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FUNCTIONAL ANALYSIS

On the Bohr-Riemann Surfaces, II

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Abstract—In this paper we introduce the notions of an analytic curve and equivalent points on the Bohr-Riemann surfaces. By means of constructive and algebraic methods we prove that the points of the Bohr-Riemann surfaces locally have the same number of equivalent points.

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1. INTRODUCTION

In this paper we continue the study of Bohr-Riemann surfaces, started in the paper [1].[1] Recall that the Bohr-Riemann surfaces are obtained by coverings of the generalized plane Δ - a locally compact space, obtained from a Cartesian product $G \times [0, \infty)$ by means of identifying the fibre $G \times \{0\}$ with a point, where G is the group of characters of everywhere dense in Euclidean topology τ of the subgroup Γ of the group of real numbers $\mathbb{R}$. The elements of Δ are the points (α, r) with $\alpha \in G$, $r > 0$ and $* = G \times \{0\}$. Notice that the punctured generalized plane $\Delta \backslash \{*\} := \Delta^0$ is a group with respect to the natural operation of coordinate-wise multiplication. The construction of the space Δ goes back to the paper by Arens and Singer [2], and Δ is canonically identified with the space $\mathcal{C} = \{\alpha r : \alpha \in G, r \in [0, \infty)\}$, which is the analog of complex plane $\mathbb{C}$, consisting of homomorphisms $\alpha r : \Gamma \to \mathbb{C} : a \mapsto \alpha(a) r^a$. The topology on Δ is the standard quotient topology $\tau_\Delta = \{U \subset \Delta : U \in k \times \tau_{[0,\infty)}\}$, where k is the topology on G, and $\tau_{[0,\infty)}$ is the contraction on $[0, \infty)$ of the Euclidean topology τ. Similarly is defined the topology $\tau_{\Delta^0} \cong k \times \tau_{(0,+\infty)}$ on Δ^0. On the space Δ the theory of generalized analytic functions is developed, allowing to obtain new results by means of methods of classical theory of analytic functions (see [3], [4]).

Now we recall the definitions of generalized analytic function and thin set in Δ, by means of which is defined the Bohr-Riemann surface. Let $\Gamma_+ = \{a \in \Gamma : a \geq 0\}$. Each character χ^a, $a \in \Gamma_+$, corresponding to an element $a \in \Gamma_+$, can be extended to a continuous function φ^a on Δ, by setting for $s = \alpha r$:

$$\varphi^a(s) = \chi^a(\alpha) r^a$$

with $\chi^a(\alpha) = \alpha(a)$.

Definition 1.1. *Let D be an open subset of Δ. A continuous on D function f is called a generalized analytic function if for any $s \in D$ there exists a neighborhood $U \subset D$, $s \in U$, such that the restriction of f on U can be uniformly approximated by linear combinations of functions φ^a, $a \in \Gamma_+$.*

The space of all generalized analytic functions on D we denote by $\mathcal{O}(D)$. In the next definition we use the fact that the space $\Delta^0 = \Delta \backslash \{*\}$ locally has a structure of the form: $V \times W$, $V \subset G_a$, $W \subset \mathbb{C}$, where $G_a = \{\alpha \in G; \alpha(a) = 1\}$ with $a \in \Gamma$ (see [3], p. 10-11).

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Definition 1.2. *Let D be an open subset of Δ. A closed subset $K \subset D$ is said to be a thin set if the following conditions hold:*

1) for each point $s \in D$, $s \neq *$, there exist a neighborhood $U \subset D$, $U = V \times W$ and a function $f \in \mathcal{O}(U)$, $f \not\equiv 0$ vanishing on $K \cap U$,

2) for each $\alpha \in V$ the restriction of f on $W_\alpha = \{\alpha\} \times W$ is not identically equal to zero,

if $* \in D$, then there exists a non-trivial function $f \in \mathcal{O}(\Delta_r)$, $\Delta_r \subset D$, vanishing on $\Delta_r \cap K$, where $\Delta_r = \{s \in \Delta : \ |s| \leq r\}$ is a generalized disc of radius r in Δ.

Now we proceed to the definition of Bohr-Riemann surface. It is known (see [5], p. 25) that a mapping of topological spaces $\pi : Y \to X$ is called (in general, branched) *covering*, if it is continuous, open and discrete, that is, for each $x \in X$ the fibre $\pi^{-1}(x)$ is a discrete set in Y. A mapping of topological spaces $\pi : Y \to X$ is said to be *unbranched covering*, if each point $x \in X$ has (the so-called smoothly covered) neighborhood U, such that

$$\pi^{-1}(U) = \bigsqcup_{i \in \mathcal{A}} U_i$$

is a disjunctive union of open sets in Y and all the contractions $\pi|_{U_i} : U_i \to U$ are homeomorphisms. If a set $\mathcal{A}$ is finite (hence, all the fibres of covering consist of the same number of points), then the (unbranched) covering is called *finite-sheeted*, and the number of fibres is called its *number of sheets*.

Definition 1.3. *A topological space X is called a Bohr-Riemann surface over Δ, if there exist a thin set $K \subset \Delta$ and a covering $\pi : X \to \Delta$, such that the contraction π on the set $X^* = X \backslash \pi^{-1}(K)$ is an unbranched finite-sheeted covering of the set $\Delta^* = \Delta \backslash K$.*

Note that questions of group structures on the Bohr-Riemann surfaces were studied in the works [1], [6] − [8]. Now we define the notion of a plane in the space Δ. Since the subgroup Γ is dense in $\mathbb{R}$, it follows that the mapping $\alpha : \mathbb{R} \to G : t \to \alpha_t$ with $\alpha_t(a) = e^{iat}$, $a \in \Gamma$ is injective and the image $\alpha(\mathbb{R})$ is dense in G (see [9], p. 55). The mapping $\alpha : \mathbb{R} \to G$ generates an imbedding

$$\varphi : \mathbb{C} \to \Delta^0 : z = t + iy \mapsto \varphi_z = \alpha_t e^{-y}.$$

The set $\Delta^0 = G \times (0, +\infty)$, which is canonically identified with the space $\{\alpha r : \alpha \in G, r \in (0, \infty)\}$, is a locally compact group with respect to the coordinate-wise multiplication by unit element $\alpha_0 = \alpha(0) = \varphi(0)$. Note that the image $\varphi(\mathbb{C})$ is dense both in Δ^0 and in Δ. A set of the form $\mathbb{C}_s = s\varphi(\mathbb{C})$ is called a *plane* in Δ^0 passing trough the point $s \in \Delta^0$; $\mathbb{C}_0 = \mathbb{C}_{\varphi(0)} (= \varphi(\mathbb{C}))$. Below, using the notion of the plane in the space Δ, we introduce the notions of an analytic curve and equivalent points on the Bohr-Riemann surfaces.

2. ANALYTIC CURVES

Let $\mathbb{C}_0$ be the above defined plane in Δ^0, passing through the unit element α_0 of the group Δ^0. As it was noticed, the set $\mathbb{C}_0$ is an everywhere dense subgroup of the group Δ^0, being the image of the additive group of complex numbers $\mathbb{C}$ under the operation of group homeomorphism $\varphi : \mathbb{C} \to \Delta^0$. Since $\alpha_0 \in \mathbb{C}_0$, then for any $s \in \Delta^0$ the set $s\mathbb{C}_0 = \mathbb{C}_s$ is a plane in Δ^0 passing through s. The set of all planes of this form decompose into cosets of the group Δ^0 by the subgroup $\mathbb{C}_0$.

Consider a curve in Δ^0, that is, the mapping $\gamma : I = [0, 1] \to \Delta^0$, which is continuous with respect to topology τ_{Δ^0} in Δ^0.

Definition 2.1. *A curve $\gamma(I) \subset \Delta^0$ is called an analytic curve, if it is completely contained in the plane $\mathbb{C}_{s_0}$, for some $s_0 \in \Delta^0$.*

Let $\gamma(I)$ be an analytic curve in Δ^0, lying in the plane $\mathbb{C}_{s_0}$, $s_0 \in \Delta^0$. Then for any $s \in \Delta^0$ the curve $\gamma_s(I)$, $\gamma_s(t) = s\gamma(t)$, $t \in I$, lies in plane $\mathbb{C}_{ss_0}$, and hence it is also an analytic curve.

