ISSN 1068-3623, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2017, Vol. 52, No. 6, pp. 276–287. © Allerton Press, Inc., 2017.
Original Russian Text © R. K. Parmar, T. K. Pogány, 2017, published in Izvestiya Natsional'noi Akademii Nauk Armenii, Matematika, 2017, No. 6, pp. 47-61.

FUNCTIONAL ANALYSIS

Extended Srivastava's Triple Hypergeometric $H_{A,p,q}$ Function and Related Bounding Inequalities

R. K. Parmar[1*] **and T. K. Pogány**[2,3**]

[1]*Government College of Engineering and Technology, Bikaner, India*

[2]*University of Rijeka, Rijeka, Croatia*[4]

[3]*Óbuda University, Budapest, Hungary*

Received September 28, 2015

Abstract—In this paper, motivated by certain recent extensions of the Euler's beta, Gauss' hypergeometric and confluent hypergeometric functions (see [4]), we extend the Srivastava's triple hypergeometric function H_A by making use of two additional parameters in the integrand. Systematic investigation of its properties including, among others, various integral representations of Euler and Laplace type, Mellin transforms, Laguerre polynomial representation, transformation formulas and a recurrence relation, is presented. Also, by virtue of Luke's bounds for hypergeometric functions and various bounds upon the Bessel functions appearing in the kernels of the newly established integral representations, we deduce a set of bounding inequalities for the extended Srivastava's triple hypergeometric function $H_{A,p,q}$.

MSC2010 numbers : 33B20, 33C20, 33B15, 33C05

DOI: 10.3103/S1068362317060036

Keywords: (p,q)*-extended Beta function;* (p,q)*-extended hypergeometric function; extended Appell function; Mellin transform; Laguerre polynomial; bounding inequality.*

Dedicated to the 75th birthday anniversary of Professor Hari M. Srivastava

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ will denote the sets of positive integers, negative integers and complex numbers, respectively. Also, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$. The definition of the *generalized hypergeometric function* with r numerator and s denominator parameters, as a series, reads as follows:

$$ {}_rF_s\big(a_1, \cdots, a_r;\, b_1, \cdots, b_s;\, z\big) = {}_rF_s\big(a_r;\, b_s;\, z\big) := \sum_{m\geq 0} \frac{(a_1)_m \cdots (a_r)_m}{(b_1)_m \cdots (b_s)_m}\, \frac{z^m}{m!}\,, $$

where $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, j = \overline{1, s}$. The series converges for all $z \in \mathbb{C}$ if $r \leq s$. It is divergent for all $z \neq 0$ when $r > s+1$, unless at least one numerator parameter is a negative integer, in which case it becomes a polynomial. Finally, if $r = s+1$, the series converges on the unit circle $|z| = 1$ when $\Re\big(\sum b_j - \sum a_j\big) > 0$. The celebrated Gauss' hypergeometric function is ${}_2F_1$, and the confluent Kummer's function is $\Phi \equiv {}_1F_1$.

Extensions, generalizations and unifications of Euler's Beta function together with related higher transcendent hypergeometric type special functions were investigated recently by a number of authors

[4]The research of T. K. Pogány has been supported in part by Croatian Science Foundation under the project No. 5435.

[*]E-mail: rakeshparmar27@gmail.com

[**]E-mail: poganj@pfri.hr

(see [2, 3], and references therein). In particular, Chaudhry *et al.* [2, p.20, Eq.(1.7)] presented the following extension of the Beta function:

$$\mathrm{B}(x,y;p) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t\,, \qquad \Re(p) > 0\,; \tag{1.1}$$

where for $p = 0$, $\min\{\Re(x), \Re(y)\} > 0$. They also obtained related connections of $\mathrm{B}(x,y;p)$ with Macdonald (or modified Bessel function of the second kind), error and Whittaker functions. Further, Chaudhry *et al.* [3] used $\mathrm{B}(x,y;p)$ to extend the Gauss' hypergeometric and the confluent (Kummer's) hypergeometric functions in the following manner:

$$F_p(a,b;c;z) = \sum_{n\geq 0}^{\infty} (a)_n \frac{\mathrm{B}(b+n,\,c-b\,;\,p)}{\mathrm{B}(b,\,c-b)}\,\frac{z^n}{n!}, \quad p \geq 0,\ |z|<1\,;\ \Re(c) > \Re(b) > 0\,, \tag{1.2}$$

$$\Phi_p(b;c;z) = \sum_{n\geq 0} \frac{\mathrm{B}(b+n,\,c-b\,;\,p)}{\mathrm{B}(b,\,c-b)}\,\frac{z^n}{n!}, \quad p \geq 0\,;\ \Re(c) > \Re(b) > 0\,, \tag{1.3}$$

respectively. More recently, Özarslan and Özergin [14] defined the *extended* first Appell function in the form:

$$F_1(a,b,b';c;x,y;p) = \sum_{m,n\geq 0} (b)_m (b')_n \,\frac{\mathrm{B}(a+m+n,c-a;p)}{\mathrm{B}(a,c-a)}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,, \qquad \Re(p) \geq 0\,, \tag{1.4}$$

provided that $\max\{|x|,|y|\} < 1$. They obtained the following integral representation (see [14, p.1826, Eq.(2.1)]):

$$F_1(a,b,b';c;x,y;p) = \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{\mathrm{B}(a,\,c-a)}(1-xt)^{-b}(1-yt)^{-b'}\,\mathrm{e}^{-\frac{p}{t(1-t)}}\,\mathrm{d}t\,, \tag{1.5}$$

for all $\Re(p) > 0$ and $\max\{|\arg(1-x)|, |\arg(1-y)|\} < \pi$; $\Re(c) > \Re(a) > 0$.

It is clear that the special cases of $(1.1) - (1.4)$ when $p = 0$ reduce to the classical Euler's Beta, Gauss' hypergeometric, confluent hypergeometric and the first Appell functions, respectively. Recently, Choi *et al.* [4] have introduced further extensions for functions $\mathrm{B}(x,y;p)$, $F_p(a,b;c;z)$ and $\Phi_p(b;c;z)$ in the following manner:

$$\mathrm{B}(x,y;p,q) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t\,, \tag{1.6}$$

when $\min\{\Re(x), \Re(y)\} > 0; \min\{\Re(p), \Re(q)\} \geq 0$, and by means of (1.6):

$$F_{p,q}(a,b;c;z) = \sum_{n\geq 0} (a)_n \frac{\mathrm{B}(b+n,\,c-b\,;\,p,q)}{\mathrm{B}(b,\,c-b)}\,\frac{z^n}{n!}, \quad |z|<1;\ \Re(c) > \Re(b) > 0, \tag{1.7}$$

$$\Phi_{p,q}(b;c;z) = \sum_{n\geq 0} \frac{\mathrm{B}(b+n,\,c-b\,;\,p,q)}{\mathrm{B}(b,\,c-b)}\,\frac{z^n}{n!}, \quad \Re(c) > \Re(b) > 0. \tag{1.8}$$

Related properties, various integral representations, Mellin transform are also given in [4].

