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FUNCTIONAL ANALYSIS

On Generalized Derivations and Centralizers of Operator Algebras with Involution

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Abstract—Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H and $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra which is closed under the adjoint operation. Let $F : \mathcal{A}(H) \to \mathcal{B}(H)$ be a linear mapping satisfying $F(AA^*A) = F(A)A^*A + Ad(A^*)A + AA^*d(A)$ for all $A \in \mathcal{A}(H)$, where the associated linear mapping $d : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfies the relation $d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A)$ for all $A \in \mathcal{A}(H)$. Then F is of the form $F(A) = SA - AT$ for all $A \in \mathcal{A}(H)$ and some $S, T \in \mathcal{B}(H)$, that is, F is a generalized derivation. We also prove some results concerning centralizers on $\mathcal{A}(H)$ and semisimple H^*-algebras.

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1. INTRODUCTION

Let $\delta : \mathbb{R} \to \mathbb{R}$ be an additive map on a ring $\mathbb{R}$. Recall that δ is called a generalized Jordan derivation if there exists a Jordan derivation $d : \mathbb{R} \to \mathbb{R}$ such that the equality

$$\delta(a^2) = \delta(a)a + ad(a) \tag{1.1}$$

holds for all $a \in \mathbb{R}$, and δ is said to be a generalized derivation if there is a derivation d on $\mathbb{R}$ satisfying

$$\delta(ab) = \delta(a)b + ad(b) \tag{1.2}$$

for all $a, b \in \mathbb{R}$.

In [7], it was proved that every generalized Jordan derivation on a 2-torsion free prime ring is a generalized derivation. This result was generalized in [14] to generalized Jordan derivations on 2-torsion free semiprime rings.

In particular, if $d = \delta$ in (1.1) and (1.2), then δ is called a Jordan derivation and derivation, respectively. The first result on Jordan derivation is due to Herstein [6] who proved that any Jordan derivation on a 2-torsion free prime ring is a Jordan derivation. Cusack [4] and Brešar [2] showed that this is also true for Jordan derivations on 2-torsion free semiprime rings. If $c \in \mathbb{R}$ is a fixed element and $\delta(a) = [c, a] = ca - ac$ for all $a \in \mathbb{R}$, then it is easy to see that δ is a derivation which is called an inner derivation determined by c. It is also well known that every linear derivation on standard operator algebra is inner (cf. [3]). Some related results on operator algebras can be found in [5], [8], [12], and references therein.

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In [13], Vukman proved that if a linear mapping d on a standard operator algebra, which is closed under the adjoint operation, or a semisimple H^*-algebra, satisfying

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A),$$

then d is a derivation.

Motivated by the above result and the concept of generalized Jordan derivations, in this paper, we aim to show that if F is a linear mapping on a standard operator algebra which is closed under the adjoint operation satisfying

$$F(AA^*A) = F(A)A^*A + Ad(A^*)A + AA^*d(A),$$

where the associated linear mapping d satisfies the relation

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A),$$

then F is a generalized derivation. A similar result is also obtained for the case of linear mappings on semisimple H^*-algebras. It should be noted that in order to prove the result on semisimple H^*-algebras, we need to have some results about left centralizers. Recall that a linear map $\phi : \mathcal{A} \to \mathcal{A}$ on an algebra $\mathcal{A}$ is called a left centralizer if $\phi(xy) = \phi(x)y$ for all $x, y \in \mathcal{A}$. The definition of a right centralizer should be self explanatory.

We now list some basic notation, definitions, and results. Throughout the paper, $\mathcal{L}(H)$ and $\mathcal{B}(H)$ will stand for the algebra of all linear operators and the algebra of all bounded linear operators on a complex Hilbert space H, respectively. By $\mathcal{F}(H) \subseteq \mathcal{B}(H)$ we denote the subalgebra of all bounded finite rank operators. We call a subalgebra $\mathcal{A}(H)$ of $\mathcal{B}(H)$ standard if it contains $\mathcal{F}(H)$. Notice that every standard operator algebra is prime. An operator $P \in \mathcal{B}(H)$ is said to be a projection if $P^* = P$ and $P^2 = P$. Each rank one operator can be expressed as $x \otimes y$, where $x \otimes y(u) = \langle u, y\rangle x$ for all $u \in H$.