Let X be a Bohr-Riemann surface over Δ, and let K be the thin set of critical points of the covering $\pi : X \to \Delta$ in Δ. Now we define the notion of an analytic curve on the subset $X^* = \pi^{-1}(\Delta^*)$ of the space X, where $\Delta^* = \Delta \backslash K$ (we assume that $* \in K$ and consider the initial covering over $\Delta^0 = \Delta \backslash \{*\}$).

By the theorem of lifting curves (see [5], §4), for each analytic curve $\gamma(I)$ in Δ^* and each point $w \in \pi^{-1}(\gamma(0))$ there exists a unique curve $\hat{\gamma}(I) \subset X^*$ with the origin at the point w, covering the curve $\gamma(I)$, that is, $\hat{\gamma}(0) = w$ and $\gamma(t) = \pi \circ \hat{\gamma}(t)$, $t \in I$. In this case the curve $\hat{\gamma}(I)$ is called the lifting of the curve $\gamma(I)$.

Definition 2.2. *A curve on X^* is called an analytic curve, if it is a lifting of some analytic curve from Δ^*.*

Thus, if $\hat{\gamma}(I) \subset X^*$ is an analytic curve, then it is a lifting of some analytic curve $\gamma(I) \subset \mathbb{C}_s^*$, $s \in \Delta^0$, where $\mathbb{C}_s^* = \mathbb{C}_s \backslash K$. Now we introduce the notion of equivalence on the fibres $\pi^{-1}(s)$, $s \in \Delta^*$.

Definition 2.3. *Two points $w_1, w_2 \in \pi^{-1}(s)$ are called equivalent, if there exists an analytic curve $\hat{\gamma}(I) \subset X^*$, such that $w_1 = \hat{\gamma}(0)$ and $w_2 = \hat{\gamma}(1)$. The equivalence of points w_1 and w_2 will be denoted by the symbol $w_1 \sim w_2$.*

It is easy to check that if $w_1 \sim w_2$ and $w_2 \sim w_3$, then $w_1 \sim w_3$. Hence, the set $\pi^{-1}(s) = \{w_1, ..., w_n\}$ is decomposed into a finite number of equivalence classes. On X^* we define a function $\nu : X^* \to \mathbb{Z}_+$, by setting for $w_0 \in X^*$:

$$\nu(w_0) = card\{w \in \pi^{-1}(\pi(w_0)) : \; w \sim w_0\}.$$

Thus, ν is a function acting on the set X^*, which to each point $w_0 \in X^*$ assigns the number of points that are equivalent to w_0.

3. LOCAL CONSTANCY OF FUNCTION ν

The main result of this section is to prove the local constancy of the counting function $\nu : X^* \to \mathbb{Z}_+$. We first examine the behavior of function ν on the set $\pi^{-1}(\mathbb{C}_s^*)$, $s \in \Delta^*$. Let $s \in \Delta^*$. Denote by $\mu(s)$ the number of equivalence classes (in the sense of Definition 2.3) over s, that is, the number of equivalence classes in the set $\pi^{-1}(s)$:

$$\mu(s) = card\{C(w) : w \in \pi^{-1}(s)\},$$

where

$$C(w) = \{u \in \pi^{-1}(\pi(w)) : u \sim w\}.$$

Thus, C is a mapping acting on X^*, which to each point $w \in X^*$ assigns the set of points that are equivalent to w. Hence, we have $cardC(w) = \nu(w)$. Now we proceed to the proof of local constancy of function ν on $\pi^{-1}(\mathbb{C}_s^*)$, $s \in \Delta^*$.

Lemma 3.1. *Let $s \in \Delta^*$. Then the function $\mu : \Delta^* \to \mathbb{Z}_+$ is constant on $\mathbb{C}_s^*$, while the function $\nu : X^* \to \mathbb{Z}_+$ is constant on the connected components of the preimage $\pi^{-1}(\mathbb{C}_s^*)$.*

Proof. First observe that since the continuous mapping $\pi_s := \pi|_{\pi^{-1}(\mathbb{C}_s^*)} : \pi^{-1}(\mathbb{C}_s^*) \to \mathbb{C}_s^*$ is a covering of a connected and locally linearly connected space $\mathbb{C}_s^*$, in general, non-connected Riemann surface $\pi^{-1}(\mathbb{C}_s^*)$, then its contraction $\pi_s|_L$ on any connected component (linear) L of the surface $\pi^{-1}(\mathbb{C}_s^*)$ is also a covering of the space $\mathbb{C}_s^*$. In particular, since π is a finite-sheeted covering, it follows that the number of such components is finite, and for any $\sigma \in \mathbb{C}_s^*$ the quantity $m(L) = card(\pi^{-1}(\sigma) \cap L)$ is constant, independent of σ and is equal to the number of sheets of the covering of space $\mathbb{C}_s^*$ by mapping $\pi_s|_L$. It is clear that the sum of all $m(L)$ over all connected components L gives n, which is the number of sheets of the covering π).

We fix an arbitrary $\sigma \in \mathbb{C}_s^*$ and consider a partition $\pi^{-1}(\sigma) = C(w_1) \cup ... \cup C(w_m)$ of the fiber $\pi^{-1}(\sigma)$ into disjunctive union of equivalence classes over σ. By the definition of equivalent points, for any $i, 1 \le i \le m$ all the points of the class $C(w_i)$ are connected by analytic curves, and hence, for any $i, 1 \le i \le m$ the class $C(w_i)$ lies in some connected component L_i of the space $\pi^{-1}(\mathbb{C}_s^*)$, containing the point $w_i \in L_i$. Also, we have $\pi^{-1}(\sigma) \cap L_i = C(w_i)$, because all the points of the fiber $\pi^{-1}(\sigma)$ lying in a single connected component with w_i are clearly equivalent to w_i. Since the contraction of the covering π_s on each connected component of the surface $\pi^{-1}(\mathbb{C}_s^*)$ is a covering of the space $\mathbb{C}_s^*$, then there is no other connected components for $\pi^{-1}(\mathbb{C}_s^*)$ different from $L_i, i = \overline{1, m}$, because, otherwise, $\mathbb{C}_s^*$ will contain points that can be covered more than σ times. But this contradicts the fact that π is a covering. Therefore m coincides with the number of connected components of $\pi^{-1}(\mathbb{C}_s^*)$, that is, does not depend on σ. Thus, for any $\sigma \in \mathbb{C}_s^*$, we have $\mu(\sigma) = m$.

Next, let $w \in \pi^{-1}(\mathbb{C}_s^*)$ and L be the connected component of $\pi^{-1}(\mathbb{C}_s^*)$, containing $w \in L$. As it was shown above, we have $C(w) = \pi^{-1}(\pi(w)) \cap L$. Therefore $\nu(w) = cardC(w) = m(L)$. Lemma 3.1 is proved.

To prove local constancy of ν on X^* we will need the following result, which can be considered as a version of the theorem on covering homotopy for covering $\pi : X^* \to \Delta^*$.

Lemma 3.2. *Let $s \in \Delta^*$ be a given point and $\gamma \subset \Delta^*$ be a given closed curve with $\gamma(0) = \gamma(1) = s$. Let $\pi^{-1}(s) = \{x_1, x_2, ..., x_n\}$ and let $\hat{\gamma} : I \to X^*$ be a curve in X^* with $\hat{\gamma}(0) = x_1$, $\hat{\gamma}(1) = x_2$, and $x_1, x_2 \in \pi^{-1}(s)$, $x_1 \ne x_2$, covering the curve γ: $\gamma(t) = \pi \circ \hat{\gamma}(t)$, $t \in I$. Further, let*

$$\pi^{-1}(U) = \bigcup_{i=1}^{n} V_i, \tag{3.1}$$

be a fixed decomposition of the preimage $\pi^{-1}(U)$ of an open smooth covering of the neighborhood U of a point s into the disjunctive union of open sets V_i that are homeomorphic to U under the mappings $\pi|_{V_i} : V_i \to U$ with inverses $\varphi_i = (\pi|_{V_i})^{-1} : U \to V_i$, and $\varphi_i(s) = x_i$, $i = \overline{1, n}$, that is, the enumeration in (3.1) is chosen so that $x_1 \in V_1$, $x_2 \in V_2$. Then there exists an open neighborhood W_0 of the unit element α_0 of the group Δ^0, such that $sW_0 \subset U$ and for any $\sigma \in W_0$ the lifting $\hat{\gamma}_\sigma : I \to X^*$ of the curve $\gamma_\sigma(t) = \sigma\gamma(t)$, $t \in I$ starting at the point $\varphi_1(\sigma s) \in V_1$ has endpoint at $\varphi_2(\sigma s) \in V_2$.