A further extension of extended Appell function (1.5), in terms of the extended beta function $\mathrm{B}(x,y;p,q)$ (1.6), we introduce as the series:

$$F_1(a,b,b';c;x,y;p,q) = \sum_{m,n\geq 0} (b)_m (b')_n \,\frac{\mathrm{B}(a+m+n,c-a;p,q)}{\mathrm{B}(a,c-a)}\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,, \tag{1.9}$$

which turns out to be a special case of the double series $\mathfrak{F}_1^{(\kappa_\ell)_{\ell\in\mathbb{N}_0}}$ when $\kappa_\ell \equiv 1$ (see [20, p. 256, Eq. (6.3)]). It should be noted that the thorough study of these functions is still an interesting open question. Note that the series (1.9) plays one of the central roles in the present paper. Also, it is clear that when

$p = q$ (resp. $p = q = 0$), the functions in (1.6) – (1.9) reduce to (1.1) – (1.4) (resp., to the classical Euler Beta, Gauss hypergeometric, confluent hypergeometric and Appell functions), respectively.

In terms of the extended beta function $\mathrm{B}(x, y; p, q)$ defined by (1.6), we now introduce the *extended Srivastava's triple hypergeometric function* for all $\alpha,\ \beta,\ \beta' \in \mathbb{C}$ and $\gamma,\ \gamma' \in \mathbb{C} \setminus \mathbb{Z}_0^-$ in the form:

$$H_{A,p,q}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \sum_{k,m,n\geq 0} \frac{(\alpha)_{k+n}(\beta)_{k+m}}{(\gamma)_k} \frac{\mathrm{B}(\beta' + m + n, \gamma' - \beta'; p, q)}{\mathrm{B}(\beta', \gamma' - \beta')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!}\,, \tag{1.10}$$

when $\min\{p, q\} \geq 0$; $|x| < r,\ \ |y| < s,\ |z| < t$, while $r = (1 - s)(1 - t)$ when $p = q = 0$.

The special case of (1.10), $H_{A,0,0} \equiv H_A$ reduces to the Srivastava's triple hypergeometric function H_A, introduced in [16] (see also [17]):

$$\begin{aligned} H_A[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] &= \sum_{k,m,n\geq 0} \frac{(\alpha)_{k+n}(\beta)_{k+m}}{(\gamma)_k} \frac{\mathrm{B}(\beta' + m + n, \gamma' - \beta')}{\mathrm{B}(\beta', \gamma' - \beta')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \\ &= \sum_{k,m,n\geq 0} \frac{(\alpha)_{k+n}(\beta)_{k+m}(\beta')_{m+n}}{(\gamma)_k(\gamma')_{m+n}} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!}\,, \end{aligned} \tag{1.11}$$

where $|x| < r, |y| < s, |z| < t\,; r = (1 - s)(1 - t)$ (compare [18, p. 43, Eq. (11)], and references therein).

Motivated essentially by the potential applications of functions $\mathrm{B}(x, y; p, q)$, $F_{p,q}(a, b; c; z)$, $\Phi_{p,q}(b; c; z)$ and the extended Appell's function $F_1(a, b, b'; c; x, y; p, q)$ in diverse areas of mathematical, physical, engineering and statistical sciences (see [4], and references therein), our aim is to introduce and investigate, in a rather systematic manner, the extended Srivastava's triple hypergeometric functions $H_{A,p,q}$, by presenting:
(i) various Euler and Laplace type integral representations, as well as, further integral representations involving the Bessel and modified Bessel functions in the kernel;
(ii) Mellin transform, Laguerre polynomial representations and certain recurrence relations;
(iii) a set of bounding inequalities, using the underlying new integral expressions, where the main tool is the Luke's rational and exponential bounds for the generalized hypergeometric functions ${}_rF_r$, and as a counterpart, diverse bounds upon the Bessel functions appearing in the kernels of integral representations.

2. ON THE EXTENDED SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTION

In this section we study three different categories of results concerning the newly defined special function $H_{A,p,q}$ that are: integral representations, recurrence and transformations formulas.

2.1. Integral representations. In this subsection, we establish a set of Euler and Laplace type integral representations for function $H_{A,p,q}$. We also obtain certain integral representations for $H_{A,p,q}$ involving the Bessel and modified Bessel functions. We begin with a simple auxiliary integral representation result, which, to the best of our knowledge, is new, and is of interest by itself.

Lemma 2.1. *For all* $\min\{\Re(p), \Re(q)\} > 0$ *and* $\max\{|\arg(1 - x)|, |\arg(1 - y)|\} < \pi$*;* $\Re(c) > \Re(a) > 0$*, we have*

$$F_1(a, b, b'; c; x, y; p, q) = \int_0^1 \frac{t^{a-1}(1 - t)^{c-a-1}}{\mathrm{B}(a, c - a)}\, (1 - xt)^{-b}\, (1 - yt)^{-b'}\, \mathrm{e}^{-\frac{p}{t} - \frac{q}{1-t}}\, \mathrm{d}t\,. \tag{2.1}$$

Proof. Applying the integral expression (1.6) to the extended Beta–function kernel, we can write

$$\begin{aligned} F_1(a, b, b'; c; x, y; p, q) &= \sum_{m,n\geq 0} (b)_m(b')_n \frac{\mathrm{B}(a + m + n, c - a; p, q)}{\mathrm{B}(a, c - a)} \frac{x^m}{m!} \frac{y^n}{n!} \\ &= \sum_{m,n\geq 0} \frac{(b)_m(b')_n}{\mathrm{B}(a, c - a)} \frac{x^m}{m!} \frac{y^n}{n!} \int_0^1 t^{a+m+n-1}(1 - t)^{c-a-1} \mathrm{e}^{-\frac{p}{t} - \frac{q}{1-t}} \mathrm{d}t \end{aligned}$$

$$= \sum_{m,n\geq 0} \binom{-b}{m}\binom{-b'}{n} \frac{(-x)^m(-y)^n}{\mathrm{B}(a,c-a)} \int_0^1 t^{a+m+n-1}(1-t)^{c-a-1}\mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\mathrm{d}t\,.$$

Taking into account the binomial series expansion $(1+u)^{-\alpha} = \sum_{k\geq 0}\binom{-\alpha}{k}u^k$; $|u|<1$, by legitimate exchange of the order of integration and summation, we obtain (2.1).

Now, we are in position to state our first main integral representation result for function $H_{A,p,q}$.

