ISSN 1068–3372, Journal of Contemporary Physics (Armenian Academy of Sciences), 2018, Vol. 53, No. 1, pp. 1–21. © Allerton Press, Inc., 2018.
Original Russian Text © V.M. Mekhitarian, 2018, published in Izvestiya Natsional'noi Akademii Nauk Armenii, Fizika, 2018, Vol. 53, No. 1, pp. 3–29.

Equations of Relativistic and Quantum Mechanics and Exact Solutions of Some Problems

V. M. Mekhitarian

Institute for Physical Research, NAS of Armenia, Ashtarak, Armenia
vm@ipr.sci.am

Received December 13, 2017

Abstract—Relativistic invariant equations are proposed for the action function and the wave function based on the invariance of the representation of the generalized momentum. The equations have solutions for any values of the interaction constant of a particle with a field, for example, in the problem of a hydrogen-like atom, when the atomic number of the nucleus Z > 137. Based on the parametric representation of the action, the expression for the canonical Lagrangian, the equations of motion and the expression for the force acting on the charge during motion in an external electromagnetic field are derived. The Dirac equation with the correct inclusion of the interaction for a particle in an external field is presented. In this form, the solutions of the equations are not limited by the value of the interaction constant. The solutions of the problem of charge motion in a constant electric field, problems for a particle in a potential well, and penetration of a particle through a potential barrier, as well as problem of a hydrogen atom are presented.

DOI: 10.3103/S1068337218010012
Keywords: quantum mechanics, relativistic invariant equations

1. INTRODUCTION

For every theory of quantum mechanics, there are two sticking points – the problem of a particle in a potential well and the problem of a hydrogen atom, the results of solution of them determine the adequacy of new approaches. The first reveals the correspondence of the solution to the uncertainty relations, which are fundamental in physics, and the second shows how accurately the results of the solution of the hydrogen atom problem correspond to the experimental data of the spectrum of the hydrogen atom within the framework of the new theory.

In 1913, Bohr, based on the Balmer empirical formulas, constructed a model of atom based on the quantization of the orbital momentum [1], which was subsequently supplemented by the more general Sommerfeld quantization rules. In those years, naturally, the presence of a spin or an intrinsic magnetic moment of the particle or, especially, spin-orbit interaction, or interaction with the nuclear spin, was not supposed. Only after 1925, one started seriously thought about these, after the publication of the work of Uhlenbeck and Gaudsmit about the spin [2].

In 1916, Sommerfeld, within the framework of relativistic approaches, derived a formula for the energy levels of a hydrogen-like atom, without taking into account the spin [3]. Sommerfeld proceeded from the model of the Bohr atom and used the relativistic relation between the momentum p and the energy E of a free particle with the mass m

$$E^2 - (\mathbf{p}c)^2 = (mc^2)^2, \tag{1}$$

where c is the speed of light.

In an external field with a four-dimensional potential (φ, **A**), it was supposed that for a particle with the charge q this relation can also be used if we subtract the components of the four-dimensional momentum of the field ($q\varphi$, $q\mathbf{A}$) from the expression for the generalized particle momentum:

$$(E - q\varphi)^2 - (\mathbf{p}c - q\mathbf{A})^2 = (mc^2)^2. \tag{2}$$

In the case of the Coulomb potential $\varphi = Z|e|/r$, where e is the charge of electron, r is the distance from the nucleus, and Z is an atomic number, we obtain in spherical coordinates

$$p_r^2 + r^2 p_\varphi^2 = p_r^2 + \frac{L^2}{r^2} = \frac{\left(E + Ze^2/r\right)^2 - \left(mc^2\right)^2}{c^2}, \tag{3}$$

where L is the angular momentum. The Bohr–Sommerfeld quantization conditions take the form

$$\begin{aligned} &\oint p_\varphi d\varphi = \hbar n_\varphi, \\ &\oint p_r dr = \oint \sqrt{\frac{\left(E + Ze^2/r\right)^2 - \left(mc^2\right)^2}{c^2} - \frac{L^2}{r^2}}\, dr = \hbar n_r, \end{aligned} \tag{4}$$

where n_φ and n_r are the orbital and radial quantum numbers, respectively.

For the energy levels, Sommerfeld obtained the formula

$$E_{n,l} = \frac{mc^2}{\sqrt{1 + \dfrac{\left(Z\alpha\right)^2}{\left(n - \dfrac{\left(Z\alpha\right)^2}{l + 1/2 + \sqrt{\left(l+1/2\right)^2 - \left(Z\alpha\right)^2}}\right)^2}}}, \tag{5}$$

where the principal quantum number $n = n_r + l + 1 = 1, 2, 3, \ldots$, $l = 0, 1, 2, 3, \ldots, n-1$ and $\alpha = e^2/\hbar c = 1/137.036$ is the fine structure constant. However, in a paper published in 1916 [3], Sommerfeld 'made a fortunate mistake' [4] and the derived formula was presented in the following form

$$E_{n,l} = \frac{mc^2}{\sqrt{1 + \dfrac{\left(Z\alpha\right)^2}{\left(n - \dfrac{\left(Z\alpha\right)^2}{l + 1 + \sqrt{\left(l+1\right)^2 - \left(Z\alpha\right)^2}}\right)^2}}}. \tag{6}$$

The formula (6) perfectly described all the peculiarities of the structure of the spectrum of hydrogen and other similar atoms with the limiting for those years accuracy of measurements, and there was no doubt about the correctness of the formula itself. Therefore, the Sommerfeld formula was perceived as empirical, and instead of the quantum number l, a 'mysterious' internal quantum number with half-integer values j = 1/2, 3/2, 5/32, ... was introduced, and formula (6) was used in the representation

$$E_{n,j} = \frac{mc^2}{\sqrt{1 + \dfrac{Z^2\alpha^2}{\left(n - \dfrac{Z^2\alpha^2}{j + 1/2 + \sqrt{\left(j+1/2\right)^2 - Z^2\alpha^2}}\right)^2}}}, \tag{7}$$

where $n = n_r + j + 1/2 = 1, 2, 3, ..., j = 1/2, 3/2, 5/2, ..., n - 1/2$, and l possess the values $l = 0$ at $j = 1/2$ and $l = j \pm 1/2$ for others. This formula coincides with the result of an exact solution of the relativistic Dirac equations in 1928 [5] for a particle with the spin 1/2 with the classical expression for the potential energy of an immobile charge in the Coulomb field of a nucleus with an atomic number Z in the form $U(r) = Ze^2/r$.

Formula (7) also indicated a strange limitation of value the charge of a nucleus with the atomic number $Z < 137$, above which the formula is losing its meaning. It was also evident that within the framework of the approaches outlined, the strong and gravitational interactions, the motions of the planets are not described. The problem $Z > 137$ or $\alpha > 1$ remains the unresolved problem of relativistic quantum mechanics. Expanding the formula (7) over the order of powers $(Z\alpha)^2$ in the Taylor series, with an accuracy of expansion up to the terms by the powers $(Z\alpha)^6$, we obtain

$$E_{n,j} = mc^2 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) + \dots . \tag{8}$$

In 1925–1926, Schrödinger worked on the derivation of the equation for the wave function of a particle describing the De Broglie waves [6]. The derivation of the equation also was based on the relativistic relation (1) between the momentum $\mathbf{p}$ and the energy E of the particle, which he presented with the help of the operators of squares of energy and momentum in the form of an equation for the wave function

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \Psi - c^2\left(-i\hbar\frac{\partial}{\partial \mathbf{r}}\right)^2 \Psi = \left(mc^2\right)^2 \Psi . \tag{9}$$

Like Sommerfeld, Schrödinger used the following representation for a particle in an external field

$$\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)^2 \Psi + c^2\left(-i\hbar\frac{\partial}{\partial \mathbf{r}} - \frac{q}{c}\mathbf{A}\right)^2 \Psi = \left(mc^2\right)^2 \Psi . \tag{10}$$

In the case of stationary states of a charged particle in the field of the Coulomb potential for a hydrogen atom it was necessary to solve the equation

$$\Delta^2\Psi + \frac{m^2c^4 - \left(E + Ze^2/r\right)^2}{\hbar^2c^2}\Psi = 0 . \tag{11}$$

Solving this equation, Schrödinger, like Sommerfeld, received the formula (5), which described the structure of the hydrogen spectrum not exactly. Moreover, from the solution of the problem for a particle in a potential well, it turns out that a particle with a wavelength $\lambda\!\!\!^{-} = \hbar/mc$ has bound states (is placed) in a well of arbitrary size and, in particular, much smaller than $\lambda/2$. This fact contradicts the fundamental principle of the quantum (wave) theory, the principle of uncertainty.

