\left\|D_x^\beta U\right\|_{L_p(\mathbb{R}^n)} \le C\|f\|_{L_p(\mathbb{R}^n)}. \tag{3.5}$$

Now we proceed to prove the uniqueness of a solution. Assume first that $U(x)$ is a finite solution of equation (3.1). Then using Fourier transform, we obtain $P(i\xi)\hat{U}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$. And since $P(i\xi) \ne 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, then by a property of Fourier transform, we get $\hat{U}(\xi) = 0$ almost everywhere in $\mathbb{R}^n$. Taking into account that $\hat{U}(\xi)$ is a continuous function, we have $\hat{U}(\xi) \equiv 0$ in $\mathbb{R}^n$, and hence $U(\xi) \equiv 0$ in $\mathbb{R}^n$. Thus, the uniqueness of a solution of equation (3.1) for finite functions from $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ is proved. In view of (3.5), we can state that for any smooth finite function $v \in W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ and for any $\beta \in \partial'\mathfrak{N}$, we have $\left\|D_x^\beta v\right\|_{L_p(\mathbb{R}^n)} \le C\|P(D)v\|_{L_p(\mathbb{R}^n)}$.

Now we consider the general case. Let $U \in W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ be a solution of the homogeneous equation $P(D)U = 0$. We show that for any bounded domain $G$, $\|U\|_{L_p(G)} = 0$. Taking into account that $U \in W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$, and that the finite functions are dense in $W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$, we conclude that there exists $U_\varepsilon \in W^{\mathfrak{N}}_{p}(\mathbb{R}^n)$ such that

$$\|U - U_\varepsilon\|_{W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)} < \varepsilon. \tag{3.6}$$

Hence, in view of (3.5), for any $\beta \in \partial'\mathfrak{N}$ the following inequality holds:

$$\left\|D_x^\beta U_\varepsilon\right\|_{L_p(\mathbb{R}^n)} \le C\|P(D_x)U_\varepsilon\|_{L_p(\mathbb{R}^n)}.$$

Taking into account that $U$ is a solution of the homogeneous equation $P(D)U = 0$, we obtain

$$\left\|D_x^\beta U\right\|_{L_p(\mathbb{R}^n)} \le \left\|D_x^\beta (U - U_\varepsilon)\right\|_{L_p(\mathbb{R}^n)} + C\|P(D)U_\varepsilon\|_{L_p(\mathbb{R}^n)}.$$

Next, since for the operator $P(D)$ the only non-zero coefficients are those $a_\alpha$ for which $\alpha \in \partial'\mathfrak{N}$, we can apply (3.6), to obtain

$$\left\|D_x^\beta U\right\|_{L_p(\mathbb{R}^n)} \le C \sum_{\beta\in\partial'\mathfrak{N}} |a_\beta| \left\|D_x^\beta (U - U_\varepsilon)\right\|_{L_p(\mathbb{R}^n)} + \varepsilon \le \varepsilon \cdot C \sum_{\beta\in\partial'\mathfrak{N}} |a_\beta| + \varepsilon.$$

By arbitrariness of $\varepsilon$, we have $\left\|D_x^\beta U\right\|_{L_p(\mathbb{R}^n)} = 0$ for any $\beta \in \partial'\mathfrak{N}$. Therefore, for any bounded domain $G$, we have $\|U\|_{L_p(G)} = 0$, and $U(x) = 0$ almost everywhere on $G$. Finally, taking into account that $G$ is an arbitrary domain, we conclude that $U(x) = 0$ almost everywhere in $\mathbb{R}^n$. Theorem 3.1 is proved.

**Theorem 3.2.** *Let* $1 \geq |\lambda| > 1 - N\lambda_{\min}$, $\sigma < \min\left\{c_0, \frac{|\lambda|}{p}\right\}$, $1 - |\lambda| + \frac{|\lambda|}{p} - (N-1)\lambda_{\min} \geq \sigma > 1 - |\lambda| + \frac{|\lambda|}{p} - N\lambda_{\min}$. *Then for any function* $f \in \mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n)$ *there exists a unique solution* $U \in W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)$ *of equation (3.1), and there exists a constant* $C > 0$ *such that for any function* $f \in \mathfrak{L}_{p,\sigma,N}(\mathbb{R}^n)$ *the following inequality holds:*

$$\|U\|_{W^{\mathfrak{N}}_{p,\sigma}(\mathbb{R}^n)} \leq C\left(\|f\|_{L_p(\mathbb{R}^n)} + \left\|(1+\rho_{\mathfrak{N}}(x))^{\sigma+N|\lambda|} f(x)\right\|_{L_1(\mathbb{R}^n)}\right). \quad (3.7)$$

The proof is similar to that of Theorem 3.1 with application of Theorem 2.2.