Theorem 2.1. *For all* $\min\{\Re(p),\Re(q)\}>0$ *and for all* $\Re(\gamma')>\Re(\beta')>0$ *when* $p=q=0$*, we have*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \int_0^1 \frac{v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}}{\mathrm{B}(\beta',\gamma'-\beta')\,(1-vy)^{\beta}(1-vz)^{\alpha}} \times {}_2F_1\left(\alpha,\beta;\gamma;\ \frac{x}{(1-vy)(1-vz)}\right)\mathrm{e}^{-\frac{p}{v}-\frac{q}{1-v}}\,\mathrm{d}v\,. \tag{2.2}$$

Proof. Observe first that the extended Srivastava's triple hypergeometric function $H_{A,p,q}$, defined by (1.10), can be expressed as a single series by the extended Appell functions (1.9) as follows:

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \sum_{k\geq 0}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\ F_1[\beta',\beta+k,\alpha+k;\gamma';y,z;p,q]\ \frac{x^k}{k!}\,. \tag{2.3}$$

Next, substituting the integral in (2.1) into (2.3), we obtain

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \frac{1}{\mathrm{B}(\beta',\gamma'-\beta')}\sum_{k\geq 0}\int_0^1 v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}(1-vy)^{-\beta} \times (1-vz)^{-\alpha}\ \exp\left(-\frac{p}{v}-\frac{q}{1-v}\right)\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}\left\{\frac{x}{(1-vy)(1-vz)}\right\}^k \mathrm{d}v. \tag{2.4}$$

Changing the order of summation and integration in (2.4), we arrive at (2.2). Theorem 2.1 is proved.

In the next theorem, we state two equivalent double integral expressions for function $H_{A,p,q}$.

Theorem 2.2. *Let the assumptions of Theorem 2.1 be fulfilled. Then*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \int_0^1\int_0^1 \frac{u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}}{\mathrm{B}(\beta,\gamma-\beta)\,\mathrm{B}(\beta',\gamma'-\beta')} \times \frac{(1-v)^{\gamma'-\beta'-1}(1-vy)^{\alpha-\beta}}{(1-ux-vy-vz+v^2yz)^{\alpha}}\ \mathrm{e}^{-\frac{p}{v}-\frac{q}{1-v}}\,\mathrm{d}u\,\mathrm{d}v\,, \tag{2.5}$$

and for all $\min\{\Re(p),\Re(q)\}>0$ *we have*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \frac{1}{\mathrm{B}(\beta,\gamma-\beta)\,\mathrm{B}(\beta',\gamma'-\beta')} \times \int_0^1\int_0^1 \frac{u^{\gamma-2}v^{\gamma'-2}(1-vy)^{\alpha-\beta}}{(1-xu-(y+z)v+yzuv)^{\alpha}}\,\mathrm{e}^{-\frac{p}{v}-\frac{q}{1-v}}\,\mathrm{d}u\,\mathrm{d}v\,. \tag{2.6}$$

If $p=q=0$*, then* (2.6) *holds when* $\Re(\gamma)>\Re(\beta)>0$ *and* $\Re(\gamma')>\Re(\beta')>0$.

Proof. Using the well-known integral formula

$${}_2F_1\,(a,\,b;\,c;\,z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,\mathrm{d}t$$

for all $\Re(c)>\Re(b)>0$; $|\arg(1-z)|\leq\pi-\epsilon, 0<\epsilon<\pi$, from (2.2) we obtain (2.5). Next, the representation (2.6) can easily be deduced from (2.5) after some simple algebraic manipulations. Theorem 2.2 is proved.

Theorem 2.3. *Let* $\min\{\Re(p),\Re(q)\}>0$ *and* $x>0$, $\max\{\Re(y),\Re(z)\}<1$, *and* $\min\{\Re(\alpha),\Re(\beta)\}>0$ *when* $p=q=0$. *Then we have*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';\mp x,y,z]=\frac{\Gamma(\gamma)\,x^{\frac{1-\gamma}{2}}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty \mathrm{e}^{-s-t}\,t^{\alpha-\frac{\gamma-1}{2}-1}s^{\beta-\frac{\gamma-1}{2}-1}$$
$$\times\left\{\begin{matrix} J_{\gamma-1}(2\sqrt{xst})\\ I_{\gamma-1}(2\sqrt{xst})\end{matrix}\right\}\Phi_{p,q}(\beta';\gamma';ys+zt)\,\mathrm{d}s\,\mathrm{d}t\,. \qquad (2.7)$$

Proof. Using the integral form of the Pochhammer symbols $(\alpha)_{k+n}$ and $(\beta)_{k+m}$ and the elementary series identity (see [18, p. 52, Eq. 1.6(2)]):

$$\sum_{m_1,m_2\ge 0}\Omega\,(m_1+m_2)\;\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}=\sum_{m\ge 0}\Omega(m)\;\frac{(x_1+x_2)^m}{m!},$$

in (1.10), and afterward applying the definition of extended confluent hypergeometric function (1.8), we obtain

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z]$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty \mathrm{e}^{-s-t}\,t^{\alpha-1}s^{\beta-1}{}_0F_1(-;\gamma;xst)\,\Phi_{p,q}(\beta';\gamma';ys+zt)\,\mathrm{d}s\,\mathrm{d}t. \qquad (2.8)$$

Next, observe that the Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ can be expressed in terms of hypergeometric functions as follows (see [21]):

$$J_\nu(z)=\frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\text{---}\,;\nu+1;-\frac{1}{4}z^2\right)\quad\text{and}\quad I_\nu(z)=\frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\text{---}\,;\nu+1;\frac{1}{4}z^2\right),\quad (2.9)$$

being $\nu\in\mathbb{C}\setminus\mathbb{Z}^-$ in both cases. Finally, combining (2.8) and (2.9), we obtain (2.7). Theorem 2.3 is proved.

2.2. Mellin transforms and representations via Laguerre polynomials. The double Mellin transforms of suitable classes of integrable functions $f(x,y)$ with indices r and s are usually defined by (see [15, p. 293, Eq. (7.1.6)]):

$$\mathcal{M}\{f(x,y)\}(r,s)=\int_0^\infty\int_0^\infty x^{r-1}y^{s-1}\;f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

provided that the improper integral exists.