In 1925 Schrödinger sent this work to the editors of 'Annalen der Physik' [7], but then took the manuscript, refused the relativistic approaches and in 1926 built a wave equation based on the classical Hamiltonian expression, the Schrödinger equation [8]

$$\mathrm{H} = \frac{\mathbf{p}^2}{2m} + U \;\rightarrow\; i\hbar\frac{\partial}{\partial t}\Psi = \left(\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial \mathbf{r}}\right)^2 + U\right)\Psi . \tag{12}$$

Equation (12) described the spectrum of the hydrogen atom only qualitatively, however, it did not have any unreasonable restrictions or singular solutions in the form of the Sommerfeld–Dirac formula. Klein [9], Fock [10] and Gordon [11] published the relativistic equation based on the wave equation for a particle without spin in 1926; it is called the Klein–Fock–Gordon equation.

With the discovery of the spin, the situation changed drastically, and in 1926 Heisenberg and Jordan [12] showed that, within the Pauli description of the spin of an electron, half the energy of the spin-orbit interaction is equal to a term with a power of α^4 in the Taylor series expansion of the Sommerfeld formula (8).

Why exactly the half, Thomas tried to explain this in 1927 by the presence of a relativistic precession of an electron in the reference frame of motion along the orbit [13]. The energy of the Thomas precession is exactly equal to half the value of the energy of the spin-orbit interaction with the inverse (positive) sign, which should be added to the energy of the spin-orbit interaction. However, the incorrect assumption that the Thomas precession frequency is identical in both frames of reference and the absence of a common and correct derivation for non-inertial (rotating) frames of reference raised doubts about the correctness of such approaches. The reason for the appearance of half the energy of the spin-orbit interaction in the Sommerfeld formula is still under investigation and is one of the unresolved problems in modern physics.

On the other hand, both in the derivation of the Sommerfeld formula and at the solution of the Klein–Fock–Gordon equation for the hydrogen atom problem [14], neither the spin nor the spin-orbit interaction energy was taken into account initially. Therefore, the obtained fine splitting can in no way be owing to the spin-orbit interaction. This is a relativistic but purely mechanical effect, when the mass (inertia) of a particle is already depends on the velocity of motion along the orbit (of the angular momentum), because of which the radial motion of the electron changes, and vice versa. Just this dependence, which results in the splitting of the energy levels of the electron, and to the impossibility of introducing only one, the principal quantum number. Nevertheless, even with this assumption, the order of splitting of the levels according to formula (8) contradicts to the logic; it turns out to be that the greater the orbital angular momentum, the lesser the energy of the splitted level.

The matrix representation of the second-order wave equation (9) by a system of equations of the first order is the Dirac construction of the relativistic electron equation [15] (the Dirac matrices are the particular representation of the Clifford–Lipschitz numbers [16]). In the standard representation the Dirac equation for a free particle has the form [17]

$$\begin{aligned} \hat{\varepsilon}\,\phi - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\chi &= mc\,\phi, \\ -\hat{\varepsilon}\,\chi + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\varphi &= mc\,\chi, \end{aligned} \tag{13}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

are the Pauli matrices (the unit matrix in the formulas is omitted).

For a particle in an external field, equation (13) is usually written in the form

$$\left(\hat{\varepsilon}-\frac{q}{c}\varphi\right)\phi-\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}}-\frac{q}{c}\mathbf{A}\right)\chi=mc\,\phi,$$
$$-\left(\hat{\varepsilon}-\frac{q}{c}\varphi\right)\chi+\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}}-\frac{q}{c}\mathbf{A}\right)\phi=mc\,\chi, \tag{15}$$

where for an invariant representation in the case of a free particle, the equations are composed for the difference between the generalized momentum and the momentum of the field.

In the case of the potential energy of an immobile charge in a Coulomb field, we obtain the Sommerfeld–Dirac formula as a result of an exact solution of this particular equation. There, again, although for a system with spin ½ the energy of the spin-orbit interaction is not taken into account initially, but the half is obtained from the exact solution of the hydrogen atom problem.

More accurate measurements of Lamb in 1947 and subsequent improvements in the spectrum of the hydrogen atom revealed that, in addition to the lines with the maximum j, all the others are also splitted and somewhat displaced (the Lamb shift). To harmonize the results of the theory with more accurate experimental data on the spectrum of the hydrogen atom, one had to propose other solutions and approaches than were laid down by the derivation of the Dirac equation.

The new theoretical approaches had yield nothing and only supplemented the theory with the illogical and non-physical proposals to overcome the emerging singularity of solutions: the renormalization, the finite difference of infinities with the desired value of the difference, and so on. The accounting for the size of the nucleus corrected only the Z value into the bigger value, but did not solved the $Z > 137$ problem. An incredible result was also obtained for the hydrogen atom problem that the electron is located, most likely, at the center of the atom, that is, in the nucleus.

The results of solution of the problem for a particle in a potential well both in the case of the Klein–Fock–Gordon equation and of the Dirac equation contradict to the basic principle of quantum mechanics, to the uncertainty principle. From the solutions, it turns out to be that a particle can be in a bound state in a well with any dimensions, in particular, with the size much smaller than the wavelength of the particle itself, $\lambda\!\!\!^{-} = \hbar/mc$ [17].

Despite Dirac himself proposed a system of linear first-degree relativistic equations in the matrix representation that described the system with spin 1/2, the contradictions did not disappear, and he himself remained unhappy with the results of his theory. As Dirac wrote in 1956 [18], the development of relativistic electron theory can now be considered as an example of how incorrect arguments sometimes lead to a valuable result. In the seventies, it became clear that the relativistic theory of quantum mechanics does not exist and new, fundamental approaches and equations should be sought for constructing a consistent theory of relativistic quantum mechanics. And in the eighties Dirac already spoke about the insuperable difficulties of existing quantum theory and the need to create a new one.

The reason for the failure of these theories is quite simple – it is in the ignoring of the dependence of the interaction energy with the field on the velocity of the particle. The generalized momentum of the system, the particle plus the external field, is the sum of the relativistic expression for the mechanical momentum of the particle and the field momentum in the case of interaction with the immobile particle

$$\mathbf{P}=\left(\varepsilon,\mathbf{p}\right)=\frac{1}{c}\left(\frac{mc^2}{\sqrt{1-\beta^2}}+q\varphi,\ \ \frac{mc^2}{\sqrt{1-\beta^2}}\boldsymbol{\beta}+q\mathbf{A}\right),\quad \mathbf{P}^2\neq\text{inv}, \tag{16}$$

which is not an invariant representation of the particle velocity. To construct some invariant from such a representation, an 'invariant' relation was used in all cases in the form of a difference between the generalized momentum of the system and the field momentum in the case of interaction with the immobile particle

$$\left(\varepsilon - q\varphi, \mathbf{p} - q\mathbf{A}\right) = \frac{1}{c}\left(\frac{mc^2}{\sqrt{1-\beta^2}}, \frac{mc^2}{\sqrt{1-\beta^2}}\boldsymbol{\beta}\right), \qquad (17)$$

$$\left(\varepsilon - q\varphi\right)^2 - \left(\mathbf{p} - q\mathbf{A}\right)^2 = \left(mc\right)^2.$$

Obviously, the permutation of the components of the generalized momentum for the construction of the invariant does not solve the posed problem. The statement that the expression (17) is the mechanical momentum of a particle and therefore is an invariant is unproven and it is necessary to apprehend the formula (17) as an empirical. Therefore, at high velocities or strong interactions, an unaccounted dependence of the energy of particle interaction with the field on the velocity of the particle motion, which results to the erroneous results or the impossibility of calculations.

In [19], an invariant representation of the generalized momentum of the system was suggested, where the dependence of the interaction energy of the particle with the field on the velocity was taken into account

$$\mathbf{P} = \left(\varepsilon, \mathbf{p}\right) = \frac{1}{c}\left(\frac{mc^2 + q\varphi + q\boldsymbol{\beta}\cdot\mathbf{A}}{\sqrt{1-\beta^2}}, \frac{\left(mc^2 + q\varphi\right)\boldsymbol{\beta} + q\mathbf{A}_{\parallel}}{\sqrt{1-\beta^2}} + q\mathbf{A}_{\perp}\right), \qquad (18)$$

$$\mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = \frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{c^2}, \qquad (19)$$

which is the four-dimensional representation of the generalized momentum of the system based on the expression for the generalized momentum of an immobile particle in a state of rest

$$\mathbf{P}_0 = \left(\varepsilon_0, \mathbf{p}_0\right) = \frac{1}{c}\left(mc^2 + q\varphi,\ q\mathbf{A}\right), \qquad (20)$$

whose invariant is always equal to the expression (19) regardless of the state of the system.

The application of variational principles to construct the relativistic and quantum theory was based on the principles of construction the mechanics with the help of the Lagrangian of the system [20], which originally was not intended for relativistic approaches. The Lagrangian construction is parametric with the one time variable $\tau = ct$, singled out from the variables of the four-dimensional space (the rest are represented by the dependence on this variable τ) and contains the total differential with respect to this variable, the velocity of the particle. Such a construction is unacceptable because of the impossibility to apply the principle of invariance of the representation of variables and the covariant representation of the action of the system.

