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On the Effect of Approximations on the Conservation of Integrals of Motion in Laguerre–Gaussian Beams

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Abstract—The effect of approximations used in description of laser beams on the results of analytical calculations and numerical simulation of an electron motion in the field of the circularly-polarized Laguerre—Gaussian mode is considered. It is shown that corrections to the paraxial approximation and the effect of the envelope should be simultaneously taken into account for correct description of the laser beam angular momentum transfer to the electron.

Keywords: orbital angular momentum, integral of motion, paraxial approximation, envelope

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1. INTRODUCTION

Currently, it becomes possible to generate high-power laser pulses with angular orbital momentum [1]. The problem of describing the interaction of such pulses with dense and rarefied plasma includes the consideration of collective effects; however, for rarefied plasma, the use of the single-particle approximation is reasonable. One of the interesting phenomena observed in numerical simulation is quasi-static magnetic field generation during the interaction of spatially structured laser beams carrying the orbital angular momentum with plasma [2—4]. In this case, a detailed analysis allows the conclusion that this process can be considered as a result of the laser beam interaction with a single particle [5].

In the general case, symmetries existing for the plane wave are broken in structured beams. However, certain spatial structures allow preservation of some of these symmetries. The latter can be used in numerical and analytical studies of the interaction of particles with waves. At the same time, under certain conditions, the parameters responsible for plane wave symmetry breaking can be determined. In such cases, the smallness of these parameters characterizes the deviations of initial symmetries and corresponding integrals of motion.

2. EQUATIONS OF ELECTRON MOTION IN THE CIRCULARLY-POLARIZED LAGUERRE—GAUSSIAN MODE

Consider the electron motion (the electron charge $-e < 0$) in an external electromagnetic field defined by the vector potential $\mathbf{A}$. Let us write the particle Lagrangian using k^{-1} as the units of lengths, where $k = \frac{2\pi}{\lambda}$ is the wavenumber, and ω^{-1} as the units of time, where ω is the wave frequency, the vector potential is given in the units of $a_0 A_0$, where $a_0 = \frac{eA_0}{mc^2}$, where m is the electron mass, A_0 is the vector potential amplitude, the momentum is given in the units of mc, and the energy is given in the units of mc^2,

$$L = -\sqrt{1 - \mathbf{v}^2} - \mathbf{Av}. \tag{1}$$

From the Lagrangian, the standard equations of motion are derived,

$$\frac{d\mathbf{p}}{dt} = -\mathbf{E} - [\mathbf{vH}], \tag{2}$$

where $\mathbf{p} = \gamma\mathbf{v}$ is the electron momentum, $\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}$ is the electron energy, $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{H} = \mathrm{curl}\mathbf{A}$ are the electric and magnetic fields, respectively.

In cylindrical coordinates, expressions for transverse components of the vector potential of the circularly polarized Laguerre—Gaussian mode take the form

$$\begin{aligned} A_r &= a_0 \,\mathrm{Re}(u_{pl}(r,\varphi,z)e^{i(t-z-\sigma_z\varphi)}),\\ A_\varphi &= a_0 \,\mathrm{Re}(-i\sigma_z u_{pl}(r,\varphi,z)e^{i(t-z-\sigma_z\varphi)}), \end{aligned} \tag{3}$$

where $\sigma_z = \mp 1$; the signs "–" and "+" correspond to right- and left-hand polarizations, respectively. The longitudinal component of the vector potential A_z is determined from the calibration condition $\mathrm{div}\mathbf{A} = 0$.

The function $u_{pl}(r, \varphi, z)$ depends only on coordinates and specifies the spatial structure of the electromagnetic beam [6],

$$\begin{aligned} u_{pl}(r,\varphi,z) = C^{(p,l)}\frac{w_0}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\\ \times\exp\left(-il\varphi - i\frac{r^2 z}{2(z^2+z_R^2)} + i(2p+|l|+1)\arctan\left(\frac{z}{z_R}\right)\right), \end{aligned} \tag{4}$$

where $C^{(p,\,l)} = \sqrt{\frac{2}{\pi}\frac{p!}{(p+|l|)!}}$ is a normalizing constant, $w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$, $L_p^{|l|}$ is the generalized Laguerre polynomial, and $z_R = \frac{w_0^2}{2}$ is the dimensionless Rayleigh length.