Let $\mathcal{A}$ be an algebra over the filed $\mathbb{C}$ of complex numbers. An involution in $\mathcal{A}$ is a map $a \mapsto a^*$ of $\mathcal{A}$ into itself such that

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (\lambda a)^* = \bar{\lambda} a^*, \quad (ab)^* = b^* a^*$$

for any $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. An algebra over $\mathbb{C}$ endowed with an involution is called an involution algebra or a $*$-algebra. Recall that a semisimple H^*-algebra is a complex semisimple Banach $*$-algebra whose norm is a Hilbert space norm such that $\langle x, yz^*\rangle = \langle xz, y\rangle = \langle z, x^*y\rangle$ is fulfilled for all elements x, y, z. Let $\mathcal{A}$ be a semisimple H^*-algebra and $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ be the collection of minimal closed ideals of $\mathcal{A}$ such that $A = \oplus_{\alpha\in\Gamma}\mathcal{A}_\alpha$. Then any element $x \in \mathcal{A}$ can be expressed as $x = \sum_{\alpha\in\Gamma} x_\alpha$ and $x_\alpha x_\beta = 0$ for $x_\alpha \in \mathcal{A}_\alpha$ and $x_\beta \in \mathcal{A}_\beta$ with $\alpha \neq \beta$. For every x and y in $\mathcal{A}$ with $x = \sum_\alpha x_\alpha$ and $y = \sum_\alpha y_\alpha$, we have $xy = \sum_\alpha x_\alpha y_\alpha$. A self-adjoint idempotent element $e \in \mathcal{A}$ is called a projection. A nonzero projection is said to be minimal if it can't be represented as a sum of two mutually orthogonal nonzero projections in $\mathcal{A}$. For more information about H^*-algebras, we refer the reader to [1] and [11].

2. MAIN RESULTS

Our first theorem is a generalization of Theorem 1 of [13].

Theorem 2.1. *Let H be a complex Hilbert space, and let $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear mapping $F : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfying the relation*

$$F(AA^*A) = F(A)A^*A + Ad(A^*)A + AA^*d(A) \tag{2.1}$$

for all $A \in \mathcal{A}(H)$, where the associated linear mapping $d : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfies the relation

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A) \tag{2.2}$$

for all $A \in \mathcal{A}(H)$. Then $F(A) = SA - AT$ for all $A \in \mathcal{A}(H)$ and some $S, T \in \mathcal{B}(H)$, which means that F is a linear generalized derivation.

It should be mentioned that in the proof below, we borrow some ideas from [10] and [13].

Proof. First we consider the restriction of F to $\mathcal{F}(H)$. Suppose $A \in \mathcal{F}(H)$. Then $A^* \in \mathcal{F}(H)$. Pick a projection $P \in \mathcal{F}(H)$ such that $AP = PA = A$ and $A^*P = PA^* = A^*$. Hence, in view of relation (2.1), we obtain

$$F(P) = F(P)P + Pd(P)P + Pd(P). \tag{2.3}$$

Right multiplication by P to (2.3) yields that $2Pd(P)P = 0$. This implies that

$$Pd(P)P = 0. \tag{2.4}$$

In view of above relation, we find that

$$Pd(P)A = 0, \ Ad(P)P = 0, \ \text{and } Ad(P)A = 0. \tag{2.5}$$

Using (2.4) in (2.3), we get

$$F(P) = F(P)P + Pd(P). \tag{2.6}$$

Replacing A by $A + P$ in (2.1) and using the fact that $A^* = (A + P)^* = A^* + P$, we obtain

$$\begin{aligned} F((A+P)(A^*+P)(A+P)) = F(A)A^*A + Ad(A^*)A \\ +AA^*d(A) + F(AA^* + A^*A + A^2) + 2F(A) + F(A^*) + F(P)P + Pd(P). \end{aligned} \tag{2.7}$$