Theorem 2.4. *For all* $\min\{\Re(r),\ \Re(s)\}>0$ *and* $\Re(\beta'+r)>0$, $\Re(\gamma'+s-\beta')>0$ *the Mellin transform of* $H_{A,p,q}$ *with respect to* $p,q\ge 0$ *is given by formula:*

$$\mathcal{M}\{H_{A,p,q}\}(r,s)=\frac{\Gamma(r)\Gamma(s)\mathrm{B}(\beta'+r,\gamma'+s-\beta')}{\mathrm{B}(\beta',\gamma'-\beta')}\,H_A[\alpha,\beta,\beta'+r;\gamma,\gamma'+r+s;x,y,z]\,. \qquad (2.10)$$

Proof. Using the definition of Mellin transform, we find from (1.10) that

$$\mathcal{M}\left\{H_{A,p,q}\right\}(r,s)=\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z]\,\mathrm{d}p\,\mathrm{d}q$$
$$=\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\left(\sum_{k,m,n\ge 0}\frac{(\alpha)_{k+n}(\beta)_{k+m}}{(\gamma)_k}\;\frac{\mathrm{B}(\beta'+m+n,\gamma'-\beta';p,q)}{\mathrm{B}(\beta',\gamma'-\beta')}\frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\right)dp\,dq$$
$$=\frac{1}{\mathrm{B}(\beta',\gamma'-\beta')}\sum_{k,m,n\ge 0}\frac{(\alpha)_{k+n}(\beta)_{k+m}}{(\gamma)_k}\;\frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}\mathrm{B}(\beta'+m+n,\gamma'-\beta';p,q)\,\mathrm{d}p\,\mathrm{d}q.$$

Next, applying the formula (see [4, p.342, Eq. (2.1)])

$$\int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \mathrm{B}(x,\, y\,;\, p,q)\, dp\, dq = \Gamma(r)\Gamma(s)\mathrm{B}(x+r,\, y+s) \qquad (\Re(r)>0,\ \Re(s)>0)$$

to the double integral, we obtain

$$\mathcal{M}\left\{H_{A,p,q}\right\}(r,s) = \Gamma(r)\Gamma(s) \sum_{k,m,n\geq 0} \frac{(\alpha)_{k+n}(\beta)_{k+m}}{(\gamma)_k} \frac{\mathrm{B}(\beta'+m+n+r, \gamma'-\beta'+s)}{\mathrm{B}(\beta',\gamma'-\beta')} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!},$$

which, in view of (1.11), gives (2.10). Theorem 2.4 is proved.

The special case of (2.10) when $r = s = 1$ yields the following relation between the function $H_{A,p,q}$ and the Srivastava's triple hypergeometric function H_A:

$$\int_0^\infty H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z]\, \mathrm{d}p\, \mathrm{d}q = \frac{\beta'(\gamma'-\beta')}{\gamma'(\gamma'+1)}\, H_A[\alpha,\beta,\beta'+1;\gamma,\gamma'+2;x,y,z]\,,$$

provided that $\Re(\gamma') > \Re(\beta') > 0$.

Theorem 2.5. *The following Laguerre polynomial representation holds for $\Re(p) > 0,\ \Re(q) > 0$:*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \frac{\mathrm{e}^{-p-q}}{\mathrm{B}(\beta',\gamma'-\beta')} \sum_{m,n\geq 0} \mathrm{B}(\beta'+n+1, \gamma'-\beta'+m+1)$$
$$\times\, H_{A,p,q}[\alpha,\beta,\beta'+n+1;\gamma,\gamma'+m+n+2;x,y,z]\ L_m(p)\ L_n(q)\,.$$

Proof. We start by recalling the following identity, *in a slightly corrected form*, due to Choi *et al.* [4, p. 350, Eq. (5.5)]:

$$\exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) = e^{-p-q} \left\{ \sum_{m,\,n=0}^{\infty} L_m(p)\, L_n(q)\, .t^{n+1}\,(1-t)^{m+1} \right\}$$

Using this identity, from (2.2) we obtain

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \frac{e^{-p-q}}{\mathrm{B}(\beta',\gamma'-\beta')} \int_0^1 v^{\beta'-1}(1-v)^{\gamma'-\beta'-1}(1-vy)^{-\beta}(1-vz)^{-\alpha}$$
$$\times\ {}_2F_1\left[\alpha,\beta;\gamma;\ \frac{x}{(1-vy)(1-vz)}\right] \left\{ \sum_{m,\,n=0}^{\infty} L_m(p)\, L_n(q)\, v^{n+1}\,(1-v)^{m+1} \right\} dv. \qquad (2.11)$$

Now, changing integration and summation order in the representation (2.11) and using (2.2), we obtain the desired result. Theorem 2.5 is proved.

2.3. Transformations and recurrence relations. In this subsection, we first derive a transformation formula and then obtain a recurrence relation for function $H_{A,p,q}$.

Theorem 2.6. *The following transformation formula for $H_{A,p,q}$ holds:*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z]$$
$$= (1-y)^{-\beta}(1-z)^{-\alpha} H_{A,q,p}\left[\alpha,\beta,\gamma'-\beta';\gamma,\gamma';\frac{x}{(1-y)(1-z)},\frac{y}{y-1},\frac{z}{z-1}\right] \qquad (2.12)$$

Proof. Applying to (2.8) the extended Kummer's transformation (see [4, p.361, Eq.(11.4)]) $\Phi_{p,q}(\beta;\gamma;z) = e^z\ \Phi_{q,p}(\gamma-\beta;\gamma;-z)$, we find that

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty e^{-s(1-y)-t(1-z)} t^{\alpha-1} s^{\beta-1}$$
$$\times\ {}_0F_1(-;\gamma;xst)\ \Phi_{q,p}(\gamma'-\beta';\gamma';-ys-zt)\, \mathrm{d}t\, \mathrm{d}s.$$

The substitution $t(1-z)=u, \quad s(1-y)=v$ leads to

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = \frac{(1-y)^{-\beta}(1-z)^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty e^{-u-v} u^{\alpha-1} v^{\beta-1}$$
$$\times \; {}_0F_1\left(-;\gamma;\frac{xuv}{(1-y)(1-z)}\right) \Phi_{q,p}\left(\gamma'-\beta';\gamma;\frac{yv}{(y-1)}+\frac{zu}{(z-1)}\right) \mathrm{d}u\,\mathrm{d}v,$$

which is exactly the same as (2.12). Theorem 2.6 is proved.

Theorem 2.7. *The following recurrence relation for $H_{A,p,q}$ holds:*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] = H_{A,p,q}[\alpha,\beta,\beta';\gamma-1,\gamma';x,y,z]$$
$$+ \frac{\alpha\beta x}{\gamma(1-\gamma)} H_{A,p,q}[\alpha+1,\beta+1,\beta';\gamma+1,\gamma';x,y,z]\,.$$

Proof. Using in the integral representation (2.8) and the contiguous relation

$${}_0F_1(-;\gamma-1;x) - {}_0F_1(-;\gamma;x) - \frac{x}{\gamma(\gamma-1)}\, {}_0F_1(-;\gamma+1;x) = 0\,,$$

we obtain the desired result. Theorem 2.7 is proved.

3. BOUNDING INEQUALITIES

In this section, we find bounding inequalities for the extended Srivastava's triple hypergeometric function $H_{A,p,q}$. We begin with a simple auxiliary lemma that gives a functional bound for function $\mathrm{B}(x,y;p,q)$, defined by (1.6).