In [21], to construct the relativistic theory on the basis of variational principles, the canonical (non-parametric) solutions of the variational problem for canonically defined integral functionals have been considered and the canonical solutions of the variational problems of mechanics in the Minkowski spaces are written. Because of unifying the variational principles of least action, flow, and hyperflow, the canonically invariant equations for the generalized momentum are obtained. From these equations, the expressions for the action function and the wave function are obtained as the general solution of the unified variational problem of mechanics.

Below, we present the generalized invariance principle and the corresponding representation of the generalized momentum of the system, the equations of relativistic and quantum mechanics, give the solutions to the problems of charge motion in a constant electric field, of particle in a potential well, of particle penetration of potential barrier, as well as the hydrogen atom problem.

2. GENERALIZATION OF THE PRINCIPLE OF INVARIANCE AND THE INVARIANT REPRESENTATION OF THE GENERALIZED MOMENTUM

The principle of invariance of the representation of a generalized pulse is applicable also in the case of motion of a particle with the velocity **v** and in the case of a transition to a reference frame moving with the velocity **V**.

The four-dimensional momentum of a particle **P** with the rest mass m moving with the velocity $\boldsymbol{\beta} = \mathbf{v}/c$ is represented in the form

$$\mathbf{P} = \left(\varepsilon, \ \mathbf{p}\right) = \left(\frac{mc}{\sqrt{1-\beta^2}}, \ \frac{mc}{\sqrt{1-\beta^2}}\boldsymbol{\beta}\right), \ \ \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = \left(mc\right)^2. \tag{21}$$

This is a property of invariance of the representation of the four-dimensional momentum **P** through the velocity of the reference system $\boldsymbol{\beta}' = \mathbf{V}/c$.

For an invariant of the system I, we have

$$I^2 = \mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = \left(\varepsilon_0\right)^2 = \left(mc\right)^2. \tag{23}$$

At $\boldsymbol{\beta} = \boldsymbol{\beta}' = 0$, we obtain

$$\mathbf{P} = \left(\varepsilon, \mathbf{p}\right)\big|_{\boldsymbol{\beta}=\boldsymbol{\beta}'=0} = \varepsilon_0\left(1, 0\right) = mc\left(1, 0\right). \tag{24}$$

Thus, the generalized momentum of a particle has an invariant representation on the particle velocity **v** and the velocity of the reference system **V**. This property should be considered because of the general principle of the relativity of motion. Accordingly, the generalized momentum of the particle **P** is an invariant regardless of the state of the system.

If a charged particle is in an external electromagnetic field with potentials (φ, **A**), then the stationary charge sees the field exactly with such potentials. If the charge has a nonzero velocity **v**, then it will interact with the field differently. To determine the interaction for a charge moving with the velocity **v**, one can start from the principle of the relativity of motion. The effective values of the force or interaction with the field of the charge moving with the velocity **v** are the same as in the case when the charge is immobile, and the field moves with the velocity –**v** (in the laboratory frame of reference).

More clearly, this can be demonstrated by an example of the Doppler effect for two atoms in the field of a resonant radiation, when one of the atoms is at rest and the other moves with the velocity **v**. The atom, which is at rest, absorbs a photon, and the moving one does not absorb or interacts weakly with the field, because of the dependence of the interaction on the velocity of the atom. It is also known that the acting field for an atom moving with the velocity **v** corresponds to the interaction with the field moving with the velocity –**v**.

Thus, for a moving charge, the effective values of the potentials (φ', **A'**) (in the laboratory frame of reference) can be written in the form [22]

$$\left(\varphi', \ \mathbf{A}'\right) = \left(\frac{\varphi + \boldsymbol{\beta}\cdot\mathbf{A}}{\sqrt{1-\beta^2}}, \ \mathbf{A}_{\perp} + \frac{\mathbf{A}_{\parallel} + \varphi\boldsymbol{\beta}}{\sqrt{1-\beta^2}}\right). \tag{25}$$

If one represents the generalized momentum of the particle in the form

$$\mathbf{P}=\frac{1}{c}\left(\frac{mc^2}{\sqrt{1-\beta^2}}+q\varphi',\ \frac{mc^2}{\sqrt{1-\beta^2}}\boldsymbol{\beta}+q\mathbf{A}'\right), \tag{26}$$

where φ' and $\mathbf{A}'$ already effective values of the interaction potentials of the particle moving with velocity $\mathbf{v}$ in a field with the potentials φ and $\mathbf{A}$, we obtain

$$\mathbf{P}=\frac{1}{c}\left(\frac{mc^2+q\varphi+q\boldsymbol{\beta}\cdot\mathbf{A}}{\sqrt{1-\beta^2}},\ \frac{\left(mc^2+q\varphi\right)\boldsymbol{\beta}+q\mathbf{A}_{\|}}{\sqrt{1-\beta^2}}+q\mathbf{A}_{\perp}\right). \tag{27}$$

The expression (27) can be represented in the form

$$\mathbf{P}=\left(\frac{mc^2+q\varphi+q\boldsymbol{\beta}\cdot\mathbf{A}}{c\sqrt{1-\beta^2}},\ \frac{mc^2+q\varphi+q\boldsymbol{\beta}\cdot\mathbf{A}}{c\sqrt{1-\beta^2}}\boldsymbol{\beta}+\frac{q}{c}\mathbf{A}-\frac{q}{c}\frac{1}{1+\sqrt{1-\beta^2}}\left(\mathbf{A}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}\right) \tag{28}$$

or

$$\mathbf{P}=\left(\varepsilon,\quad \varepsilon\boldsymbol{\beta}+\frac{q}{c}\mathbf{A}-\frac{q}{c}\frac{1}{1+\sqrt{1-\beta^2}}\left(\mathbf{A}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}\right). \tag{29}$$

For the module I of the four-dimensional vector of the generalized momentum $\mathbf{P}$, we have

$$I^2=\mathbf{P}^2=\varepsilon^2-\mathbf{p}^2=\frac{\left(mc^2+q\varphi\right)^2-\left(q\mathbf{A}\right)^2}{c^2}, \tag{30}$$

which is the four-dimensional representation of the generalized momentum of the system on the basis of the expression of the generalized momentum of a particle in the state of rest

$$\mathbf{P}_0=\left(\varepsilon_0,\mathbf{p}_0\right)=\frac{1}{c}\left(mc^2+q\varphi,\ q\mathbf{A}\right), \tag{31}$$

whose invariant is defined by the expression (30).

Thus, the generalized momentum of the particle in an external field is not only invariant relative to the transformations at the transition from one reference system to another but also has an invariant representation in terms of the velocity of motion of the particle (27); at each point of space, the value of the invariant I is determined by the expression (30). This property has not only the representation of the proper momentum of the particle (the mechanical part), but also the generalized momentum of the particle in general.

Let us generalize this result to the case of representation of the generalized momentum of any systems and interactions, arguing that, regardless of the state (the motion) of the system, the generalized four-dimensional momentum always has an invariant representation

$$\mathbf{P}=\left(\varepsilon,\mathbf{p}\right)\quad\Rightarrow\quad \mathbf{P}^2=\varepsilon^2-\mathbf{p}^2=\varepsilon_0{}^2-\mathbf{p}_0{}^2=\pm I^2=\mathrm{inv}, \tag{32}$$

where ε и $\mathbf{p}$ are the energy and momentum of the system, respectively, and the invariant is determined by the modulus of sum of the components of the generalized momentum of the system ε_0 and $\mathbf{p}_0$ at rest. If the particles interact with the field in the form ($\alpha\varphi$, $\alpha\mathbf{A}$), the invariants of the generalized momentum of the system are represented by the expressions [18]

$$\mathbf{P}_+^2 = \left(\varepsilon_0 + \alpha\varphi\right)^2 - \left(\alpha\mathbf{A}\right)^2 = \varepsilon_0^2 + 2\varepsilon_0\alpha\varphi + \left(\alpha\varphi\right)^2 - \left(\alpha A\right)^2,$$

$$\mathbf{P}_-^2 = \left(\alpha\varphi\right)^2 - \left(\varepsilon_0\mathbf{n} + \alpha\mathbf{A}\right)^2 = -\varepsilon_0^2 - 2\varepsilon_0\alpha\mathbf{n}\cdot\mathbf{A} + \left(\alpha\varphi\right)^2 - \left(\alpha A\right)^2, \tag{33}$$

$$\mathbf{P}_0^2 = \left(\varepsilon_0 + \alpha\varphi\right)^2 - \left(\varepsilon_0\mathbf{n} + \alpha\mathbf{A}\right)^2 = 2\varepsilon_0\alpha\left(\varphi - \mathbf{n}\cdot\mathbf{A}\right) + \left(\alpha\varphi\right)^2 - \left(\alpha A\right)^2.$$