On the other hand, we find that

$$\begin{aligned} &F((A+P)(A^*+P)(A+P)) = F(A)A^*A + F(A)A + F(A)A^* \\ &+F(A)P + F(P)A^*A + F(P)A^* + F(P)A + F(P)P + Ad(A^*)A + Pd(A^*)A \\ &+Ad(P)A + Pd(P)A + Ad(A^*)P + Pd(A^*)P + Ad(P)P + Pd(P)P + AA^*d(A) \\ &+A^*d(A) + Ad(A) + Pd(A) + AA^*d(P) + A^*d(P) + Ad(P) + Pd(P). \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$\begin{aligned} &F(AA^* + A^*A + A^2) + 2F(A) + F(A^*) \\ &= F(A)A + F(A)A^* + F(A)P + F(P)A^*A + F(P)A^* + F(P)A + Pd(A^*)A \\ &+Ad(P)A + Pd(P)A + Ad(A^*)P + Pd(A^*)P + Ad(P)P + Pd(P)P \\ &+A^*d(A) + Ad(A) + Pd(A) + AA^*d(P) + A^*d(P) + Ad(P). \end{aligned}$$

An application of (2.5) and (2.6) yields

$$\begin{aligned} &F(AA^* + A^*A + A^2) + 2F(A) + F(A^*) = F(A)A + F(A)A^* \\ &+F(A)P + F(P)A^*A + F(P)A^* + F(P)A + Pd(A^*)A \\ &+Ad(A^*)P + Pd(A^*)P + A^*d(A) + Ad(A) + Pd(A) + AA^*d(P) + A^*d(P) + Ad(P). \end{aligned} \tag{2.9}$$

Replacing A by $-A$ in (2.9), we get

$$\begin{aligned} F(AA^* + A^*A + A^2) - 2F(A) - F(A^*) = F(A)A^* + F(A)A \\ +F(P)A^*A - F(P)A^* - F(P)A - F(A)P + Pd(A^*)A + Ad(A^*)P \\ -Pd(A^*)P + A^*d(A) + Ad(A) - Pd(A) + AA^*d(P) - A^*d(P) - Ad(P). \end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10), we arrive at

$$\begin{aligned} &F(AA^* + A^*A + A^2) = F(A)A^* + F(A)A + F(P)A^*A \\ &+Pd(A^*)A + Ad(P)P + A^*d(A) + Ad(A) + AA^*d(P). \end{aligned} \tag{2.11}$$

Subtracting (2.10) from (2.9), we obtain

$$2F(A) + F(A^*) = F(P)A^* + F(P)A + F(A)P + Pd(A^*)P + Pd(A) + A^*d(P) + Ad(P). \tag{2.12}$$

Next, substituting iA for A into (2.10) and (2.11), we find that

$$F(A^2 - AA^* - A^*A) = F(A)A - F(A)A^* - F(P)A^*A \tag{2.13}$$

$$-Pd(A^*)A + Ad(P)A + Ad(A) - A^*d(A) - AA^*d(P)$$

and

$$2iF(A) - iF(A^*) = iF(P)A - iF(P)A^* + iF(A)P \\ -iPd(A^*)P + iPd(A) - iA^*d(P) + iAd(P). \quad (2.14)$$

This implies that

$$2F(A) - F(A^*) = F(P)A - F(P)A^* + F(A)P - Pd(A^*)P + Pd(A) - A^*d(P) + Ad(P). \quad (2.15)$$

Adding (2.12) and (2.15), we arrive at

$$2F(A) = F(A)P + Ad(P) + F(P)A + Pd(A). \quad (2.16)$$

Now adding (2.11) and (2.13), we get

$$F(A^2) = F(A)A + Ad(A) \quad (2.17)$$

for all $A \in \mathcal{A}(H)$.

