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FUNCTIONAL ANALYSIS

On the Metric Type of Measurable Functions and Convergence in Distribution

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Abstract—In the present paper, sequences of real measurable functions defined on a measure space $([0,1],\ \mu)$, where μ is the Lebesgue measure, are studied. It is proved that for every sequence f_n that converges to f in distribution, there exists a sequence of automorphisms S_n of $([0,1],\ \mu)$ such that $f_n(S_n(t))$ converges to $f(t)$ in measure. Connection with some known results is also discussed.

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1. INTRODUCTION

Let f_n, $n=1,2,\dots$ and f be real measurable functions defined on a measure space (Ω,μ), $\mu(\Omega)=1$, and let F_n, $n=1,2,\dots$ and F be the corresponding distribution functions, that is,

$$F_n(x)=\mu(\omega\in\Omega:\ f_n(\omega)\le x),\quad -\infty<x<\infty,$$

$$F(x)=\mu(\omega\in\Omega:\ f(\omega)\le x),\quad -\infty<x<\infty.$$

A sequence f_n is said to converge to f in distribution if $F_n(x)\to F(x)$ as $n\to\infty$ at each continuity point of F. We use the notation $\overset{D}{\lim}$ for this convergence. It is well known that convergence in measure implies convergence in distribution (see [1], p. 31). Clearly the converse is not true.

We also use the notions of isomorphism of measure spaces and metric type of measurable functions, introduced by V. A. Rokhlin (see [2], [3]). A mapping from one measure space to another is said to be an isomorphism if it is one-to-one and both the mapping itself and its inverse mapping map any measurable set to a measurable set of the same measure. In the case where both spaces coincide, the mapping is called an automorphism.

Two spaces admitting isomorphic mappings to each other, are called isomorphic spaces. Two functions f and g defined on the spaces M and N, respectively, are called isomorphic if there exist null sets $M_1\subset M$ and $N_1\subset N$ and an isomorphic mapping T from $M\setminus M_1$ onto $N\setminus N_1$, such that $f(t)=g(T(t))$ for any $t\in M\setminus M_1$. In this case, we also say that the functions f and g are of the same metric type. From the following chain of equalities

$$\mu\{t\in[0,1]:\ f(t)\le x\}=\mu\{t\in[0,1]:\ g(T(t))\le x\}=\mu\{(g\circ T)^{-1}(-\infty,x]\}$$

$$=\mu\{T^{-1}(g^{-1}(-\infty,x])\}=\mu\{t\in[0,1]:\ g(t)\le x\},$$

it follows that functions of the same metric type are identically distributed. The converse is not true. Here is a simple example:

$$f(t)=t,\ 0\le t\le 1,\quad g(t)=\begin{cases}2t & \text{if } 0\le t\le 1/2,\\ 2(1-t) & \text{if } 1/2\le t\le 1\ .\end{cases}$$

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A necessary and sufficient condition for two functions to be of the same metric type has been obtained by V. A. Rokhlin in his classification theorem (see [2]). In this paper we prove the following result.

Theorem 1.1. *Let f and f_n, $n = 1, 2, \ldots$, be measurable functions defined on $[0, 1]$, and let $f = \overset{D}{\lim\limits_{n\to\infty}} f_n$. Then there exists a sequence S_n, $n = 1, 2, \ldots$, of automorphisms of the space $([0, 1],\ \mu)$ such that*

$$\lim_{n\to\infty} f_n(S_n(t)) = f(t) \text{ in measure on } [0, 1]. \tag{1.1}$$

In paper [4], using the above cited sufficiently complex Rokhlin's classification theorem, it was proved the following result.

Theorem 1.2. *If a sequence of measurable functions f_n, $n = 1, 2, \ldots$, defined on $[0, 1]$, converges in measure to a function f, then there exists a sequence S_n, $n = 1, 2, \ldots$, of automorphisms of the space $([0, 1],\ \mu)$ such that*

$$\lim_{n\to\infty} f_n(S_n(t)) = f(t) \text{ almost everywhere on } [0, 1] \tag{1.2}$$

and

$$\lim_{n\to\infty} \mu\{t \in [0, 1] : S_n(t) \neq t\} = 0. \tag{1.3}$$

Note that the conditions of Theorem 1.1 do not guarantee assertion (1.3) in Theorem 1.2.