Let us represent the expression for the invariant $\varepsilon^2 - \mathbf{p}^2$ (30) in the following form

$$\varepsilon^2 = \frac{E^2}{c^2} = \mathbf{p}^2 + \frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{c^2} = \mathbf{p}^2 + m^2c^2 + 2mq\varphi + \frac{q^2}{c^2}\left(\varphi^2 - \mathbf{A}^2\right) \tag{34}$$

and divide it by $2m$. Grouping, we obtain the Hamiltonian H of the system in the form

$$\mathrm{H} = \frac{\varepsilon^2 - m^2c^2}{2m} = \frac{E^2 - m^2c^4}{2mc^2} = \frac{\mathbf{p}^2}{2m} + q\varphi + \frac{q^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right), \tag{35}$$

that is, we obtain the formula for the correspondence between the energy of the system E and the energy of the system in the classical meaning H. The correspondence in the form $\mathrm{H} = \mathbf{p}^2/2m + U(\tau, \mathbf{r})$ [20] will be complete and accurate if we determine the potential energy of interaction U and the energy of system in the classical meaning as

$$U = q\varphi + \frac{q^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right), \quad \mathrm{H} = \frac{E^2 - m^2c^4}{2mc^2} \quad \Rightarrow \quad E = \pm mc^2\sqrt{1 + \frac{2\mathrm{H}}{mc^2}}\,. \tag{36}$$

For example, the potential energy U of the electron in the field of the Coulomb potential $\varphi = Ze/r$ and in a homogeneous magnetic field $\mathbf{B}$ with the vector potential $\mathbf{A} = [\mathbf{r}\times\mathbf{B}]/2$ is

$$U = -e\varphi + \frac{e^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right) = -\frac{Ze^2}{r} + \frac{1}{2mc^2}\frac{Z^2e^4}{r^2} - \frac{e^2B^2}{8mc^2}r^2\sin^2\theta\,. \tag{37}$$

Note, whatever is the dependence of the potential φ, the possible minimum potential energy $U_{\min} = -mc^2/2$, and the potential energy as a function of the vector potential is always negative. The hard constraint of the classical potential energy value $U_{\min} = -mc^2/2$, which does not depend on the nature of the interactions, results in the fundamental changes in the description of interactions and the revision of the results of classical mechanics. At short distances, the origination of repulsion for attraction forces caused by the uncertainty principle is clearly reflected in the expression for the potential energy of the particle.

The Hamiltonian H can be called the energy and its value remains constant in the case of conservation of energy E, but the value of H and its changes differ from the true values of the energy E and changes of its quantity. Thus, the classical approaches are permissible only in the case of low velocities, when $\mathrm{H} << mc^2$ and the energy expression can be represented in the form

$$E = mc^2\sqrt{1 + \frac{2\mathrm{H}}{mc^2}} \approx mc^2 + \mathrm{H}\,. \tag{38}$$

3. EQUATIONS OF RELATIVISTIC MECHANICS AND THE CANONICAL LAGRANGIAN

Let us use the parametric representation of the Hamilton action in the form [20]

$$S = -\int_{t_1,\,\mathbf{r}_1}^{t_2,\,\mathbf{r}_2}\left(\varepsilon dt - \mathbf{p}\cdot d\mathbf{r}\right) = -\int_{\mathbf{R}_1}^{\mathbf{R}_2}\mathbf{P}\cdot d\mathbf{R} = -\int_{\mathbf{R}_1}^{\mathbf{R}_2}\mathbf{P}\cdot\frac{d\mathbf{R}}{ds}ds = -\int_{\mathbf{R}_1}^{\mathbf{R}_2}\left(\mathbf{P}\cdot\mathbf{V}\right)ds \;\rightarrow\; \min, \tag{39}$$

where ds is the four-dimensional interval and $\mathbf{V}$ is the four-dimensional generalized velocity.

The functional that takes into account the condition of the invariant representation of the generalized momentum $\mathbf{P}^2 = \varepsilon^2 - \mathbf{p}^2 = I^2 = \text{inv}$, can be composed by the method of indefinite Lagrange coefficients in the form

$$S = \int_{s_1}^{s_1}\left(-\mathbf{P}\cdot\mathbf{V} + \frac{\mathbf{P}^2 - I^2}{2\lambda}\right)ds = \int_{s_1}^{s_1}\left(\frac{\left(\mathbf{P}-\lambda\mathbf{V}\right)^2}{2\lambda}\frac{\lambda^2 - I^2}{2\lambda}\right)ds \;\to \min, \tag{40}$$

where $\lambda = \lambda(s)$ is the given parameter, determined by the condition of invariance of the representation. Because λ and I are given and they do not depend on the velocity, we have an explicit solution in the form

$$\mathbf{P} - \lambda\mathbf{V} = 0, \quad \lambda = \pm I\left(\tau, \mathbf{r}\right), \tag{41}$$

where the four-dimensional momentum is represented in the form

$$\mathbf{P} = I\mathbf{V} = \sqrt{\varepsilon^2 - \mathbf{p}^2}\left(\frac{1}{\sqrt{1-\eta^2}}, \;\frac{\boldsymbol{\eta}}{\sqrt{1-\eta^2}}\right). \tag{42}$$

Thus, the action is represented in the form

$$S = \int_{s_1}^{s_2} I ds = \int_{\tau_1}^{\tau_2}\sqrt{\varepsilon^2 - \mathbf{p}^2}\sqrt{1-\eta^2}\,d\tau \tag{43}$$

and the canonical Lagrangian of the system is given by

$$\mathrm{L} = I\sqrt{1-\eta^2} = \sqrt{\varepsilon^2 - \mathbf{p}^2}\sqrt{1-\eta^2}\,. \tag{44}$$

The correctness of the presented parametrization is confirmed by the obtained expressions for the generalized momentum and energy from the Lagrangian of the system in the form

$$\begin{aligned}\varepsilon &= \boldsymbol{\eta}\frac{\partial \mathrm{L}}{\partial \boldsymbol{\eta}} - \mathrm{L} = \frac{I}{\sqrt{1-\eta^2}} = \frac{\sqrt{\varepsilon^2 - \mathbf{p}^2}}{\sqrt{1-\eta^2}},\\ \mathbf{p} &= \frac{\partial \mathrm{L}}{\partial \boldsymbol{\eta}} = \frac{I}{\sqrt{1-\eta^2}}\boldsymbol{\eta} = \varepsilon\boldsymbol{\eta},\end{aligned} \tag{45}$$

which coincide with the initial representations of the generalized momentum and energy. Accordingly, the Lagrange equation of motion takes the form

$$\frac{d\mathbf{p}}{d\tau} = -\frac{I}{\varepsilon}\frac{\partial I}{\partial \mathbf{r}}\,. \tag{46}$$

If we multiply equation (47) by $\mathbf{p} = \varepsilon\boldsymbol{\eta}$ scalarly, after reduction to the total time differential, we obtain,

$$\frac{d\varepsilon^2}{d\tau} = \frac{\partial I^2}{\partial \tau}\,.$$

If the invariant is clearly independent of time, then the energy ε is conserved and the equation of motion is represented in the form of the Newtonian equation

$$\frac{d\boldsymbol{\eta}}{d\tau} = -\frac{I}{\varepsilon^2}\frac{\partial I}{\partial \mathbf{r}}\,. \tag{48}$$

For a particle in an external field we have

$$\mathrm{L} = -\frac{1}{c}\sqrt{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}\sqrt{1-\eta^2/c^2}\,. \tag{49}$$

The Hamilton–Jacobi equation is represented in the form

$$\left(\frac{\partial S}{\partial \tau}\right)^2 - \left(\frac{\partial S}{\partial \mathbf{r}}\right)^2 = \frac{\left(mc^2 + q\varphi\right)^2 - \left(q\mathbf{A}\right)^2}{c^2} \tag{50}$$

and it reflects the invariance of the representation of the generalized momentum. The well-known representations of the Hamilton–Jacobi equations [22] also contain the differential forms of potentials – the components of the electric **E** and the magnetic **H** fields.

Using the explicit form of the generalized momentum (29) with the accuracy of the expansion to the power of β^2, we obtain the equation of motion in the form

$$\frac{d}{d\tau}\left(\varepsilon - \frac{q}{2c}\mathbf{A}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta} = q\mathbf{E} + q\left[\boldsymbol{\beta}\times\mathbf{B}\right] - \frac{\partial}{\partial \mathbf{r}}\left(\frac{q}{c}\mathbf{A}\cdot\boldsymbol{\beta} + \frac{q^2}{2mc^2}\left(\varphi^2 - \mathbf{A}^2\right)\right), \tag{51}$$

where the velocity-dependent components of the force are present. In particular, the velocity-dependent force is present in the Faraday law of electromagnetic induction [23], which is absent in the traditional expression for the Lorentz force.