Remark: In other words, if there is a lifting of the curve γ starting and ending on the sheets V_1 and V_2, respectively, then the lifting of the "perturbed" curve γ_σ starting on the sheet V_1 also ends on V_2.

Proof of Lemma 3.2 We use the standard technique of construction of $\hat{\gamma}$ with $\hat{\gamma}(0) = x_1$, adapted to the considered case. First, for a compact $\gamma(I)$ we construct an open covering by smoothly covered sets of a special form. Namely, we show the existence of an open neighborhood $W \subset s^{-1}U$ of the unit α_0 of the group Δ^0, such that for any $t \in I$ the set $\gamma(t)W$ is smoothly covered.

Since the open smoothly covered sets form a base of the space Δ^*, then there exists a finite covering of the compact $\gamma(I)$ by such sets:

$$\gamma(I) \subset \bigcup_{i=1}^{l} U_i.$$

Let $\{W_j\}_{j \in J}$ be an open base of the locally compact space Δ^0 at the point α_0, such that for any $j \in J$ the closure $\overline{W}_j$ is compact. For each $j \in J$ we define the set

$$K_j = \{t \in I : \ \gamma(t)\overline{W}_j \subset U_i \ \ \textit{for some} \ i, \ 1 \le i \le l\}.$$

Since all $\overline{W}_j$ are closed and each of the sets $U_i, i = \overline{1, l}$ is open, then K_j also is open, $j \in J$. Next, we have $\gamma(I) \subset \bigcup\limits_{i=1}^{l} U_i$, and hence for any $t \in I$ there exists i such that $\gamma(t) \subset U_i$. Also, since $\{W_j\}_{j \in J}$ is a base at the point α_0, there exists $j \in J$, such that $\gamma(t)\overline{W}_j \subset U_i$, implying that $t \in K_j$. Thus, the family

$\{K_j\}_{j\in J}$ forms an open covering of the compact I, implying that a finite number of indices $j_1, .., j_d$ can be chosen to satisfy $I \subset \bigcup_{k=1}^{d} K_{j_k}$. Now consider the set $W = \bigcap_{k=1}^{d} W_{j_k} \cap s^{-1}U \subset s^{-1}U$. Since the sets $\{W_{j_k}\}_{k=1}^{d}$ and $s^{-1}U$ are open neighborhoods of the unit element α_0, then the set W is non-empty and is also an open neighborhoods of α_0. Let $t \in I$ be an arbitrary point. Since $I \subset \bigcup_{k=1}^{d} K_{j_k}$, there exists j_m, such that $t \in K_{j_m}$, which in view of definition of the set K_{j_m} implies existence of $i, 1 \leq i \leq m$ satisfying $\gamma(t)\overline{W}_{j_m} \subset U_i$. Taking into account that all U_i are smoothly covered, we conclude that the set $\gamma(t)W \subset \gamma(t)\overline{W}_{j_m}$ is smoothly covered. Thus, the existence of the set W with desired properties is established.

Now we outline the basic elements of construction of lifting a curve in the considered case. It is clear that the open sets $\gamma(t)W$, $t \in I$ cover the compact $\gamma(I)$. Hence there exists a finite number of points $\{t'_k\}_{k=1}^{m}$ such that the sets $\gamma(t'_k)W$, $k = \overline{1,m}$ cover $\gamma(I)$ and the intersection of the "adjacent" sets $\gamma(t'_k)W \cap \gamma(t'_{k+1})W$, $k = \overline{1, m-1}$ is non-empty. Also, it is clear that $\{t'_k\}_{k=1}^{m}$ can be chosen so that $t'_1 = 0, t'_m = 1$. Then there is a partition of the segment $I = [0,1]$ by the points $0 = t_0 < t_1 < ... < t_m = 1$, such that for any k, $k \in \overline{1,m}$ the image $\gamma([t_{k-1}, t_k])$ is completely contained in the open smoothly covered set $\gamma(t'_k)W$. It is clear that $\gamma(t_k) \in \gamma(t'_k)W \cap \gamma(t'_{k+1})W, k \in \overline{1, m-1}$. Denoting $\gamma_k := \gamma(t'_k)$, $k = \overline{1,m}$, for the preimage of the open smoothly covered set $\gamma_k W$ we obtain the representation

$$\pi^{-1}(\gamma_k W) = \bigcup_{i=1}^{n} V_i^k,$$

and for each i, $i = \overline{1,n}$, the contraction $\pi|_{V_i^k} : V_i^k \to \gamma_k W$ is a homeomorphism with the inverse $\varphi_i^k := (\pi|_{V_i^k})^{-1} : \gamma_k W \to V_i^k$, $i = \overline{1,n}$, $k = \overline{1,m}$. Now we proceed to the by steps construction of of the curve $\hat{\gamma}$. We have $\gamma = \pi \circ \hat{\gamma}$, hence on the first segment $[t_0, t_1] = [0, t_1] \subset I$ there are n possibilities to construct the first part of the curve $\hat{\gamma}$, namely: $\hat{\gamma}([0, t_1]) = \varphi_i^1 \circ \gamma([0, t_1])$, $i = \overline{1,n}$. Since for the lifting $\hat{\gamma}$ we have $\hat{\gamma}(0) = x_1$, then we choose i to satisfy $\varphi_i^1(\gamma(0)) = x_1$. Denote the chosen i by i_1. The construction of the continuous curve $\hat{\gamma}$ is continued by linking the continuous on $[t_{k-1}, t_k]$ pieces $\hat{\gamma} = \varphi_{i_k}^k \circ \gamma, k = \overline{1,m}$ at points t_k by means of selection of the next $\varphi_{i_k}^k$ by the previous $\varphi_{i_{k-1}}^{k-1}$ so that $\varphi_{i_k}^k(b_{k-1}) = \varphi_{i_{k-1}}^{k-1}(b_{k-1})$, where $b_{k-1} = \gamma(t_{k-1}) \in \gamma_{k-1}W \cap \gamma_k W$. The chain of homeomorphisms

$$\varphi_{i_k}^k : \gamma_k W \to V_{i_k}^k$$

ensures the continuity of the curve $\hat{\gamma}$ on the sequence of sheets $V_{i_k}^k$, $k = \overline{1,m}$ on which it lies. Since the curve $\hat{\gamma}, \hat{\gamma}(0) = x_1$ is uniquely determined by γ (uniqueness lifting the curve), then it does not depend on its representing construction, which we choose according to the conditions of the lemma.

Further, we have $\gamma_1 = \gamma(t'_1) = \gamma(0) = s = \gamma(1) = \gamma(t'_m) = \gamma_m$. Hence $\gamma_1 W = sW \subset U$, and the obtained first homeomorphism $\varphi_{i_1}^1 : sW \to V_{i_1}^1$ satisfies the condition $\varphi_{i_1}^1(\gamma(0)) = x_1 \in V_1$, implying that $\varphi_{i_1}^1$ is a contraction to the set sW: $\varphi_{i_1}^1 = \varphi_1|_{sW}$ of the mapping $\varphi_1 : U \to V_1$, because both φ_1 and $\varphi_{i_1}^1$ are homeomorphisms that are local inverses to π). Therefore, in view of the assumption concerning enumeration ($\hat{\gamma}(1) = x_2 \in V_2$), by similar arguments we obtain $\varphi_{i_m}^m = \varphi_2|_{sW}$. Thus, the construction of lifting a curve in the considered case is done.

Now we are going to show that for small perturbation of the initial point $x_1 \in V_1$ the corresponding (lifted) curve cannot slide from the mentioned sheets, and hence its endpoint should lie on V_2. To this end, we establish the existence of sets U_k and $\tilde{U}_k$ with specific properties.

Observe first that, since $\gamma([t_{k-1}, t_k]) \subset \gamma_k W$ is compact and $\gamma_k W$ is open, then for any $k, 1 \leq k \leq m$ there exists an open neighborhood U_k of the unit α_0, such that $U_k\gamma([t_{k-1}, t_k]) \subset \gamma_k W$.