In view of the weak dependence of potential amplitudes on the z coordinate, the electromagnetic field in the paraxial approximation is locally close to the plane wave; the corresponding approximate integral of motion is given by [7]

$$\varepsilon - P_z = \varepsilon - p_z + A_z \approx 0, \tag{5}$$

where ε, P_z, and p_z are the kinetic energy, canonical momentum conjugate to the coordinate z, and the longitudinal component of the electron momentum, respectively. For the observation time Δt, this integral gains the value $\sim\frac{a_0}{w_0^2}\Delta t$ having the smallness due to the factor $\frac{1}{w_0^2}$, where $w_0 \gg 1$ is the characteristic longitudinal beam size in dimensionless units. That is, in comparison with the energy and momentum, this value is lower by a factor of $w_0^2 \gg 1$.

This estimate can be obtained by considering the translation of the time and z coordinate to the small value ϵ in the electron action S. On the one hand, using the equations of motion, we obtain the action variation $\delta S = \epsilon(P_z - \varepsilon)|_{t_1}^{t_2}$; on the other hand, $\delta S = \epsilon\int_{t_1}^{t_2}\mathbf{v}\,\mathrm{Re}\frac{\partial \mathbf{u}}{\partial z}e^{i(t-z)}dt$, where $\mathbf{u}$ is the field amplitude weakly depending on the coordinate z. Assuming the velocity oscillation amplitude to be ~1 (in the case of weak intensity ~a_0, therefore, the estimate should be overvalued), field oscillations ~a_0 and the derivative over z ~ $\frac{1}{w_0^2}$, and assuming that the product of oscillating contributions to the velocity and vector potential components contribute to the integral, we obtain the estimate given above.

The function $u_{pl}(r, \varphi, z)$ depends on the azimuthal angle as $e^{-il\varphi}$, where $l = 0, \pm 1, \pm 2, \ldots$. Then, A_r and A_φ contain the dependence on the time and azimuthal angle only in the combination $t - (\sigma_z + l)\varphi$. Since A_z is determined from the equation $\mathrm{div}\mathbf{A} = 0$, the longitudinal component similarly depends on the time and angle. Hence, Lagrangian (1) depends only on the same combination, this leads to the integral of motion

$$\Sigma \equiv P_\varphi - (l + \sigma_z)H = \mathrm{const}, \tag{6}$$

where $P_\varphi = L_z - rA_\varphi$ is the canonical momentum conjugate to the azimuthal angle φ, $L_z = \gamma r^2\dot{\varphi}$ is the electron angular momentum and $H = \gamma = \sqrt{1 + (\mathbf{P} + \mathbf{A})^2} = \sqrt{1 + \mathbf{p}^2}$ is the particle Hamiltonian in which $\mathbf{P}$ is the electron canonical momentum.

To study the particle motion numerically, we will simulate electromagnetic field switch-on and switch-off using the approximation of the slowly varying envelope $g(t - z)$ with characteristic duration $\tau \gg 1$. Let us call such a field the quasi-monochromatic one. In calculating the electric and magnetic fields, the envelope should be considered constant in order to avoid exceeding initial accuracy. The main contribution during the interaction with a particle is made by the region near the envelope peak, where it can be expanded in the Taylor series. The first nonconstant term of this series has the smallness $\sim\frac{1}{\tau}$, which makes it possible to estimate the accuracy of this approximation as $\sim\frac{a_0}{\tau}$. Moreover, the envelope does not contain the dependence on φ, which in cases $l + \sigma_z \neq 0$ breaks the symmetry necessary for the existence of the integral of motion. In any case, the quantity Σ remains an integral of motion only to within the terms $\sim\frac{a_0}{\tau}$, which results in its change during the interaction on $\sim a_0$.

One more cause of destruction of the integral of motion is the use of the paraxial approximation in which derivatives of functions u_{pl} with respect to z coordinate are considered to be small and are omitted in a certain order. The characteristic deviation from the integrals of motion in determining fields is $\sim\frac{a_0}{w_0^2}$, which for the entire interaction time yields $\sim\frac{a_0}{w_0^2}\tau$.