3.1. Motion of an Electron in a Constant Electric Field

Let us consider the motion of an electron with the mass m and charge $-e$ in the constant electric field between the plane electrodes with the potential difference U and the distance l between them. For one-dimensional motion, taking the cathode location as the origin and anode at the point $x = l$, from (50) we have

$$\left(\frac{\partial S}{\partial \tau}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 = \frac{\left(mc^2 + eU\left(1 - x/l\right)\right)^2}{c^2}\,. \tag{52}$$

Let us represent the action S in the form

$$S = -Et + f\left(x\right), \tag{53}$$

where $E = mc^2 + eU$ is an the electron energy at the origin on the surface of the cathode under voltage $-U$; as a result, from (52) we obtain

$$S = -Et + \frac{1}{c}\int\sqrt{E^2 - \left(mc^2 + eU - qU\frac{x}{l}\right)^2}\,dx\,. \tag{54}$$

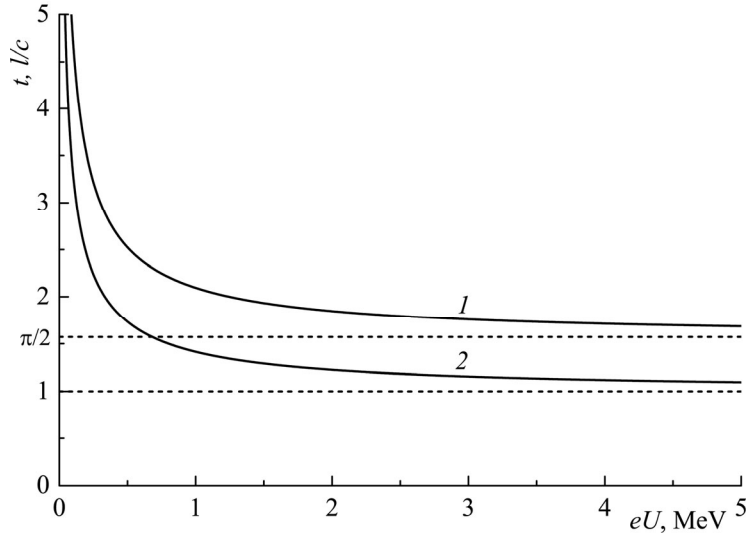
We find the solution from the condition $\partial S/\partial E = \mathrm{const}$. As a result of integration, we obtain

$$t = \frac{l}{c}\left(1 + \frac{1}{\alpha}\right)\arccos\left(1 - \frac{\alpha}{1+\alpha}\frac{x}{l}\right), \quad \alpha = qU/mc^2 \tag{55}$$

or

$$x = l\frac{1+\alpha}{\alpha}\left(1 - \cos\left(\frac{\alpha}{1+\alpha}\frac{ct}{l}\right)\right), \quad t \le \frac{l}{c}\left(\frac{1+\alpha}{\alpha}\right)\arccos\left(\frac{1}{1+\alpha}\right). \tag{56}$$

The well-known solution in the framework of the traditional theory [22] is the following:



Dependence of the flight time of the gap between the electrodes on the applied voltage according to the formula (55) (curve *1*) and (57) (curve *2*) in *l*/*c* units.

$$t=\frac{l}{\alpha c}\sqrt{\left(1+\alpha\frac{x}{l}\right)^2-1}\quad\text{or}\quad x=l\frac{\left(\alpha ct/l\right)^2}{1+\sqrt{1+\left(\alpha ct/l\right)^2}},\qquad t\le\frac{l}{c}\sqrt{1+\frac{2}{\alpha}}\,. \tag{57}$$

In the ultrarelativistic limit $eU >> mc^2$, the ratio of the flight time of the gap between the electrodes ($x = l$) is equal to $\pi/2$ according to formulas (55) and (57) (see the figure).

3.2. Problem of the Hydrogen-Like Atom

Let us consider the motion of an electron with the mass m and charge $-e$ in the field of an immobile nucleus with the charge Ze. Then the problem reduces to an investigation of the motion of the electron in the centrally symmetric electric field with the potential $-Ze^2/r$.

Choosing the polar coordinates (r, φ) in the plane of motion, we obtain the Hamilton–Jacobi equation in the form

$$\left(\frac{\partial S}{\partial\tau}\right)^2-\left(\frac{\partial S}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial S}{\partial\varphi}\right)^2-\frac{\left(mc^2-Ze^2/r\right)^2}{c^2}=0\,. \tag{58}$$

Let us represent the action S in the form

$$S=-Et+M\varphi+f\left(r\right), \tag{59}$$

where E and M are the constant energy and angular momentum of the moving particle, respectively. As a result, we obtain

$$S=-Et+M\varphi+\frac{1}{c}\int\sqrt{E^2-\left(mc^2\right)^2+2mc^2\frac{Ze^2}{r}-\frac{M^2c^2+\left(Ze^2\right)^2}{r^2}}\,dr\,. \tag{60}$$

We find trajectories from the condition $\partial S/\partial M=\text{const}$, with use of which we obtain,

$$\varphi=\int\frac{Mc}{\sqrt{E^2-\left(mc^2\right)^2+2mc^2\frac{Ze^2}{r}-\frac{M^2c^2+Ze^2}{r^2}}}\,d\frac{1}{r}\,, \tag{61}$$

which results in the solution

$$r=\frac{\left(Mc\right)^2+\left(Ze^2\right)^2}{mc^2Ze^2}\frac{1}{1+\sqrt{\left(\frac{E}{mc^2}\right)^2\left(1+\left(\frac{Mc}{Ze^2}\right)^2\right)-\left(\frac{Mc}{Ze^2}\right)^2}\cos\left(\varphi\sqrt{1+\left(\frac{Ze^2}{Mc}\right)^2}\right)}. \tag{62}$$

The coefficient of the repulsive effective potential is essentially positive, that is, $M^2c^2+\left(Ze^2\right)^2>0$ therefore, any fall of the particle onto the center is impossible. The minimum radius $r_{\min}=r_0\left(Z+1\right)$, where $r_0=e^2/mc^2$ is the classical radius of an electron.

The secular precession is found from the condition

$$\varphi\sqrt{1+\left(Ze^2/Mc\right)^2}=2\pi, \tag{63}$$

whence, we obtain

$$\Delta\varphi=2\pi-\frac{2\pi}{\sqrt{1+\left(Ze^2/Mc\right)^2}}\approx\pi\left(\frac{Ze^2}{Mc}\right)^2, \tag{64}$$

that has the opposite sign as compared with the solution in [22]. The reason for the discrepancy of the sign is the unaccounted interaction of the self-momentum with the rotating field, that is, the spin-orbit interaction.

4. EQUATIONS OF RELATIVISTIC QUANTUM MECHANICS FOR SYSTEMS WITHOUT THE SPIN

Using the principle of the invariant representation of the generalized momentum

$$\mathbf{P}^2=\varepsilon^2-\mathbf{p}^2=I^2=\text{inv}, \tag{65}$$

it is possible to compose the corresponding equation of the relativistic quantum mechanics by representing the energy and momentum variables by the corresponding operators $\hat{\varepsilon}=i\hbar\partial/\partial\tau$ and $\hat{\mathbf{p}}=-i\hbar\partial/\partial\mathbf{r}$:

$$\left(\hat{\varepsilon}\right)^2\Psi-\left(\hat{\mathbf{p}}\right)^2\Psi=\left(i\hbar\frac{\partial}{\partial\tau}\right)^2\Psi-\left(-i\hbar\frac{\partial}{\partial\mathbf{r}}\right)^2\Psi=\left(\varepsilon^2-\mathbf{p}^2\right)\Psi+i\hbar\left(\frac{\partial\varepsilon}{\partial\tau}+\text{div}\,\mathbf{p}\right)=I^2\Psi+i\hbar\left(\frac{\partial\varepsilon}{\partial\tau}+\text{div}\,\mathbf{p}\right), \tag{66}$$

$$\left(\hat{\varepsilon}\Psi\right)^2-\left(\hat{\mathbf{p}}\Psi\right)^2=\left(i\hbar\frac{\partial\Psi}{\partial\tau}\right)^2-\left(-i\hbar\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2=\left(\varepsilon^2-\mathbf{p}^2\right)\Psi^2=I^2\Psi^2. \tag{67}$$

The case of conservative systems, when any energy losses or sources in space are absent, corresponds to the relation $\partial\varepsilon/\partial\tau+\text{div}\,\mathbf{p}=0$. In this way,

$$\begin{cases}\dfrac{\partial^2\Psi}{\partial\tau^2}-\dfrac{\partial^2\Psi}{\partial\mathbf{r}^2}=-\dfrac{I^2}{\hbar^2}\Psi\\[2ex]\left(\dfrac{\partial\Psi}{\partial\tau}\right)^2-\left(\dfrac{\partial\Psi}{\partial\mathbf{r}}\right)^2=-\dfrac{I^2}{\hbar^2}\Psi^2.\end{cases} \tag{68}$$