Lemma 3.1. *For all* $\min\{p,q\} \geq 0$ *and* $\min\{\Re(x),\Re(y)\} > 0$ *we have*

$$\mathrm{B}(x,y;p,q) \leq \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\, \mathrm{B}(x,y)\,. \tag{3.1}$$

Indeed, using the sharp estimate

$$\sup_{0<t<1} \exp\left\{-\frac{p}{t}-\frac{q}{1-t}\right\} = \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\,, \quad \min\{p,q\} \geq 0\,,$$

from (1.6) we obtain (3.1).

3.1. Bounds obtained via series representations. Applying the functional bound (3.1) to all series representations of newly extended special functions involving the function $\mathrm{B}(x,y;p,q)$, such as the extended Gauss' hypergeometric $F_{p,q}$, the extended Kummer's confluent hypergeometric $\Phi_{p,q}$, the extended Appell's F_1 and the extended Srivastava's triple hypergeometric $H_{A,p,q}$ functions, given by (1.7) − (1.10), respectively, we obtain the following functional bounds.

Theorem 3.1. *For all* $\min\{p,q\} \geq 0$; $\Re(c) > \Re(b) > 0$ *and for all* $|z| < 1$ *we have*

$$F_{p,q}(a,b;c;z) \leq \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\, {}_2F_1(a,b;c;z) \tag{3.2}$$

$$\Phi_{p,q}(b;c;z) \leq \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\, \Phi(b;c;z)\,. \tag{3.3}$$

Moreover, for $\max\{|\arg(1-x)|,|\arg(1-y)|\} < \pi; \Re(c) > \Re(a) > 0$*, we have*

$$F_1(a,b,b';c;x,y;p,q) \leq \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\, F_1(a,b,b';c;x,y)\,;$$

while for $|x| < r,\ |y| < s,\ |z| < t$ *and* $t = (1-r)(1-s)$ *when* $p = q = 0$*, we have*

$$H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z] \leq \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\, H_A[\alpha,\beta,\beta';\gamma,\gamma';x,y,z].$$

Proof. To prove the inequality (3.2), observe that all parameters and expressions in (3.2) are positive, and hence we can use the series representation of the extended Gauss' hypergeometric function (1.7) and Lemma 2 to conclude that:

$$F_{p,q}(a,b;c;z) \le \frac{\mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}}{\mathrm{B}(b,c-b)} \sum_{n\ge 0} (a)_n \mathrm{B}(b+n,c-b)\, \frac{z^n}{n!}$$

$$= \frac{\mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\Gamma(c)}{\Gamma(b)} \sum_{n\ge 0} \frac{(a)_n\Gamma(b+n)}{\Gamma(c+n)}\, \frac{z^n}{n!} = \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2} \sum_{n\ge 0} \frac{(a)_n (b)_n}{(c)_n}\, \frac{z^n}{n!}.$$

From the last relation we easily obtain (3.2). The other three inequalities can be proved similarly, and so we omit the details. Theorem 3.1 is proved.

3.2. Bounds obtained via integral representations. In this subsection, we establish another type bounding inequalities for function $H_{A,p,q}$, combining its newly derived integral expressions and the bound (3.1) stated in Lemma 2. Since the integrands consist of either the exponential $\exp\{-p/t - q/(1-t)\}$ or rational functions (Theorems 1 and 2) and extended Kummer's $\Phi_{p,q}$ together with the modified Bessel functions (Theorem 3), we need auxiliary tools to bound the involved special functions.

In [11], Y. Luke, has studied, among others, the problem of two-sided inequalities for ${}_rF_r$–type generalized hypergeometric function, where the bounds consist of polynomials and/or exponential expressions. We recall some results from [11], which are usable for Kummer's function Φ. If $b_j \ge a_j > 0$, $j = \overline{1,r}$, then for all $x > 0$ we have (see [11, p. 57, Theorem 16, Eq. (5.6)]):

$$\mathrm{e}^{\theta x} < {}_rF_r\big(a_r;\, b_r;\, z\big) < 1 - \theta\big(1 - \mathrm{e}^x\big)\,, \tag{3.4}$$

where

$$\theta = \frac{\max\limits_{1\le j\le r} a_j}{\min\limits_{1\le j\le r} b_j}\,. \tag{3.5}$$

For all $c \ge b > 0$, the bilateral inequality (3.4), applied to the Kummer's confluent function $\Phi(x) = {}_1F_1(b;c;x)$, reduces to the following:

$$\mathrm{e}^{\frac{b}{c}x} \le \Phi(b;\, c;\, z) \le 1 - \frac{b}{c}\big(1 - \mathrm{e}^x\big)\,, \tag{3.6}$$

where the equality holds for $b = c$. Also, we point out some other estimates for ${}_2F_1$ from [11, p. 55, Theorem 13, Eqs. (4.21), (4.23)], which are too complicated to be used here.

For another estimation purposes we recall certain bounding inequalities for function $J_\nu(t)$ on the positive real half–axis. We first mention von Lommel's results (see [9], [10, pp. 548–549], and also [21, p. 406]):

$$|J_\nu(t)| \le 1, \qquad |J_{\nu+1}(t)| \le \frac{1}{\sqrt{2}}, \qquad \nu > 0,\, u \in \mathbb{R}, \tag{3.7}$$

and the bound obtained by Minakshisundaram and Szász (see [12, p. 37]):

$$|J_\nu(t)| \le \frac{1}{\Gamma(\nu+1)} \left(\frac{|t|}{2}\right)^\nu, \qquad t \in \mathbb{R}. \tag{3.8}$$

Another bounds were derived by Landau [8], who gave in a sense the best possible bounds for $J_\nu(t)$ with respect to ν and t. These bounds read as follows:

$$|J_\nu(t)| \le b_L\, \nu^{-1/3}, \qquad b_L = \sqrt[3]{2} \sup_{t\ge 0} \mathrm{Ai}(t), \tag{3.9}$$

$$|J_\nu(t)| \le c_L\, |t|^{-1/3}, \qquad c_L = \sup_{t\ge 0} t^{1/3} J_0(t)\,, \tag{3.10}$$

where $\mathrm{Ai}(\cdot)$ stands for the Airy function.