By Theorem 1 of [13], we see that d is an inner derivation on $\mathcal{A}(H)$. So, there exists an operator $N \in \mathcal{B}(H)$ such that

$$d(A) = NA - AN \quad (2.18)$$

for all $A \in \mathcal{F}(H)$. In view of relations (2.16) and (2.17), we conclude that F maps $\mathcal{F}(H)$ into itself. Also, from (2.17), it is clear that F is a generalized Jordan derivation on $\mathcal{F}(H)$.

Note that $\mathcal{F}(H)$ is prime and hence F is a generalized derivation on $\mathcal{F}(H)$ by Theorem 2.5 of [7]. Furthermore, Theorem 4.2 of [7] asserts that F is a generalized inner derivation on $\mathcal{F}(H)$, that is, there exist $S, T \in \mathcal{B}(H)$ such that for all $A \in \mathcal{F}(H)$,

$$F(A) = SA - AT. \quad (2.19)$$

To complete the proof, it remains to show that (2.19) holds for all $A \in \mathcal{A}(H)$. We first claim that the operators N in (2.18) and T in (2.19) differ by a scalar multiple of the identity operator I.

Indeed, for any $A, B \in \mathcal{F}(H)$, we have $F(AB) = SAB - ABT$. On the other hand, we have

$$F(A)B + Ad(B) = SAB - ATB + ANB - ABN.$$

Comparing the above two relations, we see that

$$AB(N - T) = A(N - T)B \quad (2.20)$$

holds true for all $A, B \in \mathcal{F}(H)$.

Pick $y, u \in H$ such that $\langle u, y\rangle = 1$. Now for arbitrary $x, v \in H$, the relation (2.20) becomes

$$x \otimes y \cdot u \otimes v \cdot (N - T) = x \otimes y \cdot (N - T) \cdot u \otimes v.$$

This leads to

$$(N - T)^*v = \langle (N - T)u, y\rangle v$$

for any $v \in H$. Hence, $(N - T)^* = \langle (N - T)u, y\rangle I$, or equivalently,

$$N - T = \langle y, (N - T)u\rangle I.$$

Taking $\lambda = \langle y, (N - T)u\rangle$, we get $N - T = \lambda I$.

We now define a linear map $G : \mathcal{A}(H) \to \mathcal{B}(H)$ as $G(A) = SA - AT$ for all $A \in \mathcal{A}(H)$. We set $F_0 = F - G$, and observe that $F_0(A) = 0$ for any $A \in \mathcal{F}(H)$. Thus, it remains to show that $F_0(A) = 0$ for all $A \in \mathcal{A}(H)$. For any $A \in \mathcal{A}(H)$, we can write

$$\begin{aligned}
&F_0(AA^*A) = F(AA^*A) - G(AA^*A) \\
&= F(A)A^*A + Ad(A^*)A + AA^*d(A) - SAA^*A + AA^*AT \\
&= F(A)A^*A + ANA^*A - AA^*NA + AA^*NA - AA^*AN - SAA^*A + AA^*AT \\
&= F(A)A^*A + A(T + \lambda I)A^*A - AA^*(T + \lambda I)A + AA^*(T + \lambda I)A
\end{aligned}$$

$$-AA^*A(T+\lambda I) - SAA^*A + AA^*AT = F(A)A^*A - SAA^*A + ATA^*A.$$

and

$$F_0(A)A^*A = F(A)A^*A - G(A)A^*A = F(A)A^*A - SAA^*A + ATA^*A.$$

Therefore, we have $F_0(AA^*A) = F_0(A)A^*A$ for any $A \in \mathcal{A}(H)$. Let $A \in \mathcal{A}(H)$ and P be a rank one projection. We write $K = A - AP - PA + PAP$. One can easily check that $KP = PK = K^*P = PK^* = 0$ and $F_0(K) = F_0(A)$. We have

$$F_0(A)K^*K = F_0(K)K^*K = F_0(KK^*K) = F_0(KK^*K+P) = F_0((K+P)(K+P)^*(K+P))$$
$$= F_0(K+P)(K+P)^*(K+P) = F_0(A)(K^*+P)(K+P) = F_0(A)K^*K + F_0(A)P,$$

implying that $F_0(A)P = 0$. Since P is arbitrary, it follows that $F_0(A) = 0$ for all $A \in A(H)$. This completes the proof of the theorem.