For the charged particle in an external field with an invariant in the form of (30), the equations will take the form

$$\begin{cases} \dfrac{\partial^2\Psi}{\partial\tau^2}-\dfrac{\partial^2\Psi}{\partial\mathbf{r}^2}=-\dfrac{\left(mc^2+q\varphi\right)^2-\left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi \\ \left(\dfrac{\partial\Psi}{\partial\tau}\right)^2-\left(\dfrac{\partial\Psi}{\partial\mathbf{r}}\right)^2=-\dfrac{\left(mc^2+q\varphi\right)^2-\left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi^2. \end{cases} \tag{69}$$

For stationary states we obtain

$$\begin{cases} \dfrac{\partial^2\Psi}{\partial\mathbf{r}^2}+\dfrac{E^2-\left(mc^2+q\varphi\right)^2+\left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi=0 \\ \left(\dfrac{\partial\Psi}{\partial\mathbf{r}}\right)^2+\dfrac{E^2-\left(mc^2+q\varphi\right)^2+\left(q\mathbf{A}\right)^2}{\hbar^2c^2}\Psi^2=0. \end{cases} \tag{70}$$

Rewriting the equations taking into account the formulas of the classical correspondence (36)

$$U=q\varphi+\frac{q^2}{2mc^2}\left(\varphi^2-\mathbf{A}^2\right),\quad \mathrm{H}=\frac{E^2-m^2c^4}{2mc^2}\ \Rightarrow\ E=\pm mc^2\sqrt{1+\frac{2\mathrm{H}}{mc^2}}\,, \tag{71}$$

we will obtain the equations for the wave function in the traditional representation

$$\begin{gathered} \Delta\Psi+\frac{2m}{\hbar^2}\left(\mathrm{H}-U\right)\Psi=0, \\ \left(\frac{\partial\Psi}{\partial\mathbf{r}}\right)^2+\frac{2m}{\hbar^2}\left(\mathrm{H}-U\right)\Psi^2=0, \end{gathered} \tag{72}$$

the first of which formally coincides with the Schrödinger equation for the wave function of stationary states.

For the action function S associated with the wave function by the representation $\Psi = A\exp(-iS/\hbar)$ or $S = i\hbar\ln\Psi + i\hbar\ln A$, we will obtain

$$\begin{cases} \dfrac{\partial^2S}{\partial\mathbf{r}^2}=0 \\ \left(\dfrac{\partial S}{\partial\mathbf{r}}\right)^2-\dfrac{E^2-\left(mc^2+q\varphi\right)^2+\left(q\mathbf{A}\right)^2}{c^2}=0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2S}{\partial\mathbf{r}^2}=0 \\ \left(\dfrac{\partial S}{\partial\mathbf{r}}\right)^2-2m\left(\mathrm{H}-U\right)=0, \end{cases} \tag{73}$$

which represents the exact classical correspondence instead of the quasiclassical approximation [24]. Note, the equations similar to (73) also follow from the equation (46.2) in [24] if we demand for an exact correspondence and equate to zero the real and imaginary parts.

4.1. Particle in the one-dimensional Potential Well

Let us consider the particle of mass m in a one-dimensional rectangular potential well of the form

$$V\left(x\right)=\begin{cases} 0, & 0\ge x\ge a \\ -V_0, & 0\le x\le a. \end{cases} \tag{74}$$

From the first equation of system (70) we have

$$\frac{d^2\Psi}{dx^2}+\frac{E^2-\left(mc^2+V(x)\right)^2}{\hbar^2c^2}\Psi=0\,. \tag{75}$$

Then, $U_0 = -V_0 + V_0^2/(2mc^2)$ corresponds to the potential energy of the particle in the well in the classical meaning. In the latter case, it is known [3] that the bound state with the energy H = 0 ($E = mc^2$) arises under the conditions

$$U_0=-\frac{\pi^2\hbar^2}{2ma^2}n^2=-V_0+\frac{V_0^2}{2mc^2}\geq-\frac{mc^2}{2},\quad a\geq\frac{\pi\hbar}{mc}n=\frac{\lambda}{2}n\,, \tag{76}$$

$$E_n=mc^2\left(1-\sqrt{1-\frac{\pi^2\hbar^2}{m^2c^2a^2}n^2}\right)=mc^2\left(1-\sqrt{1-\left(\frac{\lambda}{2a}n\right)^2}\right)=mc^2\frac{\left(\frac{\lambda}{2a}n\right)^2}{1+\sqrt{1-\left(\frac{\lambda}{2a}n\right)^2}}, \tag{77}$$

where $\lambda=2\pi\lambda\!\!\!^{-}=2\pi\hbar/mc=h/mc$. Maximum depth of the classic well is equal to $U_0 = -mc^2/2$ at $V_0 = mc^2$. The condition for the existence of the bound state with an energy H = 0 ($E = mc^2$) in a potential well of size a is expressed by the relation

$$a\ =\ \lambda n/2,\quad n=1,2,3\ldots. \tag{78}$$

In the three-dimensional case, the bound state with the energy H = 0 ($E = mc^2$) arises under the same conditions [24] for a spherical well with a diameter d and depth V_0 with the $d\ =\lambda n/2,\ n=1,2,3\ldots$.

The solution of this simple example is fundamental and accurately represents the uncertainty principle $\Delta x\Delta p \geq \hbar/2$. It clearly represents the wave property of the particle, clearly showing that the standing wave exists only at the condition $a\geq\lambda/2$ when the geometric dimensions of the well are greater than half the wavelength of the particle.

4.2. Penetration of a Particle through a Potential Barrier

Let us consider the problem of penetration of a particle through the rectangular potential barrier [24] with the height V_0 and width a. Then, $U_0 = V_0 + V_0^2/(2mc^2)$ corresponds to the potential energy of the particle in the well in the classical meaning, and $H = (E^2 - m^2c^4)/2mc^2$ corresponds to the energy. Substituting these expressions into the solution of the Schrödinger equation for the rectangular potential barrier, we obtain for the transmission coefficient D of the particle penetrating through the potential barrier at $E > |V_0 + mc^2|$

$$D=\left[1+\frac{\left(\left(\frac{V_0}{mc^2}+1\right)^2-1\right)^2}{4\left(\left(\frac{E}{mc^2}\right)^2-1\right)\left(\left(\frac{E}{mc^2}\right)^2-\left(\frac{V_0}{mc^2}+1\right)^2\right)}\sin^2\left(\frac{a}{\lambda\!\!\!^{-}}\sqrt{\left(\frac{E}{mc^2}\right)^2-\left(\frac{V_0}{mc^2}+1\right)^2}\right)\right]^{-1}, \tag{79}$$

and at $E < |V_0 + mc^2|$

$$D=\left[1+\frac{\left(\left(\frac{V_0}{mc^2}+1\right)^2+1-2\left(\frac{E}{mc^2}\right)^2\right)^2}{4\left(\left(\frac{E}{mc^2}\right)^2-1\right)\left(\left(\frac{V_0}{mc^2}+1\right)^2-\left(\frac{E}{mc^2}\right)^2\right)}\sinh^2\left(\frac{a}{\lambda\!\!\!^{-}}\sqrt{\left(\frac{V_0}{mc^2}+1\right)^2-\left(\frac{E}{mc^2}\right)^2}\right)\right]^{-1}, \tag{80}$$

where $\lambda\!\!\!^{-} = \hbar/mc$ is the de Broglie wavelength of the particle. As can be seen, the barrier is formed only in the energy range $-2mc^2 > V_0 > mc^2$.

4.3. Motion of an Electron in a Constant Electric Field

Let us consider the motion of an electron with the mass m and charge $-e$ in a constant electric field F. From equation (50), we have

$$\frac{d^2\Psi}{dx^2} + \frac{E^2 - \left(mc^2 + eFx\right)^2}{\hbar^2 c^2}\Psi = 0\,. \tag{81}$$

Rewriting the equation in the form

$$\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}\left(\frac{E^2}{2mc^2} - \frac{m}{2}\left(\frac{eF}{mc}\right)^2\left(x + \frac{mc^2}{eF}\right)^2\right)\Psi = 0\,, \tag{82}$$

we obtain the equation of a linear oscillator with the frequency $\omega = eF/mc$, from which the discrete levels of E_n are determined by the formula

$$E_n = \sqrt{\hbar eFc\left(2n+1\right)} = mc^2\sqrt{\frac{eF\lambda\!\!\!^{-}}{mc^2}\left(2n+1\right)}, \quad \lambda\!\!\!^{-} = \hbar/mc, \quad n = 0,1,2,3,\ldots\,. \tag{83}$$

In the framework of the traditional theory, the energy spectrum of a charge in a constant electric field F is continuous [24]. In this case, for the particle with the energy $E_n = mc^2$, and $n = \dfrac{mc^2}{2eF\lambda\!\!\!^{-}} - \dfrac{1}{2}$ the distance between nearest levels is $\Delta E_{mc^2} = eF\lambda\!\!\!^{-}$.

In general case

$$\Delta E_n = mc^2\sqrt{\frac{eF\lambda\!\!\!^{-}}{mc^2\left(2n+1\right)}}\,. \tag{84}$$

4.4. Problem of the Hydrogen Atom

The motion of a charged particle in the Coulomb field can be described as a motion in the field of an atomic nucleus (without the spin and magnetic moment) with the potential energy $-Ze^2/r$.