Combining Theorems 1.1 and 1.2, we obtain the following result.

Theorem 1.3. *If a sequence of measurable functions f_n, $n = 1, 2, \ldots$, defined on $[0, 1]$ converges in distribution to a function f, then there exists a sequence S_n, $n = 1, 2, \ldots$, of automorphisms of the space $([0, 1],\ \mu)$ such that*

$$\lim_{n\to\infty} f_n(S_n(t)) = f(t) \text{ almost everywhere on } [0, 1]. \tag{1.4}$$

Since functions of the same metric type are identically distributed, Theorem 1.3 implies Skorohod's well-known representation theorem (see [5]).

Corollary 1.1 (Skorohod)**.** *Let X_n, $n = 1,\ 2,\ \ldots$ and X be random variables defined on the probability space $([0, 1],\ \mu)$, and let $X = \overset{D}{\lim\limits_{n\to\infty}} X_n$. Then there exists a sequence of random variables Y_n, $n = 1, 2, \ldots$, such that*
a) for any $n = 1,\ 2,\ \ldots$ the random variables X_n and Y_n have the same distributions;
b) $\lim_{n\to\infty} Y_n(t) = X(t)$ almost surely on $[0, 1]$.

The next result, which is an immediate consequence of Theorem 1.3, shows that identically distributed measurable functions in a sense are close by the metric type.

Corollary 1.2. *If f and g are identically distributed measurable functions on $[0, 1]$, then there exists a sequence S_n, $n = 1, 2, \ldots$, of automorphisms of the space $([0, 1],\ \mu)$ such that*

$$\lim_{n\to\infty} f(S_n(t)) = g(t) \text{ almost everywhere on } [0, 1].$$

2. AUXILIARY RESULTS

We use two auxiliary results, which we state as lemmas.

Lemma 2.1. *Let A and B be measurable sets in $[0,1]$ such that $\mu(A)=\mu(B)>0$. Then the spaces (A,μ) and (B,μ) are isomorphic.*

Proof. It is enough to prove that (A,μ) is isomorphic to $([0,\mu(A)],\mu)$. We can assume that all the points of A are density points. Consider the function

$$f(t)=\int_0^t \chi_A d\mu=\mu([0,t]\cap A),$$

where χ_A stands for the characteristic function of the set A.

Observe that the function f is increasing and absolutely continuous on $[0,1]$. Therefore the image $f(E)$ of every measurable set $E\subset[0,1]$ is measurable. It is clear that f is one-to-one on A. We show that f preserves the measures of the subsets of A. We check this for A.

Indeed, let $\varepsilon>0$ be an arbitrary number. Since $f'(x)=1$ on A, there exists a countable system of intervals Δ_k, $k=1,2,\ldots$, such that

$$A\subset\bigcup_{k=1}^{\infty}\Delta_k,\quad \mu(A)\le\sum_{k=1}^{\infty}\mu(\Delta_k)<\mu(A)+\varepsilon, \tag{2.1}$$

$$(1-\varepsilon)\mu(\Delta_k)<\mu(f(\Delta_k\cap A))<(1+\varepsilon)\mu(\Delta_k)\ ,\ \ k=1,\ 2,\ldots \tag{2.2}$$

Summing over k all terms in (2.2), we obtain

$$(1-\varepsilon)\sum_{k=1}^{\infty}\mu(\Delta_k)\le\mu(f(A))\le(1+\varepsilon)\sum_{k=1}^{\infty}\mu(\Delta_k),$$

which, in view of (2.1), implies that

$$(1-\varepsilon)\mu(A)\le\mu(f(A))\le\mu(A)+\varepsilon(\mu(A)+1+\varepsilon). \tag{2.3}$$

Finally, taking into account that ε is arbitrary, from (2.3) we obtain $\mu(f(A))=\mu(A)$, and the result follows.

We also use the following obvious assertion.