Next, for any k, $2 \leq k \leq m$ there is a neighborhood $\tilde{U}_k$ of the unit α_0, such that $\varphi_{i_k}^k(\beta) = \varphi_{i_{k-1}}^{k-1}(\beta)$ for all $\beta \in b_{k-1}\tilde{U}_k$. Indeed, we have $b_{k-1} = \gamma(t_{k-1}) \in \gamma_{k-1}W \cap \gamma_k W$ and $\varphi_{i_{k-1}}^{k-1}(b_{k-1}) = \varphi_{i_k}^k(b_{k-1})$.

Since $\gamma_{k-1}W$ and $\gamma_k W$ are open sets, the set $\gamma_{k-1}W \cap \gamma_k W \ni b_{k-1}$ is also open, and hence, there is a neighborhood $\tilde{U}_k$ of the unit α_0, such that $b_{k-1}\tilde{U}_k \subset \gamma_{k-1}W \cap \gamma_k W$. This implies the equality $\varphi^{k-1}_{i_{k-1}}(\beta) = \varphi^k_{i_k}(\beta)$, $\beta \in b_{k-1}\tilde{U}_k$, because $\varphi^{k-1}_{i_{k-1}}$ and $\varphi^k_{i_k}$ are homeomorphisms that are local inverses to π.

Finally, we prove that under the above conditions, for any σ from the open neighborhood $W_0 = \bigcap_{k=2}^{m} (U_k \cap \tilde{U}_k) \cap U_1$ of the unit α_0 the lifting $\hat{\gamma}_\sigma$ of the curve γ_σ with initial point $\varphi_1(\sigma s)$ has endpoint at $\varphi_2(\sigma s)$. To this end, consider the mapping

$$v(t) = \varphi^k_{i_k}(\sigma\gamma(t)), t \in [t_{k-1}, t_k],\ \ k = \overline{1, m},$$

and show that v is a continuous curve that coincides with $\hat{\gamma}_\sigma$. Clearly, it is enough to establish continuity of v at the points t_k, $k = \overline{1, m-1}$. We have

$$v(t) = \begin{cases} \varphi^k_{i_k}(\sigma\gamma(t)), & t \in [t_{k-1}, t_k], \\ \varphi^{k+1}_{i_{k+1}}(\sigma\gamma(t)), & t \in [t_k, t_{k+1}]. \end{cases}$$

Since $\sigma \in W_0 \subset \tilde{U}_{k+1}$, we have $b_k\sigma \in b_k\tilde{U}_{k+1}$. Therefore, $\varphi^k_{i_k}(b_k\sigma) = \varphi^{k+1}_{i_{k+1}}(b_k\sigma)$, that is, $\varphi^k_{i_k}(\gamma(t_k)\sigma)) = \varphi^{k+1}_{i_{k+1}}(\gamma(t_k)\sigma)$. Thus, the continuity of v at t_k is proved, implying that $v(t)$, $t \in I$ is a continuous curve. Since each $\varphi^k_{i_k}$, $k = \overline{1, m}$, on its domain of definition is the inverse of π, it follows from the definition of v that $\pi \circ v(t) = \sigma\gamma(t) = \gamma_\sigma(t)$, $t \in I$, showing that v is a lifting of the curve γ_σ. Further, we have $v(0) = \varphi^1_{i_1}(\sigma\gamma(0)) = \varphi^1_{i_1}(\sigma s)$. Since $\sigma \in W_0 \subset U_m$, it follows from the definition of the set U_m that $\sigma\gamma([t_{m-1}, t_m]) \subset \gamma_m W = sW$, implying that $\sigma s = \sigma\gamma(t_m) \in sW$. Hence using $\varphi^1_{i_1} = \varphi_1|_{sW}$, we obtain $v(0) = \varphi^1_{i_1}(\sigma s) = \varphi_1(\sigma s)$. Thus, v is the lifted curve $\hat{\gamma}_\sigma$ mentioned in the statement of the lemma. Now we show that the endpoint of the curve $\hat{\gamma}_\sigma$ lies on the sheet V_2. We have $\hat{\gamma}_\sigma(1) = v(1) = \varphi^m_{i_m}(\sigma\gamma(1)) = \varphi^m_{i_m}(\sigma s)$, and since $\sigma s \in sW$ and $\varphi^m_{i_m} = \varphi_2|_{sW}$, we obtain $\hat{\gamma}_\sigma(1) = \varphi^m_{i_m}(\sigma s) = \varphi_2(\sigma s) \in V_2$. Lemma 3.2 is proved.

Corollary 3.1. *For each element $w \in X^*$ there exists a neighborhood V, such that for any $z \in V$ the inequality holds: $\nu(z) \geq \nu(w)$.*

Proof. Let $w \in X^*$ and $\pi(w) = s \in \Delta^*$. Let U be a smoothly covered neighborhood of the point s, such that $\pi^{-1}(U) = \bigcup_{i=1}^{n} V_i$ and all $\pi : V_i \to U$ are homeomorphisms with inverses $\varphi_i : U \to V_i$. Assume that $w \in V_1$, and take some $u \neq w$ from $C(w)$. Then $\pi(u) = s$ and from homeomorphism of π on each V_i we obtain $u \notin V_1$. Let $u \in V_2$. Since $u \in C(w)$, by the definition of the set $C(w)$ there exists an analytic curve with initial point and endpoint at w and u respectively. That is, there exists an analytic curve $\gamma \subset \Delta^*$ with $\gamma(0) = \gamma(1) = s$, whose lifting $\hat{\gamma} \subset X^*$ satisfies $\hat{\gamma}(0) = w, \hat{\gamma}(1) = u$. Now let $W_0^{(2)}$ be the set W_0 from Lemma 3.2 for the considered case (we write index 2 since we assume that $u \in V_2$). Denoting $V_1^{(2)} = V_1 \cap \pi^{-1}(sW_0^{(2)}) = \varphi_1(sW_0^{(2)})$, we can apply Lemma 3.2 to conclude that for any $x \in V_1^{(2)}$ there exists an analytic curve with initial point x and endpoint on the set $\varphi_2(sW_0^{(2)}) \subset V_2$. Therefore, on the sheet V_2 the points from $V_1^{(2)}$ have the same number of equivalent points as that of w (namely, one equivalent point).

Next, considering in turn the sheets $V_3, ..., V_n$ and taking into account that w can have equivalent points only on the sheets V_i, $i = \overline{2, n}$ (namely, at most one equivalent point on each sheet), we obtain the sets $V_1^{(3)}, ..., V_1^{(n)}$. Now it is easy to see that the set $V = \bigcap_{i=2}^{n} V_1^{(i)}$ will satisfy the requirements of the corollary. Corollary 3.1 is proved.

Now we are in position to state and prove the main result of this section.

Theorem 3.1. *The function $\nu : X^* \to \mathbb{Z}_+$ is locally constant on X^*.*

Proof. We first prove that the function $\mu : \Delta^* \to \mathbb{Z}_+$ is constant on Δ^*. We have $\mu(\sigma) = card\{C(w), w \in \pi^{-1}(\sigma)\}$. According to Corollary 3.1 for $w_1 \in \pi^{-1}(\sigma)$ there exists a neighborhood V_1 such that $\nu(z) \geq \nu(w_1), z \in V_1$, meaning that the number of equivalent points for z is greater than or equal to that of point w_1. Let $\pi^{-1}(\sigma) = (w_1, ..., w_n)$ and let $V_1, ..., V_n$ be the corresponding neighborhoods of these points. Define $U = \bigcap_{i=1}^{n} \pi(V_i)$. Assume that $\xi \in U$ and consider $\mu(\xi) = card\{C(z), z \in \pi^{-1}(\xi)\}$. Take an arbitrary $z \in \pi^{-1}(\xi)$ and assume that $z \in V_i$ for some $i, 1 \leq i \leq n$. Then by the definition of the set V_i we have that the number of equivalent points for point $z \in \pi^{-1}(\xi)$ is greater than or equal to that of point $w_i \in \pi^{-1}(\sigma)$: $\nu(z) \geq \nu(w_i)$, implying that the number of equivalence classes of points from $\pi^{-1}(\xi)$ is not greater than that of points from $\pi^{-1}(\sigma)$, that is, $\mu(\xi) \leq \mu(\sigma)$.