In spherical coordinates, the equation (70) for the wave function takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{1}{\hbar^2c^2}\left(E^2 - \left(mc^2 - \frac{Ze^2}{r}\right)^2\right)\Psi = 0. \tag{85}$$

Separating the variables

$$\Psi = \Phi_m\left(\varphi\right)Y_{l,m}\left(\theta\right)R_{n_R,l}\left(r\right) \tag{86}$$

and introducing the notations [22]

$$\begin{gathered}\alpha = \frac{e^2}{\hbar c}, \quad \rho = \frac{mZe^2}{\hbar^2}\frac{2r}{N} = Z\alpha\frac{mc}{\hbar}\frac{2r}{N}, \quad \mathrm{H}_n = \frac{E_n{}^2 - m^2c^4}{2mc^2} = -\frac{mZ^2e^4}{\hbar^2}\frac{1}{2N^2} = -mc^2Z^2\alpha^2\frac{1}{2N^2}, \\ M^2 = \hbar^2 l\left(l+1\right), \quad s\left(s+1\right) = l\left(l+1\right) + Z^2\alpha^2 \quad \Rightarrow \quad s = -1/2 + \sqrt{\left(l+1/2\right)^2 + Z^2\alpha^2}\end{gathered} \tag{87}$$

(only the positive root is taken for s), for stationary states we have

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi,$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) - \frac{m^2}{\sin^2\theta}Y = -l(l+1)Y, \tag{88}$$

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} - \frac{s(s+1)}{\rho^2}R = -\left(\frac{n_r}{\rho} - \frac{1}{4}\right)R,$$

where $m = \pm 0, \pm 1, \pm 2, \ldots$, $l = 0, 1, 2, 3, \ldots$, $|m| < l$ and $s = -1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}$.

The solution of equations (88) formally coincides with the well-known Fuse solution for the molecular Kratzer potential in the form $U = \frac{A}{r^2} - \frac{B}{r} = \frac{Z^2e^4}{2mc^2}\frac{1}{r^2} - Ze^2\frac{1}{r}$ at the condition, that $n - s - 1 = n_r$ must be a positive integer or zero. According to (87), we obtain the energy levels

$$\mathrm{H}_{n,j} = -mc^2\frac{Z^2\alpha^2}{2\left(n_r + 1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}\right)^2},$$

$$E_{n,j} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n_r + 1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}\right)^2}}, \tag{89}$$

where the radial quantum number $n_r = 0, 1, 2, \ldots$. Introducing the principal quantum number $n = n_r + l + 1/2$, $l < n$ (n = 1, 2, 3, …), we finally obtain

$$E_{n,j} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n + \frac{Z^2\alpha^2}{l + 1/2 + \sqrt{(l+1/2)^2 + Z^2\alpha^2}}\right)^2}}. \tag{90}$$

For the ground state with the $l = 0$ and $n = 1$, we have

$$E_0 = \frac{mc^2}{\sqrt{1/2 + \sqrt{1/4 + Z^2\alpha^2}}}, \quad s = \frac{Z^2\alpha^2}{1/2 + \sqrt{1/4 + Z^2\alpha^2}} \tag{91}$$

without any restrictions for the value of Z. In this case, $1 - s > 0$ and there is no fall of the particle on the center [24], and the probability of finding the particle at the center (in the nucleus) is always equal to zero.

In this case, the obtained fine splitting is in no way connected with the spin-orbit interaction and is due to the relativistic dependence of the mass on the orbital and radial velocity of motion, which results to the splitting of the levels.

5. EQUATIONS OF RELATIVISTIC QUANTUM MECHANICS FOR SYSTEMS WITH THE SPIN ½

In the standard representation, the Dirac equations in compact notation for a particle have the form [15]

$$\hat{\varepsilon}\phi - \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi = mc\phi,$$

$$\hat{\varepsilon}\chi + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\phi = mc\chi. \tag{92}$$

In addition, for the particle in an external field they can be represented in the form

$$\begin{aligned}\hat{\varepsilon}\phi-\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\chi&=\left(mc+\frac{q}{c}\varphi\right)\phi+\frac{q}{c}\boldsymbol{\sigma}\cdot\mathbf{A}\chi,\\ \hat{\varepsilon}\chi+\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\,\phi&=\left(mc+\frac{q}{c}\varphi\right)\chi-\frac{q}{c}\boldsymbol{\sigma}\cdot\mathbf{A}\phi.\end{aligned}\tag{93}$$

By writing the wave equations for the wave functions, we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\mathbf{r}^2}\right)\phi&=-\frac{\left(mc^2+q\varphi\right)^2-\left(q\mathbf{A}\right)^2}{\hbar^2c^2}\phi-\frac{q}{\hbar c}\boldsymbol{\sigma}\cdot\left(\mathbf{B}-i\mathbf{E}\right)\chi,\\ \left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\mathbf{r}^2}\right)\chi&=-\frac{\left(mc^2+q\varphi\right)^2-\left(q\mathbf{A}\right)^2}{\hbar^2c^2}\chi+\frac{q}{\hbar c}\boldsymbol{\sigma}\cdot\left(\mathbf{B}-i\mathbf{E}\right)\phi,\end{aligned}\tag{94}$$

where we used the properties of the Pauli matrices.

In the case of a stationary state, the standard representation of the wave equations (92) has the form

$$\begin{aligned}\left(\varepsilon-mc-\frac{q}{c}\varphi\right)\varphi&=\boldsymbol{\sigma}\cdot\left(\mathbf{p}+\frac{q}{c}\mathbf{A}\right)\chi,\\ \left(\varepsilon+mc+\frac{q}{c}\varphi\right)\chi&=\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{q}{c}\mathbf{A}\right)\varphi.\end{aligned}\tag{95}$$

5.1. Problem of the Hydrogen Atom

For a charge in a potential field with the central symmetry [17], we have

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}\dfrac{f(r)}{r}\Omega_{jlm}\\ \\ (-1)^{1+l-l'}\dfrac{g(r)}{r}\Omega_{jl'm}\end{pmatrix}.\tag{96}$$

After substituting (96) into (95), we obtain

$$\begin{cases}f'+\dfrac{\chi}{r}f-\left(\varepsilon+mc-\dfrac{Ze^2}{c}\dfrac{1}{r}\right)g=0\\ g'-\dfrac{\chi}{r}g+\left(\varepsilon-mc+\dfrac{Ze^2}{c}\dfrac{1}{r}\right)f=0,\end{cases}\qquad\begin{cases}j=\left|l\pm1/2\right|,\quad j_{\max}=l_{\max}+1/2\\ \chi=-1,\quad l=0\\ \chi=\pm\left(j+1/2\right).\end{cases}\tag{97}$$

Let us represent the functions f and g in the form

$$\begin{aligned}f&=\sqrt{mc+\varepsilon}e^{-\rho/2}\rho^{\gamma}\left(Q_1+Q_2\right),\\ g&=\sqrt{mc-\varepsilon}e^{-\rho/2}\rho^{\gamma}\left(Q_1-Q_2\right),\end{aligned}\tag{98}$$

where

$$\rho=2\lambda r/\hbar,\quad\lambda=\sqrt{\left(mc\right)^2-\varepsilon^2},\quad\gamma=\sqrt{\chi^2+Z^2\alpha^2},\quad\alpha=\frac{e^2}{\hbar c}.\tag{99}$$

Substituting (98) into the equations (97), for the sum and difference of the equations we have

$$\begin{aligned}\rho Q_1' + (\gamma - Z\alpha\, mc/\lambda) Q_1 + (\chi - Z\alpha\,\varepsilon/\lambda) Q_2 = 0,\\ \rho Q_2' + (\gamma + Z\alpha\, mc/\lambda - \rho) Q_2 + (\chi + Z\alpha\,\varepsilon/\lambda) Q_1 = 0.\end{aligned} \tag{100}$$

Close to $\rho = 0$, the system of equations always has a solution, because

$$\gamma^2 - (Z\alpha mc/\lambda)^2 = \chi^2 - (Z\alpha\varepsilon/\lambda)^2. \tag{101}$$

Then

$$Q_2 = -\frac{\gamma - Z\alpha\, mc/\lambda}{\chi - Z\alpha\,\varepsilon/\lambda} Q_1 = -\frac{\chi + Z\alpha\,\varepsilon/\lambda}{\gamma + Z\alpha\, mc/\lambda} Q_1. \tag{102}$$

Forming equations of the second order and solving with respect to Q_1 and Q_2, we obtain

$$\begin{aligned}\rho Q_1'' + (2\gamma + 1 - \rho) Q_1' - (\gamma - Z\alpha\, mc/\lambda) Q_1 = 0,\\ \rho Q_2'' + (2\gamma + 1 - \rho) Q_2' - (\gamma + 1 - Z\alpha\, mc/\lambda) Q_2 = 0.\end{aligned} \tag{103}$$