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 2.1 ([13], Theorem 1)**.** *Let H be a complex Hilbert space, and let $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear mapping $d : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfying the relation*

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A)$$

for all $A \in \mathcal{A}(H)$. Then $d(A) = TA - AT$ for all $A \in \mathcal{A}(H)$ and some $T \in \mathcal{B}(H)$, which means that d is an inner derivation.

The proof of the following theorem is similar to that of Lemma of [10]. For the sake of completeness, we include it here.

Theorem 2.2. *Let H be a complex Hilbert space, and let $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra, which is closed under the adjoint operation. Further, let $\phi : \mathcal{A}(H) \to \mathcal{B}(H)$ be a linear mapping satisfying*

$$\phi(AA^*A) = \phi(A)A^*A \tag{2.21}$$

for all $A \in \mathcal{A}(H)$. Then ϕ is a left centralizer and there exists a linear operator $C \in \mathcal{L}(H)$ such that for all $A \in \mathcal{A}(H)$, $\phi(A) = CA$.

Proof. Let $A \in \mathcal{F}(H)$ and P be a finite rank projection such that $AP = PA = A$. Substituting $A + P$ for A in relation (2.21), we obtain

$$\phi(A^2 + A^*A + AA^* + 2A + A^*) = \phi(A)A + \phi(P)A^*A + \phi(A)A^* + \phi(A)P + \phi(P)A + \phi(P)A^*.$$

Replacing A by $A + P$ and $A - P$ respectively in the above relation, we can get

$$\phi(2A + A^*) = \phi(A)P + \phi(P)A + \phi(P)A^*. \tag{2.22}$$

Replacing A by iA in (2.22), we get

$$\phi(2iA - iA^*) = i\phi(A)P + i\phi(P)A - i\phi(P)A^*.$$

It follows that

$$\phi(-2A + A^*) = -\phi(A)P - \phi(P)A + \phi(P)A. \tag{2.23}$$

Equalities (2.22) and (2.23) yield that $\phi(A^*) = \phi(P)A^*$. Replacing A^* by A results in

$$\phi(A) = \phi(P)A. \tag{2.24}$$

We now show that ϕ is a left centralizer on $\mathcal{F}(H)$, that is, $\phi(AB) = \phi(A)B$ for all $A, B \in \mathcal{F}(H)$. If H is finite dimensional, the choosing $P = I$, we get $\phi(AB) = \phi(I)AB = \phi(A)B$. If H is of infinite dimension, then we fix an element $x \in H$, and claim that for any $y \in H$, there exists an element $x_y \in H$ such that $\phi(x \otimes y) = x_y \otimes y$. Let $y_1, y_2 \in H$. If y_1 and y_2 are linearly independent, then

$$\phi(x \otimes (y_1 + y_2)) = x_{y_1+y_2} \otimes (y_1 + y_2) = x_{y_1+y_2} \otimes y_1 + x_{y_1+y_2} \otimes y_2.$$

On the other hand, we have

$$\phi(x \otimes y_1) + \phi(x \otimes y_2) = x_{y_1} \otimes y_1 + x_{y_2} \otimes y_2.$$

It follows that $x_{y_1} = x_{y_1+y_2} = x_{y_2}$. In the case where y_1 and y_2 are linearly dependent, we may find a $y_3 \in H$ such that y_1, y_3 as well as y_2, y_3 are linearly independent. Therefore, $x_{y_1} = x_{y_3} = x_{y_2}$.