Thus, for any $\sigma \in \Delta^*$ there exists a neighborhood U of the point σ, such that

$$\mu(\xi) \leq \mu(\sigma), \xi \in U. \tag{3.2}$$

We set $\mu = \min_{\sigma \in \Delta^*} \mu(\sigma)$ and $D = \{\sigma \in \Delta^* : \mu(\sigma) = \mu\}$. Since the function μ takes values from $\mathbb{Z}_+$, we have $D \neq \varnothing$. Now we show that $D = \Delta^*$, that is, $\mu(s) = \mu$ on Δ^*.

We fix an arbitrary $s \in \Delta^*$ and any $\sigma \in D$. Then in view of (3.2), there exists a neighborhood $U \ni \sigma$, such that $\mu|_U \leq \mu(\sigma) = \mu \leq \mu(s)$. Since the set $\mathbb{C}_s^*$ is everywhere dense in Δ^*, we have $U \cap \mathbb{C}_s^* \neq \varnothing$. So, by Lemma 3.1, we obtain

$$\mu|_{\mathbb{C}_s^*} = \mu|_{U \cap \mathbb{C}_s^*} \leq \mu(\sigma) = \mu \leq \mu(s) = \mu|_{\mathbb{C}_s^*},$$

implying $\mu(s) = \mu(\sigma) = \mu$, and hence $s \in D$. Thus, $D = \Delta^*$ and the function μ is constant on Δ^*.

The constancy of function μ on Δ^* implies the equality $\nu(z) = \nu(w)$ for any z from the neighborhood V of point w (see Corollary 3.1). Indeed, assuming the opposite, that is, existence of $z \in V$ with $\nu(z) > \nu(w)$, by the first part of the proof, yields the strong inequality $\mu(\pi(z)) < \mu(\pi(w))$, which is a contradiction. Thus, the local constancy of function ν on X^* is established. Theorem 3.1 is proved.

4. LOCAL CONSTANCY OF FUNCTION ν. ALGEBRAIC VERSION

In Section 3 it was proved that Corollary 3.1 implies local constancy of function ν on X^* (Theorem 3.1). Also, Corollary 3.1 was proved by means of constructive lifting of curves from Δ^* (Lemma 3.2). In this section we prove Corollary 3.1 using algebraic methods. We first prove a technical result.

Lemma 4.1. *Let K be a compact set and let $p(t, x) = x^n + g_1(t)x^{n-1} + ... + g_{n-1}(t)x + g_n(t)$, $t \in K$ be a polynomial with continuous coefficients: $g_i \in C(K)$, $i = \overline{1, n}$. Further, let the function $f \in C(K)$ satisfy the condition $p(t, f(t)) = 0$, $t \in K$ and let $C = \max_{1 \leq i \leq n}\{||g_i||\}$. Then $||f|| := \sup_{t \in K} |f(t)| < 1 + C$.*

If $C = 0$, then all g_i are equal to 0. This means that $p(x, t) = x^n$, implying $f = 0$. Thus, we have $||f|| = 0 < 1 + 0 = 1 + C$. For $C > 0$ and $||f|| \leq 1$ the conclusion is trivial: $||f|| < 1 + C$.

Now let $C > 0$ and $||f|| > 1$. Then there exists $t_0 \in K$, such that $|f(t_0)| = ||f|| > 1$. Since $f(t_0)^n = -g_1(t_0)f(t_0)^{n-1} - ... - g_n(t_0)$, we have

$$|f(t_0)| \leq C(1 + \frac{1}{|f(t_0)|} + ... + \frac{1}{|f(t_0)|^{n-1}}) < C\frac{|f(t_0)|}{|f(t_0)| - 1},$$

implying that $||f|| = |f(t_0)| < 1 + C$. Lemma 4.1 is proved.

The example of polynomial $q(x) = x^2 - C$ shows that for sufficiently small C $(C < 1/4)$, the equality in Lemma 4.1 cannot be improved to obtain $||f|| \leq 2C$.

The next result apparently concerns to mathematical folklore, and hence we provide its complete proof.

Lemma 4.2. *Let $K = [0,1]$ and $p(t,x) = x^n + g_1(t)x^{n-1} + ... + g_n(t)$ be a polynomial with continuous coefficients: $g_i \in C(K)$, $i = \overline{1,n}$ and with discriminant: $d_p(t) \neq 0$ for all $t \in K$. Then there exist exactly n functions $h_i \in C(K)$, $i = \overline{1,n}$, that are pairwise different for all points of K and representing the set of solutions of equation $p(t,x) = 0$ over K, that is,*

$$p(t, h_i(t)) = 0, \ t \in K, \ i = \overline{1,n}.$$

Remark: Notice that since for each point $t_0 \in K$ the equation $p(t_0, x) = 0$ has exactly n solutions, then the mutually distinct values $h_i(t_0)$, $i = \overline{1,n}$, represent *all* solutions of equation $p(t_0, x) = 0$, that is, by the values $\{h_i(t)\}_{i=1}^n$, $t \in K$, is exhausted the set of *all* solutions of equation $p(t,x) = 0$, $t \in K$.

Proof of Lemma 4.2. Define the set

$$K_p = \{(t,x) \in K \times \mathbb{C} : p(t,x) = 0\}.$$

We have to find continuous mutually non-coinciding functions $h_i \in C(K)$, $i = \overline{1,n}$, such that

$$K_p = \{(t,x) \in K \times \mathbb{C} : p(t,x) = 0\} = \bigcup_{i=1}^{n} \{(t, h_i(t)) : t \in K\}.$$

By Hurwitz-Rouché's theorem, the projection $\pi : K_p \to K : (t,x) \mapsto t$ on the first coordinate is unbranched n-sheets covering, and by continuity of function $g_i \in C(K)$, $i = \overline{1,n}$, the projection on the second coordinate $\eta : K_p \to \mathbb{C} : (t,x) \mapsto x$ is a continuous mapping.

Consider the curve $u : I \to K$, $u(t) = t$, $t \in I (= K)$ and the fiber $\pi^{-1}(0) = \{(0, x_1), ..., (0, x_n)\}$ over the point $0 \in K$. By the lifting theorem, there exist n liftings $\hat{u}_i : I \to K_p$, $i = \overline{1,n}$ of the curve u, such that $u = \pi \circ \hat{u}_i$ and $\hat{u}_i(0) = (0, x_i)$, $i = \overline{1,n}$.

We set $h_i = \eta \circ \hat{u}_i$, $i = \overline{1,n}$, and show that they are the desired functions. To this end, observe first that the functions h_i are continuous as superpositions of continuous functions η and $\hat{u}_i$, $i = \overline{1,n}$. Further, by the definition of the mapping η, the function $h_i(t)$ is the second "coordinate" of the point $\hat{u}_i(t)$. It follows from the relation $\pi \circ \hat{u}_i(t) = u(t) = t$ that the first "coordinate" of the point $\hat{u}_i(t)$ is t. Hence, we have

$$\hat{u}_i(t) = (t, h_i(t)), \quad t \in K, \tag{4.1}$$

implying $(t, h_i(t)) \in K_p$, $t \in K$, $i = \overline{1,n}$.

Now we show that for any $t \in K$ the points $h_i(t)$ are mutually distinct. Assume the opposite, that is, existence of an element $t_0 \in K$ and indices $i \neq j$, such that $h_i(t_0) = h_j(t_0)$. Consider the set $T = \{t \in K : h_i(t) = h_j(t)\}$. According to the above assumption, T is non-empty. From continuity of functions h_i and h_j it follows that T is a closed set. We show that T is also open in K.

Let $t' \in T$. Then by (4.1) we have $\hat{u}_i(t') = \hat{u}_j(t')$. Since π is a covering, there exists an open set $U \ni \pi(\hat{u}_i(t')) = t'$ in K, for which there is an open set $V \ni \hat{u}_i(t') = \hat{u}_j(t')$ in K_p, such that $\pi : V \to U$ is a homeomorphism, and hence is a bijection on V. On the other hand, since $\hat{u}_i, \hat{u}_j$ are continuous, there exists $\delta > 0$, such that $t \in K$ and $|t - t'| < \delta$ imply that $\hat{u}_i(t)$ and $\hat{u}_j(t)$ belong to V. Since on the set V, π is a bijection, the relation $\pi(\hat{u}_i(t)) = t = \pi(\hat{u}_j(t))$ implies that for $t \in K$, $|t - t'| < \delta$ the following equality of liftings holds:

$$\hat{u}_i(t) = \hat{u}_j(t), \tag{4.2}$$

that is, $h_i(t) = h_j(t)$, implying $K \cap (t' - \delta, t' + \delta) \subset T$, and hence the set T is open. Since K is connected, we have $T = K$. This means that the equality (4.2) is fulfilled on the entire K, which is impossible, because $\hat{u}_i(0) = (0, x_i) \neq (0, x_j) = \hat{u}_j(0)$. Thus, the points $h_i(t), i = \overline{1,n}$ are mutually distinct on $t \in K$. Lemma 4.2 is proved.