Lemma 2.2. *Let $m>1$ be a natural number and $\varepsilon>0$. Then for any two systems of positive numbers $a_1,\ldots,a_m$ and $b_1,\ldots,b_m$, satisfying the conditions*

$$\sum_{i=1}^{m}a_i=\sum_{i=1}^{m}b_i=1\quad\textit{and}\quad|a_i-b_i|\le\varepsilon,\quad i=1,\ldots,m,$$

we have $\sum\limits_{i=1}^{m}\min\{a_i,b_i\}\ge1-m\varepsilon$.

3. PROOF OF THEOREM 1.1

Let F_n and F be the distribution functions corresponding to f_n and f, respectively. Denote by $C(F)$ the set of continuity points of F. By the assumption of the theorem we have $\lim_{n\to\infty}F_n(x)=F(x)$ for all $x\in C(F)$.

We first consider the case where the sequence f_n is uniformly bounded on $[0,1]$. We take a segment $[a,b]$ so that $a,b\in C(F)$ and $a<f_n(t)\le b$ for all $n=1,2,\ldots$ and $t\in[0,1]$.

We construct a sequence of partitions of the segment $[a,b]$:

$$Q_k=\{a=x_{k,0}<x_{k,1}<\cdots<x_{k,m_k}=b\},\quad k=1,\ 2,\ldots$$

such that for all $k = 1, 2, \ldots$ and $i = 0, 1, \ldots, m_k$ the following conditions are fulfilled: $Q_k \subset Q_{k+1} \subset C(F)$ and

$$\max\{(x_{k,i+1} - x_{k,i}) : 0 \le i \le m_k - 1\} < \frac{1}{k}.$$

Now we construct a sequence S_n of automorphisms of the space $([0,1], \mu)$. To this end, we first take numbers $\varepsilon_k > 0$ to satisfy

$$m_k \varepsilon_k \to 0 \quad \text{as} \quad k \to \infty. \tag{3.1}$$

Next, we take a sequence of natural numbers $1 < n_1 < n_2 < \cdots < n_k < \cdots$, so that for all $k = 1, 2, \ldots$; $n \ge n_k$ and $i = 0, 1, \ldots, m_k$ the inequality is satisfied:

$$|F_n(x_{k,i}) - F(x_{k,i})| < \varepsilon_k. \tag{3.2}$$

The automorphisms S_n we construct by groups. First for $1 \le n < n_1$, then for $n_1 \le n < n_2$ and so on. For $1 \le n < n_1$ we set $S_n = I$, where I is the identity automorphism of the space $([0,1], \mu)$.

Let the automorphisms $S_1, \ldots, S_{n_k - 1}$ be constructed, and let $n_k \le n < n_{k+1}$. Consider the sets

$$E^n_{k,i} = \{t \in [0,1] : x_{k,i} < f_n(t) \le x_{k,i+1}\}, \quad i = 0, 1, \ldots, m_k - 1\,,$$

$$E_{k,i} = \{t \in [0,1] : x_{k,i} < f(t) \le x_{k,i+1}\}, \quad i = 0, 1, \ldots, m_k - 1\,,$$

and observe that since $\mu(E^n_{k,i}) = F_n(x_{k,i+1}) - F_n(x_{k,i})$ and $\mu(E_{k,i}) = F(x_{k,i+1}) - F(x_{k,i})$, then in view of (3.2) we have $|\mu(E^n_{k,i}) - \mu(E_{k,i})| < 2\varepsilon_k$, $n \ge n_k$, $i = 0, 1, \ldots, m_k - 1$. For each pair $E^n_{k,i}$ and $E_{k,i}$, we take the sets $A^n_{k,i} \subset E^n_{k,i}$ and $A_{k,i} \subset E_{k,i}$ to satisfy

$$\mu(A^n_{k,i}) = \mu(A_{k,i}) = \min\{\mu(E^n_{k,i}), \mu(E_{k,i})\}.$$

Then, by Lemma 2.1, there exists an isomorphism of the set $A_{k,i}$ onto $A^n_{k,i}$, which we denote by $S^n_{k,i}$, $n_k \le n < n_{k+1}$, $i = 0, 1, \ldots, m_k - 1$.