Pick an element $u \in H$ such that $\langle u, y\rangle \neq 0$. Let $v \in H$ be arbitrary. We have

$$\phi(x \otimes y \cdot u \otimes v) = \phi(\langle u, y\rangle x \otimes v) = x_{\langle u,y\rangle v} \otimes \langle u, y\rangle v$$
$$= \langle u, y\rangle x_{\langle u,y\rangle v} \otimes v = \langle u, y\rangle x_y \otimes v = x_y \otimes y \cdot u \otimes v = \phi(x \otimes y)u \otimes v.$$

If $\langle u, y\rangle = 0$, we have $\phi(x \otimes y \cdot u \otimes v) = 0$ and, by (2.24),

$$\phi(x \otimes y \cdot u \otimes v) = \phi(P)x \otimes y \cdot u \otimes v = 0$$

for some finite rank projection P. Now, we can conclude that for any $A, B \in \mathcal{F}(H)$ $\phi(AB) = \phi(A)B$. This implies that ϕ is a left centralizer on $\mathcal{F}(H)$. Next, we pick $y, u \in H$ with $\langle y, u\rangle = 1$, and define $Cx = \phi(x \otimes u)y$ for any $x \in H$. Obviously, C is linear. Now for any $A \in \mathcal{F}(H)$ and $x \in H$,

$$CAx = \phi(Ax \otimes u)y = \phi(A)x \otimes u(y) = \phi(A)(\langle y, u\rangle x) = \phi(A)x.$$

Thus, $\phi(A) = CA$ for all $A \in \mathcal{F}(H)$.

To complete the proof, it remains to show that $\phi(A) = CA$ for all $A \in \mathcal{A}(H)$. Define Φ by $\Phi(A) = CA$ for all $A \in \mathcal{A}(H)$ and let $\phi_0 = \phi - \Phi$. It is obvious that $\phi_0(A) = 0$ for all $A \in \mathcal{F}(H)$. One can check that $\phi_0(AA^*A) = \phi_0(A)A^*A$ for all $A \in \mathcal{A}(H)$.

Let $A \in \mathcal{A}(H)$. Suppose that P is a finite rank projection and let $K = A - AP - PA + PAP$. We have

$$\phi_0(K)K^*K = \phi_0(KK^*K) = \phi_0(KK^*K + p)$$
$$= \phi_0((K + P)(K + P)^*(K + P)) = \phi_0(K + P)(K + P)^*(K + P).$$

This leads to $\phi_0(K)P = 0$. Observing that $\phi_0(K) = \phi_0(A)$, we get $\phi_0(A)P = 0$ for any finite rank projection P. Hence, $\phi_0(A) = 0$ for all $A \in \mathcal{A}(H)$. Theorem 2.2 is proved.

The proof of the next result is just a modification of that of Theorem of [10]. We present the proof for the reader's convenience.

Theorem 2.3. *Let $\phi : \mathcal{A} \to \mathcal{A}$ be a linear mapping on a semisimple H^*-algebra $\mathcal{A}$ satisfying*

$$\phi(xx^*x) = \phi(x)x^*x \tag{2.25}$$

for all $x \in \mathcal{A}$. Then ϕ is a left centralizer.

Proof. Let $e \in \mathcal{A}$ be a projection. Replacing x by $x + e$ and $x - e$ in (2.25), respectively, and comparing the resulting equalities, we arrive at

$$\phi(ee^*x + xe^*e + ex^*e) = \phi(e)e^*x + \phi(x)e^*e + \phi(e)x^*e. \tag{2.26}$$

Let $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ be a collection of minimal closed ideals of $\mathcal{A}$ such that their orthogonal direct sum is $\mathcal{A}$. For $\alpha \in \Gamma$ and $x \in \mathcal{A}_\alpha$, let e be a minimal projection with $e \in \mathcal{A}_\beta$ $(\alpha \neq \beta)$. It follows from (2.26) that $\phi(x)e = 0$. Thus, $\phi(x) \in \mathcal{A}_\alpha$, which implies that $\mathcal{A}_\alpha$ is invariant under ϕ. By Theorem 2.2, we conclude that ϕ is a left centralizer on $\mathcal{A}_\alpha$ for each $\alpha \in \Gamma$. Furthermore, it follows from Theorem 2.2 and Remark 1 of [9] that ϕ is continuous on $\mathcal{A}_\alpha$ for every $\alpha \in \Gamma$.