Note that the assertion proved in the Lemma 4.2 can be reformulated as non-existence of a continuous on K function $g \neq h_i$, $i = \overline{1,n}$ coinciding with one of the functions h_i at each point from K.

Our basic tool in this section is the following lemma.

Lemma 4.3. *Under the conditions of Lemma 4.2 for any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$, such that for any collection of functions $\varepsilon_i \in C(K), \varepsilon_i : K \to \mathbb{C}$ with $||\varepsilon_i|| < \varepsilon$, $i = \overline{1,n}$ can be found n functions $\tilde{h}_i \in B_\delta(h_i)$, $i = \overline{1,n}$ for which for each $t \in K$ the points $\tilde{h}_i(t)$, $i = \overline{1,n}$, represent n distinct zeros of the "perturbed" polynomial*

$$p_\varepsilon(t,x) := x^n + \sum_{i=1}^{n}(g_i(t) + \varepsilon_i(t))x^{n-k}. \tag{4.3}$$

Here $B_\delta(h) = \{f \in C(K) : ||f - h|| < \delta\}$.

Proof. We first prove existence of $\varepsilon_0 > 0$, such that for each $\varepsilon \le \varepsilon_0$ any polynomial of the form (4.3) with $||\varepsilon_i|| < \varepsilon$, $i = \overline{1,n}$, satisfies the conditions of Lemma 4.2, that is, it has everywhere different from zero discriminant on K. To this end, we use the known interpretation of $\mathbb{C}^n$ as a space of the coefficients of the polynomials over the field $\mathbb{C}$. Let $D = \{w \in \mathbb{C}^n : d(w) = 0\}$ be the set of zeros of the discriminant mapping $d : \mathbb{C}^n \to \mathbb{C}$, assigning to the vector $w \in \mathbb{C}^n$ of the coefficients of a polynomial the value $d(w)$ of its discriminant. Consider the mapping

$$G : K \to \mathbb{C}^n : t \mapsto (g_1(t), ..., g_n(t)) \cong x^n + g_1(t)x^{n-1} + ... + g_n(t) = p(t,x).$$

Then the image $G(K) = g_1(K) \times ... \times g_n(K)$ is a compact, and by the assumption, we have $G(K) \cap D = \varnothing$, because the discriminant of the polynomial $p(t,x)$ is everywhere different from zero on K. Denote by $d_0 = d(G(K), D)$ the distance between the sets $G(K)$ and D. Since these sets are closed, and in addition, the first is also compact, we have $d_0 > 0$. We show that as ε_0 can be taken the constant $d_0/2\sqrt{n}$. Indeed, for any collection $\tilde{G} = (\tilde{g}_1, ..., \tilde{g}_n)$ with $||\tilde{g}_i - g_i|| < \varepsilon \le \varepsilon_0$, $i = \overline{1,n}$, we have $d(\tilde{G}(t), G(t)) < \varepsilon_0\sqrt{n} = d_0/2$ for any $t \in K$. Hence, using the inequality $|d(G(t), D) - d(\tilde{G}(t), D)| < d(G(t), \tilde{G}(t))$, $t \in K$ (see, e.g., [10], p. 377), for any $t \in K$ we obtain the following chain of inequalities:

$$d(\tilde{G}(t), D) \ge d(G(t), D) - d(\tilde{G}(t), G(t)) > d_0 - d_0/2 > 0,$$

which implies that $\tilde{G}(K) \cap D = \varnothing$.

Thus, under the above conditions, for any polynomial of the form (4.3) by Lemma 4.2 there exist n functions $\tilde{h}_i$, $i = \overline{1,n}$, representing the zeros of this polynomial for each fixed $t \in K$ with $\varepsilon_i = \tilde{g}_i - g_i$. Now we show existence of $\varepsilon > 0$, such that for $||\varepsilon_i|| < \varepsilon$, $i = \overline{1,n}$ the continuous solutions of the equation $p_\varepsilon(t,x) = 0$ are contained in $B_\delta(h_i)$, $i = \overline{1,n}$.

Observe first that for any choice of $\tilde{G}$ with $||\tilde{g}_i - g_i|| < \varepsilon_0$ we have $||\tilde{g}_i|| < ||g_i|| + \varepsilon_0, i = \overline{1,n}$. Then, we have $\tilde{C} := \max_i ||\tilde{g}_i|| < C + \varepsilon_0$, where $C = \max_i ||g_i||$. By Lemma 4.1 we obtain $||\tilde{h}_i|| < 1 + \tilde{C} < 1 + C + \varepsilon_0$ for all $i = 1, .., n$.

Next, let $\delta_0 = \min\limits_{1 \le i < j \le n} \inf\limits_{t \in K} |h_i(t) - h_j(t)|$. By Lemma 4.2 we have $\delta_0 > 0$. Since for $\delta_1 < \delta_2$ clearly $B_{\delta_1}(h) \subset B_{\delta_2}(h)$, then, without loss of generality, we can assume that the arbitrary chosen δ satisfies the condition $\delta < \delta_0/2$. Then by Hurwitz-Rouché's theorem, there exists a constant $\varepsilon_1 > 0$, such that for $|b_i - g_i(0)| < \varepsilon_1$, $i = \overline{1,n}$ the polynomial $P(x) = x^n + b_1x^{n-1} + .. + b_n$ in each of the circles $|x - h_i(0)| < \delta, i = \overline{1,n}$, has exactly one zero (of multiplicity 1).

Further, we fix an arbitrary $\varepsilon > 0$ satisfying the following conditions:

a) $\varepsilon < \varepsilon_0$; then by the definition of ε_0, from $||\tilde{g}_i - g_i|| < \varepsilon$ follows existence of mutually non-coinciding functions $\tilde{h}_i \in C(K)$, $i = \overline{1,n}$, representing the zeros of the polynomial (4.3),

b) $\varepsilon < \varepsilon_1$; then by the definition of ε_1, if $|\tilde{g}_i(0) - g_i(0)| < \varepsilon$, then $\tilde{h}_i$ can be enumerated so that $|\tilde{h}_i(0) - h_i(0)| < \delta, i = \overline{1,n}$,

c) $\varepsilon[(1 + C + \varepsilon_0)^n - 1]/(C + \varepsilon_0) < \delta^n$; then for any $i \in \{1, .., n\}$, in view of equality $p_\varepsilon(t, \tilde{h}_i(t)) = 0$, we have

$$|p(t, \tilde{h}_i(t))| = |p(t, \tilde{h}_i(t)) - p_\varepsilon(t, \tilde{h}_i(t))|$$

$$= |(\tilde{h}_i(t)^n + g_i(t)\tilde{h}_i(t)^{n-1} + .. + g_n(t)) - (\tilde{h}_i(t)^n + \tilde{g}_i(t)\tilde{h}_i(t)^{n-1} + .. + \tilde{g}_n(t))| \tag{4.4}$$

$$= |\varepsilon_1(t)\tilde{h}_i(t)^{n-1} + ... + \varepsilon_n(t)| < \delta^n$$

on K for $||\varepsilon_i|| < \varepsilon$, where $\varepsilon_i = \tilde{g}_i - g_i$.