Krasikov [5] established uniform bounds for $|J_\nu(t)|$. Let $\nu > -1/2$, then

$$J_\nu^2(t) \le \frac{4\big(4t^2 - (2\nu+1)(2\nu+5)\big)}{\pi\big((4t^2-\lambda)^{3/2}-\lambda\big)} =: \mathfrak{K}_\nu(t),$$

for all $t > \frac{1}{2}\sqrt{\lambda + \lambda^{2/3}}$, $\quad \lambda := (2\nu+1)(2\nu+3)$. This estimate is sharp in a certain sense (see [5, Theorem 2]). In turn, Krasikov recently has obtained a set of more precise and simpler bounds for $|J_\nu(t)|$ (see [6, 7]). More precisely, for all $\nu \ge 1/2$ and for all $t \ge 0$ the following inequality holds (see [6, p. 210, Theorem 3]):

$$\left|t^2 - \left|\nu^2 - \frac{1}{4}\right|\right|^{1/4} |J_\nu(t)| \le \sqrt{\frac{2}{\pi}}, \tag{3.11}$$

where the constant on the right-hand side is sharp. Next, Theorem 4 from [6, p. 210] implies the following inequality:

$$|J_\nu(t)| \le \sqrt{\frac{2}{\pi t}} + \rho\, c \left|\nu^2 - \frac{1}{4}\right| t^{-3/2}, \qquad t > 0,\ |\rho| < 1, \tag{3.12}$$

where

$$c = \begin{cases} \left(\frac{2}{\pi}\right)^{3/2}, & x \ge 0,\ |\nu| \le 1/2 \\ \frac{4}{5}, & 0 < x < \sqrt{|\nu^2 - 1/4|},\ \nu > 1/2 \\ \frac{2}{\pi}, & x \ge \sqrt{|\nu^2-1/4|},\ \nu > 1/2. \end{cases}$$

Here the constant c cannot be less then $1/\sqrt{2\pi}$. For another kind of bounds for function $J_\nu(t)$ consult [6, Theorems 2, 5, 6] and [7, Theorems 2, 4].

It is worth to mention that Olenko [13, Theorem 1] established the following upper bound:

$$\sup_{t\ge 0} \sqrt{t}|J_\nu(t)| \le b_L \sqrt{\nu^{1/3} + \frac{\alpha_1}{\nu^{1/3}} + \frac{3\alpha_1^2}{10\nu}} = d_O, \qquad \nu > 0, \tag{3.13}$$

where α_1 is the smallest positive zero of the Airy–function Ai and b_L is the Landau's constant from above. In this respect we also point out Krasikov's result [6, p. 211, Eq. (7)].

Further considerable upper bounds are listed, for instance, in [1, 19].

Finally, a different approach to estimate the function $|J_\mu(t)|$ was used by Srivastava and Pogány in [19]. Let us denote by $\chi_S(x)$ the characteristic (or indicator) function of a set S, that is, $\chi_S(x) = 1$ for $x \in S$ and $\chi_S(x) = 0$ elsewhere. In this approach, the integration interval is the positive real half-axis, therefore we need an efficient bound for $|J_\mu(t)|$ on $(0, A]$, $A > \sqrt{\lambda + \lambda^{2/3}}/2$. So, we use the bounding function

$$|J_\mu(t)| \le \mathcal{V}_\mu(t) := \frac{d_O}{\sqrt{t}}\chi_{(0,A_\lambda]}(t) + \sqrt{\mathfrak{K}_\mu(t)}\left(1 - \chi_{(0,A_\lambda]}(t)\right), \tag{3.14}$$

where, by simplicity reasons, we choose $A_\lambda = \frac{1}{2}\left(\lambda + (\lambda+1)^{2/3}\right)$, because $\mathfrak{K}_\nu(t)$ is positive and monotone decreasing for $t \in \frac{1}{2}\big((\lambda + \lambda^{2/3}), \infty\big)$ (cf. [19, §3]). Notice that as A_λ can be taken any function of the form $\frac{1}{2}\big(\lambda + (\lambda+\eta)^{2/3}\big)$ with $\eta > 0$. (The interested reader is referred to [1], too). Obviously, combining (3.11), (3.12) in $\mathcal{V}_\mu(t)$ replacing Olenko's result and/or $\mathfrak{K}_\nu(t)$ in (3.14), we can define further bounding functions for $|J_\nu(t)|$.

Since the integral representation (2.7) can also be expressed in terms of modified Bessel function I_ν, we can apply the Luke's estimate to bound $H_{A,p,q}$. This inequality reads as follows (see [11]):

$$I_\mu(t) < \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\mu+1)}\cosh t, \qquad t > 0, \mu + 1 > 0. \tag{3.15}$$

Now we are ready to state and prove our second set of bounding inequality results.

Theorem 3.2. *Let* $\min\{\Re(p),\Re(q)\}>0$ *and for all* $\max\{\Re(y),\Re(z)\}<1$, $\min\{\Re(\alpha),\Re(\beta)\}>0$ *when* $p=q=0$. *Then under* $2\min\{\alpha,\beta\}+1>\gamma>0$ *we have*

$$\big|H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';-x,y,z]\big| \le \frac{\Gamma(\gamma)\,\Gamma(\alpha-\frac{\gamma-1}{2})\,\Gamma(\beta-\frac{\gamma-1}{2})\,|x|^{\frac{1-\gamma}{2}}}{\sqrt{2}\,\Gamma(\alpha)\Gamma(\beta)\,\mathrm{e}^{(\sqrt{p}+\sqrt{q})^2}} \times\left\{1-\frac{\beta'}{\gamma'}\Big(1-(1-y)^{-\beta+\frac{\gamma-1}{2}}\,(1-z)^{-\alpha+\frac{\gamma-1}{2}}\Big)\right\}. \tag{3.16}$$

In the same parameter range for all $x>0$ *we have*

$$\big|H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';-x,y,z]\big| \le \frac{\Gamma(\gamma)\,\Gamma(\alpha-\frac{\gamma-1}{2})\,\Gamma(\beta-\frac{\gamma-1}{2})\,b_L\,|x|^{\frac{1-\gamma}{2}}}{\sqrt[3]{\gamma-1}\,\Gamma(\alpha)\Gamma(\beta)\,\mathrm{e}^{(\sqrt{p}+\sqrt{q})^2}} \times\left\{1-\frac{\beta'}{\gamma'}\Big(1-(1-y)^{-\beta+\frac{\gamma-1}{2}}\,(1-z)^{-\alpha+\frac{\gamma-1}{2}}\Big)\right\}. \tag{3.17}$$

For all $\min\{\alpha,\beta,\gamma\}>0$ *and for* $x>0;\ y,z<1$, *we have*

$$\big|H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';-x,y,z]\big| \le \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\left\{1-\frac{\beta'}{\gamma'}\Big(1-(1-y)^{-\alpha}\,(1-z)^{-\beta}\Big)\right\}. \tag{3.18}$$