Finally, applying Theorem 2.3, for an ordinary multianisotropic space we have the following theorem.

**Theorem 3.3.** *The following assertions hold.*

*(a) If* $|\lambda| - \frac{|\lambda|}{p} > 1$, *then for any function* $f \in L_p(\mathbb{R}^n) \bigcap L_1(\mathbb{R}^n)$ *the equation (3.1) has a unique solution* $U \in W^{\mathfrak{N}}_p(\mathbb{R}^n)$, *for which the inequality (3.2) is satisfied for* $\sigma = 0$.

*(b) If* $|\lambda| - \frac{|\lambda|}{p} \leq 1$, $|\lambda| - \frac{|\lambda|}{p} + N\lambda_{\min} > 1 > |\lambda| - \frac{|\lambda|}{p} + (N-1)\lambda_{\min}$, *then for* $f \in \mathfrak{L}_{p,0,N}(\mathbb{R}^n)$ *the equation (3.1) has a unique solution in* $W^{\mathfrak{N}}_p(\mathbb{R}^n)$, *for which (3.7) is satisfied for* $\sigma = 0$.

The proof is similar to that of Theorem 3.1 with application of Theorem 2.3.

## REFERENCES


1. G. V. Demidenko, "Integral operators determined by quasielliptic equations. I", Sibirsk. Mat. Zh., **34** (6), 52-67, 1993.
2. G. V. Demidenko, "On quasi-elliptic operators in $\mathbb{R}^n$", Sibirsk. Mat. Zh., **39** (5), 1028-1037, 1998.
3. G. V. Demidenko, "Quasielliptic operators and Sobolev type equations", Sibirsk. Mat. Zh., **49** (5), 1064-1076, 2008.
4. S. V. Uspenskii, "The representation of functions defined by a certain class of hypoelliptic operators", Trudy Mat. Inst. Steklov, **117**, 292-299, 1972.
5. S. M. Nikol'skii, "Stable boundary-value problems of a differentiable function of several variables", Mat. Sb., **61(103)** (2), 224-252, 1963.
6. V. Mikhailov, "Behavior at infinity of certain class of polynomials" Trudy Mat. Inst. Steklov, **91**, 58-81, 1967.
7. S. Gindikin, L. Volovich, "The Method of Newton's Polyhedron in the theory of Partial Differential Equations", K.A.P., London, **86**, 266, 1992.
8. G. A. Karapetyan, "On stabilization at infinity to polynomial of solutions of a certan class of regular equations", Trudy Mat. Inst. Steklov, **187**, 116-129, 1989.
9. G. A. Karapetyan, "Integral representation of functions and embedding theorems for multianisotropic spaces for the three-dimensional case", Eurasian Mathematical Journal, **7** (2), 19-39, 2016.
10. G. A. Karapetyan, "Integral representations and embedding theorems for $n$-dimensional multianisotropic spaces with one anisotripy vertex", Sibirsk. Mat. Zh., **58** (3), 573-590, 2017.
11. G. A. Karapetyan, M.K. Arakelyan, "Embedding theorems for general multianisotropic spaces", Mat. Zametki (submited).
12. S.L. Sobolev, *Some applications of functional analysis in mathematical physics* (Nauka, Novosibirsk, 1962).
13. V.P. Il'in, "Integral representations of differentiable functions of classes $W^l_p(G)$", Sib. Math. Journal, **8** (3), 573-586, 1967.
14. O.V. Besov, "On coercitivity in nonisotropic Sobolev spaces", Mat. Sb., **73(115)** (4), 585-599, 1967.
15. O.V. Besov, V.P. Il'in, S.M. Nikol'skii, *Integral representations of functions and embedding theorems* (Nauka, Moscow, 1975).
16. O.V. Besov, V. P. Il'in, L.D. Kudryavcev, P.I. Lizorkin, S. M. Nikol'skii, "Theory of embeddings of differentiable functions of many variables", Partial differential equations, 38-63, 1970.
17. L. A. Bagirov, V.A. Kondrat'ev, "On elliptic equations in $\mathbb{R}^n$", Dif. uravneniya, **11** (3), 498-504, 1975.
18. L. A. Bagirov, "A priori estimates, existence theorems, and the behavior at infinity of solutions of quasielliptic equations in $\mathbb{R}^n$", Mat. Sb., **110** (4), 475-492, 1979.
19. P. I. Lizorkin, "$(L_q, L_q)$-multipliers of Fourier integrals", DAN SSSR, **152** (4), 808-811, 1963.
20. G. G. Kazaryan, "Estimates of differential operators, and hypoelliptic operators", Trudy Mat. Inst. Steklov, **140**, 130-161, 1976.
21. L.E. Frankel, "Formula for higher derivatives of composite functions", Math. Proc. of the Cambridge Phil. Society, **83** (2), 159-165, 1978.