3. ELECTRON MOTION IN THE CIRCULARLY POLARIZED LAGUERRE—GAUSSIAN MODE WITH ENVELOPE

Consider the interaction of the initially resting electron with the circularly polarized Laguerre—Gaussian mode, defined by the vector potential

$$\begin{aligned} A_r &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(u_{pl} e^{i(t - z - \sigma_z\varphi)}\right), \\ A_\varphi &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(-i\sigma_z u_{pl} e^{i(t - z - \sigma_z\varphi)}\right), \\ A_z &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(-i\frac{\partial u_{pl}}{\partial r} e^{i(t - z - \sigma_z\varphi)} + \frac{il}{r}\sigma_z u_{pl} e^{i(t - z - \sigma_z\varphi)}\right). \end{aligned} \tag{7}$$

The electric and magnetic fields should be determined with the same accuracy as the vector potential. That is the terms $\sim\frac{a_0}{\tau}$ arising from envelope derivatives and terms $\sim\frac{a_0}{w_0^2}$ arising from u_{pl} derivatives should be ommited. Thus, for the electric and magnetic fields, we obtain

$$\begin{aligned} E_r = H_\varphi &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(-iu_{pl} e^{i(t - z - \sigma_z\varphi)}\right), \\ E_\varphi = -H_r &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(-\sigma_z u_{pl} e^{i(t - z - \sigma_z\varphi)}\right), \\ E_z &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(-\frac{\partial u_{pl}}{\partial r} e^{i(t - z - \sigma_z\varphi)} + \frac{l}{r}\sigma_z u_{pl} e^{i(t - z - \sigma_z\varphi)}\right), \\ H_z &= \frac{a_0}{\sqrt{2}} g(t - z)\,\mathrm{Re}\left(-i\sigma_z\frac{\partial u_{pl}}{\partial r} e^{i(t - z - \sigma_z\varphi)} + \frac{il}{r} u_{pl} e^{i(t - z - \sigma_z\varphi)}\right). \end{aligned} \tag{8}$$

Let us analyze Eq. (2) using numerical simulation for an initially resting electron in field (8) with $p = 0$ and $l = 1$. We take an envelope in the form $g(t) = \cos^2\left(\frac{t - \tau/2}{\tau}\pi\right)$ at $0 \le t \le \tau$ and $g(t) = 0$ otherwise.

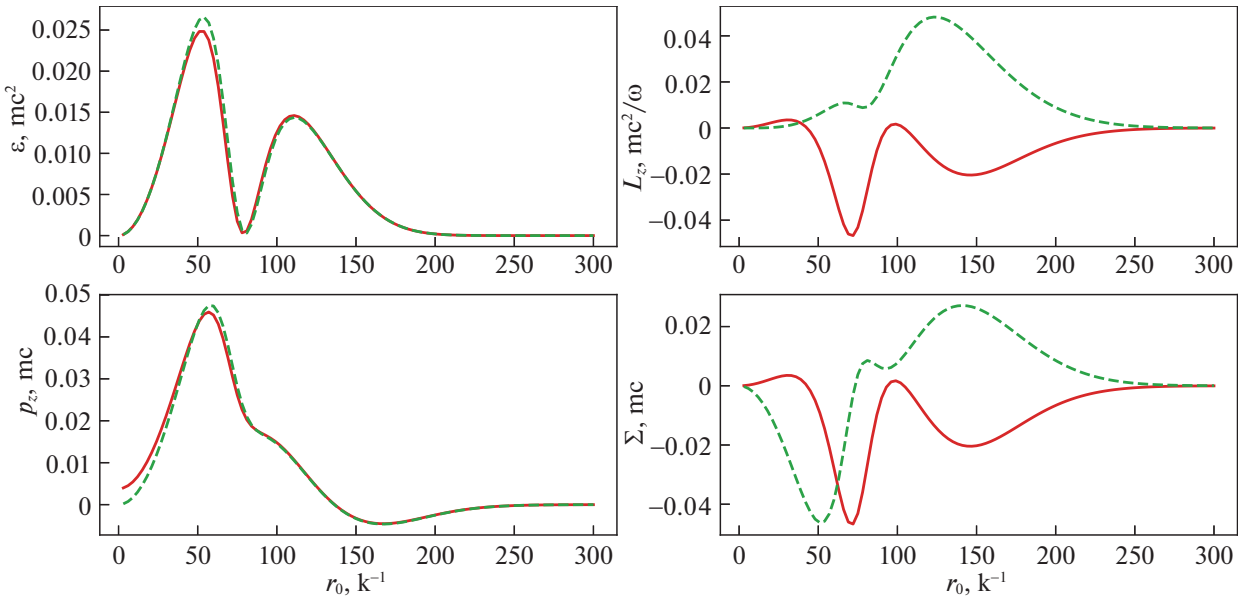

Fig. 1. Dependence of the kinetic energy ε, longitudinal momentum p_z, angular momentum L_z, and Σ transferred to the electron on the initial distance r_0 of the electron from the beam axis during the motion in the Laguerre—Gaussian quasi-monochromatic mode with $w_0 = 10$ and $n = 12$, obtained by numerical simulation for left-hand (green broken curve) and right-hand (red solid curve) polarized modes.