For $\min\{\alpha,\beta\}>0;\ \gamma>1$ *and for* $x>0;\ y,z<1$, *we have*

$$\Big|H_{A,p,q}(\alpha,\beta,\beta';\gamma,\gamma';-x,y,z]\Big| \le \frac{\Gamma(\gamma)\,\mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}}{\Gamma(\alpha)\,\Gamma(\beta)} \times\begin{cases} \frac{c_L\,x^{-\frac{\gamma}{2}+\frac13}}{\sqrt[3]{2}}\,\Gamma\left(\alpha-\frac{\gamma}{2}+\frac13\right)\Gamma\left(\beta-\frac{\gamma}{2}+\frac13\right)\left(1-\frac{\beta'}{\gamma'}+\frac{\frac{\beta'}{\gamma'}}{(1-y)^{\beta-\frac{\gamma}{2}+\frac13}(1-z)^{\alpha-\frac{\gamma}{2}+\frac13}}\right)\\[2ex] \frac{d_O\,x^{-\frac{\gamma}{2}+\frac14}}{\sqrt{2}}\,\Gamma\left(\alpha-\frac{\gamma}{2}+\frac14\right)\Gamma\left(\beta-\frac{\gamma}{2}+\frac14\right)\left(1-\frac{\beta'}{\gamma'}+\frac{\frac{\beta'}{\gamma'}}{(1-y)^{\beta-\frac{\gamma}{2}+\frac14}(1-z)^{\alpha-\frac{\gamma}{2}+\frac14}}\right);\end{cases}$$

here the bound above holds if $6\alpha-3\gamma+2>0$, *while expression below appears for* $4\alpha-2\gamma+1>0$. *Moreover, for all* $\min\{\alpha,\beta,\gamma\}>0$ *and for all* $y,z\in(0,1-\sqrt{x}),\ x>0$, *we have*

$$\big|H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';x,y,z]\big| \le \mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}\left\{\frac{1-\frac{\beta'}{\gamma'}}{(1-\sqrt{x})^{\alpha+\beta}}+\frac{\frac{\beta'}{\gamma'}}{(1-\sqrt{x}-y)^{\beta}(1-\sqrt{x}-z)^{\alpha}}\right\}. \tag{3.19}$$

Proof. From the double integral representation (2.7) and the estimate (3.3), we obtain

$$\big|H_{A,p,q}[\alpha,\beta,\beta';\gamma,\gamma';-x,y,z]\big| \le \frac{\Gamma(\gamma)\,\mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}}{\Gamma(\alpha)\Gamma(\beta)\,|x|^{\frac{\gamma-1}{2}}}\int_0^\infty\int_0^\infty \mathrm{e}^{-s-t}\,t^{\alpha-\frac{\gamma-1}{2}-1}s^{\beta-\frac{\gamma-1}{2}-1} \times\left\{\begin{matrix}\big|J_{\gamma-1}(2\sqrt{xst})\big|\\ I_{\gamma-1}(2\sqrt{xst})\end{matrix}\right\}\Phi(\beta';\gamma';ys+zt)\,\mathrm{d}s\,\mathrm{d}t =: \frac{\Gamma(\gamma)\,\mathrm{e}^{-(\sqrt{p}+\sqrt{q})^2}}{\Gamma(\alpha)\Gamma(\beta)\,|x|^{\frac{\gamma-1}{2}}}\,R_1\,. \tag{3.20}$$

Next, using the second inequality in (3.6), we get

$$R_1 \le \int_0^\infty\int_0^\infty \mathrm{e}^{-s-t}\,t^{\alpha-\frac{\gamma-1}{2}-1}s^{\beta-\frac{\gamma-1}{2}-1}\left\{\begin{matrix}\big|J_{\gamma-1}(2\sqrt{xst})\big|\\ I_{\gamma-1}(2\sqrt{xst})\end{matrix}\right\}\left\{1-\frac{\beta'}{\gamma'}\left(1-\mathrm{e}^{ys+zt}\right)\right\}\mathrm{d}s\,\mathrm{d}t =: R_2\,. \tag{3.21}$$

Now, we bound the modulus of the Bessel functions in the integrand of R_2 for each of the cases of the theorem. First, using the von Lommel's uniform bound (3.7), valid for all $\gamma>0$, from (3.20) and (3.21) we obtain (3.16). In similar manner, (3.17) follows from the Landau's first inequality (3.9) and (3.20), (3.21).

The bound (3.8) due to Minakshisundaram and Szász is of magnitude $|J_{\gamma-1}(t)| \le C_\gamma\, t^\kappa$, and so do the second Landau's (3.10) and Olenko's (3.13) inequalities, where $C_\gamma > 0$, $\kappa \in \{\nu, -\frac{1}{3}, -\frac{1}{2}\}$, respectively. Thus, by these three inequalities, we get

$$\begin{aligned} R_2 &\le C_\gamma |x|^{\frac{\kappa}{2}} \int_0^\infty \int_0^\infty \mathrm{e}^{-s-t}\, t^{\alpha+\frac{\kappa-\gamma+1}{2}-1} s^{\beta+\frac{\kappa-\gamma+1}{2}-1} \left\{1 - \frac{\beta'}{\gamma'}\left(1-\mathrm{e}^{ys+zt}\right)\right\} \mathrm{d}s\,\mathrm{d}t \\ &= C_\gamma |x|^{\frac{\kappa}{2}} \left\{ \left(1-\frac{\beta'}{\gamma'}\right) \int_0^\infty \int_0^\infty \mathrm{e}^{-s-t} t^{\alpha+\frac{\kappa-\gamma+1}{2}-1} s^{\beta+\frac{\kappa-\gamma+1}{2}-1}\, \mathrm{d}s\,\mathrm{d}t \right. \\ &\qquad \left. + \frac{\beta'}{\gamma'} \int_0^\infty \int_0^\infty \mathrm{e}^{-(1-y)s-(1-z)t} t^{\alpha+\frac{\kappa-\gamma+1}{2}-1} s^{\beta+\frac{\kappa-\gamma+1}{2}-1}\, \mathrm{d}s\,\mathrm{d}t \right\} \\ &= C_\gamma |x|^{\frac{\kappa}{2}}\, \Gamma\left(\alpha + \frac{\kappa-\gamma+1}{2}\right) \Gamma\left(\beta + \frac{\kappa-\gamma+1}{2}\right) \\ &\qquad \times \left\{ 1 - \frac{\beta'}{\gamma'} + \frac{\beta'}{\gamma'} \frac{1}{(1-y)^{\beta+\frac{\kappa-\gamma+1}{2}} (1-z)^{\alpha+\frac{\kappa-\gamma+1}{2}}} \right\}. \end{aligned}$$

Now, choosing $\kappa = \gamma - 1$ we arrive at (3.18), then for $\kappa = -\frac{1}{3}, -\frac{1}{2}$ we realize the bounds affiliated to the second Landau's and Olenko's estimates, respectively.