Now we show that for such ε the following implication holds:

$$||\varepsilon_i|| < \varepsilon \Rightarrow \tilde{h}_i \in B_\delta(h_i),\ \ i = 1, ..., n.$$

To this end, we choose an arbitrary $i_0 \in \{1, ..., n\}$ and consider the quantity

$$t_0 := \sup\{\tau \in [0,1] : |h_{i_0}(t) - \tilde{h}_{i_0}(t)| < \delta \ for\ t \in [0, \tau]\}.$$

We have $t_0 > 0$, because the function $r(t) = |h_{i_0}(t) - \tilde{h}_{i_0}(t)|$ is continuous and is strictly less than δ for $t = 0$ (see part b)). It is clear that $t_0 \le 1$. Assume that $t_0 < 1$. By the definition of t_0 we have $r(t) < \delta$ for $t \in [0, t_0)$. Next, we have $r(t_0) = \delta$. Indeed, the assumption $r(t) < \delta$ contradicts the precision of upper bound $t_0 < 1$, and $r(t) > \delta$ - the continuity of function $r(t)$. Finally, using the definition of δ_0, we obtain for any $j \ne i_0$

$$|h_j(t_0) - \tilde{h}_{i_0}(t_0)| = |h_j(t_0) - h_{i_0}(t_0) + h_{i_0}(t_0) - \tilde{h}_{i_0}(t_0)| \ge$$

$$\ge |h_j(t_0) - h_{i_0}(t_0)| - r(t_0) \ge \delta_0 - \delta > 2\delta - \delta = \delta. \tag{4.5}$$

Since the pairs $(t_0, h_j(t_0))$, $j = \overline{1,n}$, are the roots of polynomial $p(t,x)$, we can write $p(t_0, x) = \prod_{j=1}^{n} (x - h_j(t_0))$, which in view of (4.4) and (4.5) implies

$$\delta^n > |p(t_0, \tilde{h}_{i_0}(t_0))| = r(t_0) \prod_{j=1, j\ne i_0}^{n} |\tilde{h}_{i_0}(t_0) - h_j(t_0)| > \delta\delta^{n-1} = \delta^n. \tag{4.6}$$

The obtained contradiction shows that t_0 should be 1.

However, the substitution $t_0 = 1$ into (4.5) and (4.6) demonstrates also the contradictoriness of the assumption $r(1) = \delta$. Thus, $|h_{i_0}(t) - \tilde{h}_{i_0}(t)| < \delta$ for $t \in K$, and since the function h_{i_0} and $\tilde{h}_{i_0}$ are continuous, then $||\tilde{h}_{i_0} - h_{i_0}|| < \delta$, implying $\tilde{h}_{i_0} \in B_\delta(h_{i_0})$. Taking into account that i_0 is arbitrary, this completes the proof of Lemma 4.3.

Now we turn to the study of the algebraic version of the theory developed in this paper. Let

$$p(s, x) = x^n + f_1(s)x^{n-1} + ... + f_n(s)$$

be a polynomial with generalized analytic coefficients $f_i \in \mathcal{O}(\Delta^0)$, $i = \overline{1,n}$ and discriminant d_p. It is clear that d_p also is a generalized analytic function: $d_p \in \mathcal{O}(\Delta^0)$. Denote by $N_p = N(d_p)$ the set of zeros of the discriminant d_p. Then either N_p is nowhere dense (discrete) in Δ^0, or $N_p = \Delta^0$. We assume the first case, that is, N_p is nowhere dense in Δ^0, and the null-set N_p will play the role of a thin set. Consider the space

$$\Delta_p^0 = \{(s, x) \in \Delta^0 \times \mathbb{C} : p(s, x) = 0\},$$

and the covering

$$\pi : \Delta_p^0 \to \Delta^0 : (s, x) \mapsto s.$$

Observe that the contraction $\pi|_{\Delta_p^*} : \Delta_p^* = \pi^{-1}(\Delta^*) \to \Delta^*$ will be an unbranched covering over $\Delta^* = \Delta^0 \setminus N_p$, which we also denote by π. Thus, Δ_p^0 becomes a Bohr-Riemann surface. We denote $\mathbb{C}_s^* = \mathbb{C}_s \cap \Delta^* = \mathbb{C}_s \setminus N_p$, $\mathbb{C}_{p,s}^* = \pi^{-1}(\mathbb{C}_s^*)$ and $\mathbb{C}_{p,s} = \pi^{-1}(\mathbb{C}_s)$. Recall that a curve $u : I \to \Delta^0$ is called analytic, if $u(I) \subset \mathbb{C}_s$ for some $s \in \Delta^0$ (as s can be taken $u(0)$).

Definition 4.1. *A curve $\hat{u} : I \to \Delta_p^*$ m Δ_p^* is called analytic, if its projection $u = \pi \circ \hat{u}$ under the covering π is an analytic curve.*

Lemma 4.4. *The following conditions are equivalent:*

1) $\hat{u} : I \to \Delta_p^*$ is an analytic curve,

2) there exists $s \in \Delta^0$, such that $\hat{u}(I) \subset \mathbb{C}_{p,s}^*$.

Proof. Assume that $\hat{u} : I \to \Delta_p^*$ is an analytic curve. Then there exist $s \in \Delta^0$ and a curve $u(I) \subset \mathbb{C}_s$, such that $u = \pi \circ \hat{u}$. Since $\hat{u}(I) \subset \Delta_p^* = \pi^{-1}(\Delta^*)$, then $u(I) = \pi \circ \hat{u}(I) \subset \Delta^*$, implying $u(I) \subset \mathbb{C}_s \cap \Delta^* = \mathbb{C}_s^*$, and hence $\hat{u}(I) \subset \pi^{-1}(\mathbb{C}_s^*) = \mathbb{C}_{p,s}^*$. Now assume that there exists $s \in \Delta^0$, such that $\hat{u}(I) \subset \mathbb{C}_{p,s}^*$. Then $u(I) = \pi \circ \hat{u}(I) \subset \mathbb{C}_s^*$, that is, u is an analytic curve, and hence the curve $\hat{u}$ is also analytic. Lemma 4.4 is proved.

As it was shown above, the locally compact abelian group structure given on Δ^0 allows for each $s \in \Delta^0$ and an analytic curve $u : I \to \Delta^0$ to define a curve $u_s : I \to \Delta^0$, by setting $u_s(t) = su(t), t \in I$, which also will be an analytic curve.

Lemma 4.5. *Let $u : I \to \Delta^*$ be a (analytic) curve. Then there is a neighborhood U of the unit element of the group Δ^0, such that for any $s \in U$ the (analytic) curve $u_s(I)$ is contained in Δ^*.*

Proof. We have $u(I) \subset \Delta^*$, implying that $u(I)$ does not contain points from N_p. Since the set N_p is discrete, there is a non-overlapping with N_p neighborhood of the curve $u(I)$, that is, there is a neighborhood U of the unit element α_0, such that $u(I)U \cap N_p = \varnothing$. Then, it is clear that, for any $s \in U$ the curve $u_s(I) = su(I)$ does not intersect N_p, implying that $u_s(I) \subset \Delta^*$. Lemma 4.5 is proved.

Similar to the Definition 2.3, two points $w, w' \in \Delta_p^*$ will be called *equivalent*, and denoted by $w \sim w'$, if $\pi(w) = \pi(w')$ and there exists an analytic curve $\hat{u} : I \to \Delta_p^*$, such that $\hat{u}(0) = w$, $\hat{u}(1) = w'$. Again, if $w \sim w'$ and $w' \sim w''$, then $w \sim w''$. Let, as before, $C(w)$ be the set of all points (including w) that are equivalent to w. Taking into account that the covering is n-sheeted, we have $cardC(w) \leq n$. Also, it follows from the transitivity of the equivalence relation that for any $w \in \Delta_p^*$ there exists an analytic curve $\hat{u}(I)$, such that $\hat{u}(0) = w$ and $C(w) \subset \hat{u}(I)$.

Now we examine the local behavior on Δ_p^* of the function $\nu : \Delta_p^* \to \mathbb{Z}_+$, $\nu(w) = cardC(w)$. As it was mentioned above, Corollary 3.1 implies the local constancy of function ν on the Bohr-Riemann surface (Theorem 3.1). In the next theorem, using an algebraic method, we prove the assertion of Corollary 3.1 for the considered case, which again yields the local constancy of function ν on Δ_p^*.

Theorem 4.1. *For each element $w \in \Delta_p^*$ there exists a neighborhood V, such that for any $z \in V$ the inequality holds: $\nu(z) \geq \nu(w)$.*

Proof. We fix an arbitrary $w_0 \in \Delta_p^*$ with $\pi(w_0) = s_0 \in \Delta^*$. Let $\nu(w_0) = k$ and $C(w_0) = (w_0, w_1, ..., w_{k-1})$. Further, let $\hat{u} : I \to \Delta_p^*$ be an analytic curve with $\hat{u}(0) = w_0$ and $C(w_0) \subset \hat{u}(I)$. Then there exist $0 = t_0 < t_1 < ... < t_{k-1} \leq 1$ such that $\hat{u}(t_i) = w_i, i = \overline{0, k-1}$ and $\pi \circ \hat{u}(t_i) = \pi(w_i) = s_0, i = \overline{0, k-1}$. Denoting by $u(t) = \pi \circ \hat{u}(t), t \in I$ the projection of the analytic curve $\hat{u} \subset \Delta_p^*$, we obtain $u(I) \subset \Delta^*$ and $u(t_i) = \pi \circ \hat{u}(t_i) = s_0, i = \overline{0, k-1}$.