To determine the conservation accuracy of the integral of motion, we will compare the values of Σ with the kinetic energy ε, since the latter is not the integral of motion and enters the Σ definition. Therefore, it is reasonable to assume that integrals of motion can be considered as conserving approximately if their values are small in comparison with ε, i.e., $\Sigma \ll \varepsilon$.

At parameters $a_0 = 1$, $w_0 = 10$, and $n = 12$, where $n = \frac{\tau}{2\pi}$, the effect of the paraxial approximation and envelope is strong. The parameter $\frac{a_0}{w_0^2}\tau$ takes the value of ~0.1 which due to numerical coefficients can appear insufficiently small for conserving Σ; therefore, it appears comparable to the kinetic energy ε (see Fig. 1). In this case, the values of Σ are not conserved, which is caused by an insufficient accuracy of the paraxial approximation and the approximation of the slowly varying envelope.

Let us improve the applicability conditions of the paraxial approximation, taking $w_0 = 100$. The results of calculations without and with regard to derivatives of Laguerre—Gaussian amplitudes with respect to z, shown in Fig. 2, differ slightly, which suggests that the paraxial approximation is satisfied with high accuracy.

However, we can see that Σ is not conserved in this case as well, which means the importance of taking into consideration the dependence of the envelope on $t - z$. Let us add terms with derivatives of the envelope, calculated through the vector potential to electric and magnetic fields. Such additions are strictly speaking over the accuracy of the approximations, since their consideration also requires leaving corrections to the vector potential $\sim\frac{a_0}{\tau}$ when in solving Maxwell's equations.

The dependences calculated taking into account the derivative of the envelope are shown in Fig. 3. The value of Σ became ten times smaller, but yet comparable to the electron energy. In this case, Σ nonconservation is caused only by the paraxial approximation which makes a contribution of $\sim\frac{a_0}{w_0^2}\tau \sim 10^{-3}$ at given parameters. Due to numerical coefficients, the contribution can become smaller, which is seen in the figure.

The consideration of derivatives of field amplitudes with respect to z coordinate makes Σ small in comparison with ε, which is seen in Fig. 4. However, correct consideration of these corrections necessitates