As to the use of the bound (3.15) to estimate R_2, we remark that $\cosh u \le e^u, u > 0$, and hence by the arithmetic mean-geometric mean inequality, we get

$$\begin{aligned} R_2 &\le \frac{|x|^{\frac{\gamma-1}{2}}}{\Gamma(\gamma)} \int_0^\infty \int_0^\infty \mathrm{e}^{-s-t+2\sqrt{xst}}\, t^{\alpha-1}\, s^{\beta-1} \left\{1 - \frac{\beta'}{\gamma'}\left(1-\mathrm{e}^{ys+zt}\right)\right\} \mathrm{d}s\,\mathrm{d}t \\ &\le \frac{|x|^{\frac{\gamma-1}{2}}}{\Gamma(\gamma)} \int_0^\infty \int_0^\infty \mathrm{e}^{-(1-\sqrt{x})s-(1-\sqrt{x})t}\, t^{\alpha-1}\, s^{\beta-1} \left\{1 - \frac{\beta'}{\gamma'}\left(1-\mathrm{e}^{ys+zt}\right)\right\} \mathrm{d}s\,\mathrm{d}t =: R_3. \end{aligned}$$

Thus, we have

$$R_3 = \frac{\Gamma(\alpha)\,\Gamma(\beta)\,|x|^{\frac{\gamma-1}{2}}}{\Gamma(\gamma)} \left\{ \frac{1-\frac{\beta'}{\gamma'}}{(1-\sqrt{x})^{\alpha+\beta}} + \frac{\frac{\beta'}{\gamma'}}{(1-\sqrt{x}-y)^\beta (1-\sqrt{x}-z)^\alpha} \right\},$$

which proves the upper bound in (3.19). Theorem 3.2 is proved.

4. CONCLUDING REMARKS AND OBSERVATIONS

In the present paper, we have introduced the extended Srivastava triple hypergeometric functions $H_{A,p,q}$ with the help of the extended Beta function $\mathrm{B}(x,\, y\,;\, p, q)$. The special cases of (2.2) − (2.8) and (2.12) for $p = q = 0$, reduce to the already known results for the triple hypergeometric function H_A (see [16–18]).

To refine the bounds presented above, we can also apply Luke's companion estimate to (3.4) (see [11, p. 57, Theorem 16, Eq. (5.8)])

$$1 + \theta x \mathrm{e}^{\frac{\psi}{2}x} < {}_rF_r\big(a_r;\, b_r;\, z\big) < 1 + \theta x \left(1 - \frac{\psi}{2} + \frac{\psi}{2}\mathrm{e}^x\right), \tag{4.1}$$

where θ is given by (3.5), and

$$\psi = \frac{1 + \max\limits_{1\le j\le r} a_j}{1 + \min\limits_{1\le j\le r} b_j}\,.$$

These notations simplify (4.1) to

$$1 + \frac{b}{c} x \exp\left\{\frac{1+b}{2(1+c)}\, x\right\} < \Phi(b;\, c;\, z) < 1 + \frac{b}{c} x \left(1 - \frac{1+b}{2(1+c)}\left(1 - \mathrm{e}^x\right)\right).$$

Now, following the procedure, used in the proof of Theorem 3.2 with this upper bound and/or replacing some of bounds for the Bessel J_ν used there either by Krasikov's results (3.11), (3.12) or another kind bounds exposed in [6, 7], we can obtain a new set of bounding inequalities for function $H_{A,p,q}$. However, this approach will be exploited in some future work.

Acknowledgements. The authors are indebted to the anonymous referee for constructive comments and suggestions, which substantially encompassed the present paper. We are also thankful for drawing our attention to important publications [6, 7].

REFERENCES

1. Á. Baricz, P.L. Butzer, T.K. Pogány, "Alternating Mathieu series, Hilbert-Eisenstein series and their generalized Omega functions", in T. Rassias, G. V. Milovanović (Eds.), *Analytic Number Theory, Approximation Theory, and Special Functions* - In Honor of Hari M. Srivastava, Springer, New York, 2014, 775–808.
2. M.A. Chaudhry, A. Qadir, M. Rafique and S.M.Zubair, "Extension of Euler's Beta function", J. Comput. Appl. Math., **78**, 19–32, 1997.
3. M.A. Chaudhry, A. Qadir, H.M. Srivastava and R.B. Paris, "Extended hypergeometric and confluent hypergeometric functions", Appl. Math. Comput., **159**, 589–602, 2004.
4. J. Choi, A.K. Rathie and R.K. Parmar, "Extension of extended beta, hypergeometric and confluent hypergeometric functions", Honam Math. J., **36** (2), 357–385, 2014.
5. I. Krasikov, "Uniform bounds for Bessel functions", J. Appl. Anal., **12** (1), 83–91, 2006.
6. I. Krasikov, "Approximations for the Bessel and Airy functions with an explicit error term", LMS J. Comput. Math., **17** (1), 209–225, 2014.
7. I. Krasikov, "On the Bessel function $J_\nu(x)$ in the transition region", LMS J. Comput. Math., **17** (1), 273–281, 2014.
8. L. Landau, "Monotonicity and bounds on Bessel functions", in *Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory* (Berkeley, CA, 1999), 147–154 *Electron. J. Differ. Equ. Conf.*, 4, Southwest Texas State Univ., 2000.
9. E.C.J. von Lommel, "Die Beugungserscheinungen einer kreisrunden Öffnung und eines kreisrunden Schirmchens theoretisch und experimentell bearbeitet", Abh. der math. phys. Classe der k.b. Akad. der Wiss. (München), **15**, 229–328, 1884-1886.
10. E.C.J. von Lommel, "Die Beugungserscheinungen geradlinig begrenzter Schirme", Abh. der math. phys. Classe der k.b. Akad. der Wiss. (München), **15**, 529–664, 1884–1886.
11. Y.L. Luke, "Inequalities for generalized hypergeometric functions", J. Approx. Theory, **5**, 41–65, 1974.
12. S. Minakshisundaram and O. Szász, "On absolute convergence of multiple Fourier series", Trans. Amer. Math. Soc., **61** (1), 36–53, 1947.
13. A.Ya. Olenko, "Upper bound on $\sqrt{x}J_\nu(x)$ and its applications", Integral Transforms Spec. Functions, **17** (6), 455–467, 2006.
14. M.A. Özarslan and E. Özergin, "Some generating relations for extended hypergeometric function via generalized fractional derivative operator", Math. Comput. Modelling, **52**, 1825–1833, 2010.
15. R.B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals* (Cambridge University Press, Cambridge, 2001).
16. H.M. Srivastava, "Hypergeometric functions of three variables", Gaṇita, **15**, 97–108, 1964.
17. H.M. Srivastava, "Some integrals representing triple hypergeometric functions", Rend. Circ. Mat. Palermo, **16** (2), 99–115, 1967.
18. H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series* (Wiley, New York, 1985).
19. H.M. Srivastava and T.K. Pogány, "Inequalities for a unified Voigt functions in several variables", Russian J. Math. Phys., **14** (2), 194–200, 2007.
20. H.M. Srivastava, R.K. Parmar and P. Chopra, "A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions", Axioms, **1** (3), 238–258, 2012.
21. G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, London, 1944).