Now for each n satisfying $n_k \le n < n_{k+1}$, we construct an automorphism S^n_k of the space $([0,1], \mu)$ as follows. We set

$$S^n_k(t) = S^n_{k,i}(t) \quad \text{for } t \in A^n_{k,i};\ i = 0, 1, \ldots, m_k - 1.$$

On the complementary set $[0,1] \setminus \bigcup\limits_{i=0}^{m_k-1} A^n_{k,i}$, in view of Lemma 2.1, as S^n_k we take an arbitrary isomorphism of $[0,1] \setminus \bigcup\limits_{i=0}^{m_k-1} A^n_{k,i}$ onto $[0,1] \setminus \bigcup\limits_{i=0}^{m_k-1} S^n_k(A^n_{k,i})$.

Next, for each n satisfying $n_k \le n < n_{k+1}$, we set $S_n = S^n_k$. Continuing this process infinitely, we construct the desired sequence of automorphisms S_n of the space $([0,1], \mu)$.

Now we prove that $f_n(S_n(t))$ converges to $f(t)$ in measure on $[0,1]$. Indeed, if $n_k \le n < n_{k+1}$, then according to the construction of S_n and Lemma 1.2, we have

$$\mu\left\{t \in [0,1] : |f_n(t) - f(t)| \ge \frac{1}{k}\right\} < 2 m_k \varepsilon_k. \tag{3.3}$$

It follows from (3.1) and (3.3) that $f_n \circ S_n \to f$ in measure. Thus, Theorem 1.1 is proved in the special case where the sequence f_n is uniformly bounded on $[0,1]$.

Now we prove the theorem in the general case. Let f_n, $n = 1, 2, \ldots$ be an arbitrary (not necessarily uniformly bounded on $[0,1]$) sequence of measurable functions. Let $a, b \in C(F)$, $a < b$ be arbitrary points, and let φ be a continuous, strictly increasing function that maps $(-\infty, \infty)$ onto (a, b). For instance, we can take

$$\varphi(x) = \frac{b-a}{2} \cdot \frac{x}{1+|x|} + \frac{a+b}{2}.$$

Thus, φ and the inverse mapping φ^{-1} are continuous and preserve the order. Let, as above, F_n and F be the distribution functions corresponding to f_n and f, respectively. We prove that the superposition $\varphi \circ f_n$ converges in distribution to $\varphi \circ f$. Indeed, let G_n and G be the distribution functions corresponding to $\varphi \circ f_n$ and $\varphi \circ f$, respectively. Then we have

$$\begin{aligned} G_n(x) &= \mu\{t \in [0,1] : \varphi(f_n(t)) \le x\} = \mu\left\{f_n^{-1}\left(\varphi^{-1}(-\infty, x]\right)\right\} = \mu\left\{f_n^{-1}(a, \varphi^{-1}(x)]\right\} \\ &= \mu\left\{t \in [0,1] : a < f(t) \le \varphi^{-1}(x)\right\} = F_n(\varphi^{-1}(x)) - F_n(a). \end{aligned} \tag{3.4}$$

Similarly, we get

$$G(x) = F(\varphi^{-1}(x)) - F(a). \tag{3.5}$$

If $x \in C(G)$, then by the continuity of φ^{-1}, we have $\varphi^{-1}(x) \in C(F)$. Hence, in view of (3.4), (3.5) and condition $a \in C(F)$, we obtain $\lim_{n\to\infty} G_n(x) = G(x)$ for all $x \in C(G)$. Therefore, according to the proved case, there exists a sequence of automorphisms S_n, such that $\varphi \circ f_n \circ S_n \to \varphi \circ f$ in measure. Taking into account that continuous functions preserve convergence in measure (see [1], p. 39), we obtain

$$f_n \circ S_n = \varphi^{-1}(\varphi \circ f_n \circ S_n) \to \varphi^{-1}(\varphi \circ f) = f \text{ in measure.}$$

This completes the proof of Theorem 1.1.

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