Let $\{x_n\} \subseteq \mathcal{A}$ and $y \in \mathcal{A}$ be such that

$$\lim_{n\to\infty} x_n \to 0 \quad \text{and} \quad \lim_{n\to\infty} \phi(x_n) \to y.$$

If $e \in \mathcal{A}$ is a minimal projection, from (2.26) we see that

$$0 = \lim_{n\to\infty} [\phi(e)ex_n + \phi(x_n)e + \phi(e)x_n^*e] = ye,$$

implying that $y = 0$. By Closed Graph Theorem, ϕ is continuous.

For any $x, y \in \mathcal{A}$, we write $x = \sum_{\alpha\in\Gamma} x_\alpha$ and $y = \sum_{\alpha\in\Gamma} y_\alpha$, where $x_\alpha, y_\alpha \in \mathcal{A}_\alpha$ ($\alpha \in \Gamma$). We have

$$\phi(xy) = \phi\Big(\sum_{\alpha\in\Gamma} x_\alpha \sum_{\alpha\in\Gamma} y_\alpha\Big) = \phi\Big(\sum_{\alpha\in\Gamma} x_\alpha y_\alpha\Big) = \sum_{\alpha\in\Gamma} \phi(x_\alpha y_\alpha) = \sum_{\alpha\in\Gamma} \phi(x_\alpha) y_\alpha$$
$$= \Big(\sum_{\alpha\in\Gamma} \phi(x_\alpha)\Big)\Big(\sum_{\alpha\in\Gamma} y_\alpha\Big) = \phi\Big(\sum_{\alpha\in\Gamma} x_\alpha\Big)\Big(\sum_{\alpha\in\Gamma} y_\alpha\Big) = \phi(x)y.$$

Thus, $\phi(xy) = \phi(x)y$ for all $x, y \in \mathcal{A}$. This completes the proof of Theorem 2.3.

We conclude our paper by proving an analog of Theorem 2.1 on semisimple H^*-algebras.

Theorem 2.4. *Let $\mathcal{A}$ be a semisimple H^*-algebra. Suppose there exists a linear mapping $F : \mathcal{A} \to \mathcal{A}$ satisfying the relation*

$$F(xx^*x) = F(x)x^*x + xd(x^*)x + xx^*d(x)$$

for all $x \in \mathcal{A}$, where the associated linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfies the relation

$$d(xx^*x) = d(x)x^*x + xd(x^*)x + xx^*d(x)$$

for all $x \in \mathcal{A}$. Then F is a generalized derivation.

Proof. By Theorem 2 of [13], d is a linear derivation. Now, for any $x \in \mathcal{A}$, we have

$$(F-d)(xx^*x) = F(xx^*x) - d(xx^*x) = \big(F(x)x^*x + xd(x^*)x + xx^*d(x)\big)$$
$$-\big(d(x)x^*x + xd(x^*)x + xx^*d(x)\big) = F(x)x^*x - d(x)x^*x = (F-d)(x)x^*x.$$

In view of Theorem 2.3, we conclude that $F - d$ is a left centralizer. Therefore, for any $x, y \in \mathcal{A}$, using the fact that d is a derivation, we obtain

$$F(xy) = (F-d)(xy) + d(xy) = (F-d)(x)y + d(x)y + xd(y) = F(x)y + xd(y).$$

Hence, F is a generalized derivation. Theorem 2.4 is proved.

Corollary 2.2 ([13], Theorem 2)**.** *Let $\mathcal{A}$ be a semisimple H^*-algebra. Suppose there exists a linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfying the relation*

$$d(xx^*x) = d(x)x^*x + xd(x^*)x + xx^*d(x)$$

for all $x \in \mathcal{A}$. Then d is a derivation.

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