Clearly, to complete the proof, it is enough to show that for any sequence $w_\lambda \to w_0$ there exists λ_0, such that $\nu(w_\lambda) \geq \nu(w_0) = k$ for $\lambda > \lambda_0$. From the convergence $w_\lambda \to w_0$ it follows that $s_\lambda := \pi(w_\lambda) \to s_0$. Denote $s_\lambda^0 = s_0^{-1} s_\lambda$ and observe that $s_\lambda^0 \to \alpha_0$, where α_0 is the unit of the group Δ^0. Define the curves $u_\lambda : I \to \Delta^0$ by $u_\lambda(t) = s_\lambda^0 u(t), t \in I$. Then, by Lemma 4.5 there is λ_1 such that for $\lambda > \lambda_1$ the curves $u_\lambda(I)$ are contained in Δ^*. Next, consider the polynomials

$$p(u(t), x) = x^n + f_1(u(t))x^{n-1} + ... + f_n(u(t)),$$

$$p(u_\lambda(t), x) = x^n + f_1(u_\lambda(t))x^{n-1} + ... + f_n(u_\lambda(t)).$$

Since the curve $u(t)$, $t \in I$ belongs to the set Δ^*, by Lemma 4.2 the equation $p(u(t), x) = 0$, $t \in I$ has exactly n continuous mutually distinct solutions. It is clear that for any $\varepsilon > 0$ there is λ_ε, such that for $\lambda \geq \lambda_\varepsilon$ the inequality holds:

$$\max_{1\leq i\leq n} ||f_i(u(t)) - f_i(u_\lambda(t))||_{C(I)} < \varepsilon.$$

Applying Lemma 4.3, we conclude that for $\lambda > \max\{\lambda_1, \lambda_\varepsilon\}$ the equation $p(u_\lambda(t), x) = 0$, $t \in I$ also has exactly n continuous mutually distinct solutions, close (uniformly on [0,1]) to the solutions of the equation $p(u(t), x) = 0$, $t \in I$.

Let $\hat{u}(t) = (\hat{s}(t), \hat{x}(t))$, $t \in I$. It follows from the definition of covering π that $u(t) = \pi \circ \hat{u}(t) = \hat{s}(t)$, $t \in I$, that is, $\hat{u}(t) = (u(t), \hat{x}(t))$, $t \in I$. In particular, we have $w_i = \hat{u}(t_i) = (u(t_i), \hat{x}(t_i)) = (s_0, \hat{x}(t_i))$, $i = \overline{0, k-1}$. Since $\hat{u}(t) \subset \Delta_p^*$, $t \in I$, from the definition of the set Δ_p^* we obtain

$$\hat{x}^n(t) + f_1(u(t))\hat{x}^{n-1}(t) + ... + f_n(u(t)) = 0,\ t \in I,$$

implying that the function $\hat{x}(t)$ is one of the solutions of equation $p(u(t), x) = 0$. Hence, according to Lemma 4.3, for $\lambda > \lambda_{\varepsilon(\delta)}$ among the solutions of equation $p(u_\lambda(t), x) = 0$ there is $\hat{x}_\lambda(t)$ to satisfy

$$||\hat{x}_\lambda - \hat{x}||_{C(I)} < \delta, \tag{4.7}$$

where

$$\delta < \min_{1\leq i<j\leq k-1} |\hat{x}(t_i) - \hat{x}(t_j)|/2, \tag{4.8}$$

with $w_i = (s_0, \hat{x}(t_i))$, $i = \overline{0, k-1}$. Next, since the curve u is analytic, u_λ also will be an analytic curve, and taking into account the relation $u_\lambda = \pi(u_\lambda, \hat{x}_\lambda)$, we conclude that the curve $\hat{u}_\lambda : I \to \Delta_p^*$ is analytic with $\hat{u}_\lambda(t) = (u_\lambda(t), \hat{x}_\lambda(t))$, $t \in I$. By the construction we have $u_\lambda(t_i) = s_\lambda^0 u(t_i) = s_0^{-1} s_\lambda s_0 = s_\lambda$, $i = \overline{0, k-1}$. Therefore the points $\hat{u}_\lambda(t_i) = (s_\lambda, \hat{x}_\lambda(t_i))$, $i = \overline{0, k-1}$ belong to the curve $\hat{u}_\lambda(I)$. Since $\pi(\hat{u}_\lambda(t_0)) = s_\lambda = \pi(w_\lambda)$ and $w_\lambda \to w_0 = (s_0, \hat{x}(t_0))$, $s_\lambda \to s_0$, then taking δ in (4.7) sufficiently small and λ sufficiently large ($\lambda > \lambda_0 > \max\{\lambda_1, \lambda_{\varepsilon(\delta)}\}$), we obtain $w_\lambda = \hat{u}_\lambda(t_0)$. Besides, using (4.7) and (4.8), for $i \neq j$ we obtain

$$\begin{aligned}|\hat{x}_\lambda(t_i) - \hat{x}_\lambda(t_j)| &= |(\hat{x}(t_i) - \hat{x}(t_j)) - (\hat{x}(t_i) - \hat{x}_\lambda(t_i)) - (\hat{x}_\lambda(t_j) - \hat{x}(t_j))| \\ &\geq |(\hat{x}(t_i) - \hat{x}(t_j))| - |(\hat{x}(t_i) - \hat{x}_\lambda(t_i))| - |\hat{x}_\lambda(t_j) - \hat{x}(t_j)| > 2\delta - \delta - \delta = 0,\end{aligned}$$

implying that $\hat{x}_\lambda(t_i) \neq \hat{x}_\lambda(t_j)$, and hence $\hat{u}_\lambda(t_i) \neq \hat{u}_\lambda(t_j)$, $i \neq j$.

Thus, we have constructed an analytic curve $\hat{u}_\lambda$ in Δ_p^*, for which $\hat{u}_\lambda(0) = \hat{u}_\lambda(t_0) = w_\lambda$, $\pi(\hat{u}_\lambda(t_i)) = s_\lambda$, $i = \overline{0, k-1}$, and $\hat{u}_\lambda(t_i) \neq \hat{u}_\lambda(t_j)$, $i \neq j$. This means that w_λ has at least k equivalent points $\hat{u}_\lambda(t_i)$, $i = \overline{0, k-1}$, implying that $\nu(w_\lambda) \geq k = \nu(w_0)$. Theorem 4.1 is proved.

Corollary 4.1. *The function $\nu : \Delta_p^* \to \mathbb{Z}_+$ is locally constant on Δ_p^*.*

REFERENCES

1. A. F. Beknazaryan, S. A. Grigoryan, "On the Bohr-Riemann surfaces" [in Rusian], Izv. NAN Armenii, Matematika, **49** (5), 76-88, 2014.
2. R. Arens, I.M. Singer, "Generalized analytic functions", Trans. Amer. Math. Soc., **81** (2), 379-393, 1956.
3. S. A. Grigoryan, "Generalized analytic functions", Uspekhi Mat. Nauk [Russian Math. Surveys], **49** (2), 3-43, 1994.
4. S. A. Grigoryan, "The divisor of a generalized analytic function", Matem. zametki, **61** (5), 655-661, 1997.
5. O. Forster, *Riemann Surfaces* (Mir, Moscow, 1980).
6. S. A. Grigorian, R. N. Gumerov, A. V. Kazantsev, "Group structure in finite coverings of compact solenoidal groups", Lobachevskii Journal of Mathematics, **6**, 39-46, 2000.
7. S. A. Grigorian, R. N. Gumerov, "On the structure of finite coverings of compact connected groups", Topology and Its Applications, **153**, 3598-3614, 2006.
8. R. N. Gumerov, "Weierstrass polynomials and coverings of compact groups", Sib. Mat. Journal, **54** (2), 320-324, 2013.
9. S. Morris, *Pontryagin's Duality and Construction of Locally Compact Abelian Groups* (Mir, Moscow, 1980).
10. R. Engelking, *General Topology* (Mir, Moscow, 1986).