With allowance for (101), the solution of these equations is

$$\begin{aligned}Q_1 = AF(\gamma - Z\alpha\, mc/\lambda,\ 2\gamma + 1,\ \rho),\\ Q_2 = -A\frac{\gamma - Z\alpha\, mc/\lambda}{\chi - Z\alpha\,\varepsilon/\lambda} F(\gamma + 1 - Z\alpha\, mc/\lambda,\ 2\gamma + 1,\ \rho),\end{aligned} \tag{104}$$

where $F(\alpha, \beta, z)$ is the degenerate hypergeometric function and A is the normalization constant of the wave function. The function $F(\alpha, \beta, z)$ reduces to a polynomial, if the parameter α is equal to an integer negative number or zero. Therefore, finite solutions for the functions f and g are

$$\gamma - \frac{Z\alpha mc}{\lambda} = -n_r. \tag{105}$$

From the expressions (104), we obtain

$$\begin{aligned}f = A\sqrt{mc + \varepsilon}\, e^{-\rho/2}\rho^{\gamma-1}\left(F(-n_r,\ 2\gamma + 1,\ \rho) + \frac{n_r}{\chi - Z\alpha\,\varepsilon/\lambda} F(1 - n_r,\ 2\gamma + 1,\ \rho)\right),\\ g = A\sqrt{mc - \varepsilon}\, e^{-\rho/2}\rho^{\gamma-1}\left(F(-n_r,\ 2\gamma + 1,\ \rho) - \frac{n_r}{\chi - Z\alpha\,\varepsilon/\lambda} F(1 - n_r,\ 2\gamma + 1,\ \rho)\right),\end{aligned} \tag{106}$$

where $n_r = 0, 1, 2, \ldots$ is the radial quantum number. For the energy levels, we obtain from the condition (105)

$$\frac{\varepsilon_{p,\chi}}{mc} = \sqrt{1 - \frac{Z^2\alpha^2}{\left(n_r + \sqrt{\chi^2 + Z^2\alpha^2}\right)^2}}$$

and taking into account the obtained values of χ, we finally have

$$E_{n,j} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n_r + \sqrt{(j + 1/2)^2 + Z^2\alpha^2}\right)^2}} = mc^2\sqrt{1 - \frac{Z^2\alpha^2}{\left(n + \frac{Z^2\alpha^2}{j + 1/2 + \sqrt{(j + 1/2)^2 + Z^2\alpha^2}}\right)^2}}, \tag{107}$$

where the principal quantum number $n = n_r + j + 1/2$. Besides $j = n - 1/2$, all other levels with $j < n - 1/2$ are degenerated twice in the orbital angular momentum $l = |j \pm 1/2|$. The ground state energy for $n = 1$ and $j = 1/2$ is

$$E_0 = \frac{mc^2}{\sqrt{1+Z^2\alpha^2}} \tag{108}$$

without any limitations for the value of Z. In this case $\gamma - 1 = -1 + \sqrt{1+Z^2\alpha^2} > 0$, and no falling of particle on the center is observed, and the probability to find the particle in the center (in the nucleus) is always equal to zero.

In the resulting formula (107), the order of sequence of the fine splitting levels is inverse relative to the order of sequence in the well-known Sommerfeld–Dirac formula. If to compare the expansions in a series in the degree of the fine-structure constant of two formulas

$$\frac{E_n}{mc^2} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)\alpha^2 + \left(\frac{1}{8} + \frac{3}{8n^4} - \frac{1}{2n^3\left(j+1/2\right)}\right)\alpha^4, \quad \frac{\Delta E_{3/2,1/2}}{mc^2} = \frac{\alpha^4}{32}, \tag{109}$$

$$\frac{E_n}{mc^2} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)\alpha^2 + \left(-\frac{3}{8} - \frac{1}{8n^4} + \frac{1}{2n^3\left(j+1/2\right)}\right)\alpha^4, \quad \frac{\Delta E_{3/2,1/2}}{mc^2} = -\frac{\alpha^4}{32}, \tag{110}$$

then the difference will be equal to

$$\frac{\Delta E_n}{mc^2} = \frac{\alpha^4}{2} - \frac{\alpha^4}{2n^4} - \frac{\alpha^4}{n^3\left(j+1/2\right)}, \tag{111}$$

where the last term is the expression for the spin-orbit interaction energy. Thus, to obtain the true value of the energy levels of the hydrogen atom, it is necessary to add the energy of the spin-orbit interaction in formula (107) in the form (111). This is completely justified, because such an interaction was not initially included in equation (97) and was not reflected in the final result.

6. CONCLUSION

The principle of invariance is generalized and the corresponding representation of the generalized momentum of the system is proposed; the equations of relativistic and quantum mechanics are proposed which are devoid of the above-mentioned shortcomings and contradictions. The equations have solutions for any values of the interaction constant of the particle with the field, for example, in the problem of a hydrogen-like atom, when the atomic number of the nucleus $Z > 137$. The equations are applicable for different types of particles and interactions.

Based on the parametric representation of the action and the canonical equations, the corresponding relativistic mechanics based on the canonical Lagrangian is constructed and the equations of motion and expression are derived for the force acting on the charge moving in an external electromagnetic field.

The matrix representation of equations of the characteristics for the action function and the wave function results in the Dirac equation with the correct enabling of the interaction. In this form, the solutions of the Dirac equations are not restricted by the value of the interaction constant and have a spinor representation by scalar solutions of the equations for the action function and the wave function.

The solutions are presented for the problem of charge motion in a constant electric field, the problem of a particle in a potential well, the problem of the penetration of a particle through a potential barrier, and the problems of the hydrogen atom.

The analysis of the solutions shows the full compliance with the principles of the relativistic and quantum mechanics, and the solutions are devoid of any restrictions on the nature and magnitude of the interactions.

REFERENCES

1. Bohr, N., *Philos. Mag.*, 1913, vol. 26, pp. 1; 476; 857.
2. Uhlenbeck, G.E. and Goudsmit, S., *Naturwissenschaften*, 1925, vol. 47, p. 953; *Nature*, 1926, vol. 117, p. 264.
3. Sommerfeld, A., *Ann. Phys.*, 1916, vol. 51, p. 1; *Phys. Z.*, 1916, vol. 17, p. 491.
4. Granovskii, Y.I., *Physics-Uspekhi*, 2004, vol. 47, p. 523.
5. Dirac, P.A.M., *Proc. Royal Society A*, 1928, p. 117, p. 610.
6. De Broglie, M.L., *Comptes Rendus*, 1923, vol. 177, p. 507.
7. Hoffman, D., *Erwin Schrödinger*, Leipzig: Teubner, 1984.
8. Schrödinger, E., *Ann. Physik.*, 1926, vol. 79, p. 389.
9. Klein, O., *Z. Phys.*, 1926, vol. 37, p. 895.
10. Fock, V., *Z. Phys.*, 1926, vol. 38, p. 242.
11. Gordon, W., *Z. Phys.*, 1926, vol. 40, p. 117.
12. Heisenberg, W. and Jordan, P., *Zeits. Phys.*, 1926, vol. 37, p. 263.
13. Thomas, L.H., *Nature*, 1926, vol. 117, p. 514; *Phil. Mag.*, 1927, vol. 3, p. 1.
14. Sokolov, A.A., Ternov, I.M., and Kilmister, C.W., *Radiation from Relativistic Electrons*, New York: American Institute of Physics, 1986.
15. Berestetskii, V.B., Lifshitz, E.M., and Pitaevski, L.P*., Quantum Electrodynamics*, Oxford: Pergamon, 1984.
16. Madelung, E., *Mathematische Hilfsmtittel des Physikers*, 3rd ed., Berlin: Springer, 1936.
17. Akhiezer, A.I., and Berestetskii, V.B., *Quantum electrodynamic*, John Wiley & Sons, 1965.
18. Dirac, P.A.M., *Trudy instituta istorii estestvoznaniya i tekhniki*, 1959, vol. 22, p. 32.
19. Mekhitarian, V.M., *J. Contemp. Phys. (Armenian Ac. Sci.)*, 2012, vol. 47, p. 249.
20. Landau, L.D. and Lifshitz, E.M., *Mechanics,* vol. 1, 3rd ed., Butterworth–Heinemann, 1976.
21. Mekhitarian, V.M., *J. Contemp. Phys. (Armenian Ac. Sci.)*, 2013, vol. 48, p. 1.
22. Landau, L.D. and Lifshitz, E.M., *The Classical Theory of Fields*, Butterworth–Heinemann, 1975.
23. Mekhitarian, V.M., *J. Contemp. Phys. (Armenian Ac. Sci.)*, 2016, vol. 51, p. 108.
24. Landau, L.D. and. Lifshitz, E.M., *Quantum Mechanics: Non-Relativistic Theory*, vol. 3, 3rd ed., Pergamon Press, 1977.