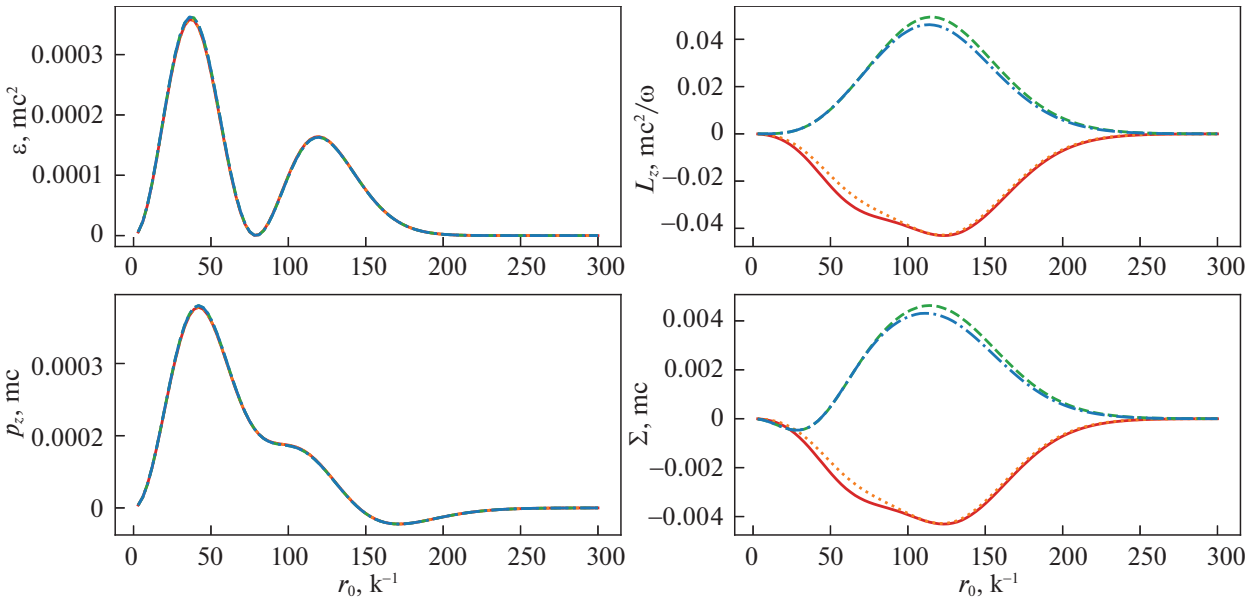

Fig. 2. Dependence of the transferred kinetic energy ε, longitudinal momentum p_z, angular momentum L_z, and Σ on the initial distance r_0 of the electron on the beam axis during the motion in the Laguerre—Gaussian quasi-monochromatic mode with $w_0 = 100$ and $n = 12$, obtained by numerical simulation for left-hand (green broken curve) and right-hand (red solid curve) polarized modes without regard to derivatives of amplitudes with respect to z and with regard to them (orange doted curve and blue dash-dotted curve for right- and left-hand polarized modes, respectively).

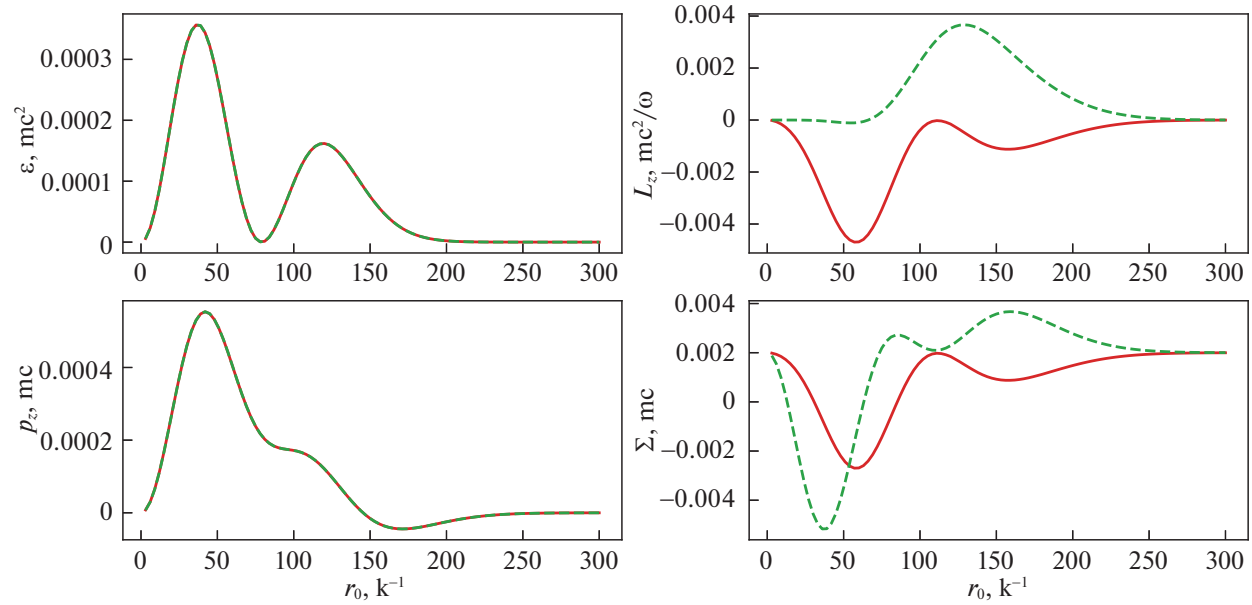

Fig. 3. Dependence of the transferred kinetic energy ε, longitudinal momentum p_z, angular momentum L_z, and Σ on the initial distance r_0 of the electron on the beam axis during the motion in the Laguerre—Gaussian quasi-monochromatic mode with $w_0 = 100$ and $n = 12$, with regard to the derivative of the envelope, obtained by numerical simulation for left-hand (green broken curve) and right-hand (red solid curve) polarized modes.

the determination fields (7) themselves with the same accuracy. In the case, of left-hand polarization, Σ is not an integral of motion, since the envelope does not contain angle φ. Its addition will restore Σ conservation, if terms with derivatives of the envelope with respect to angle φ will also be added in this case.

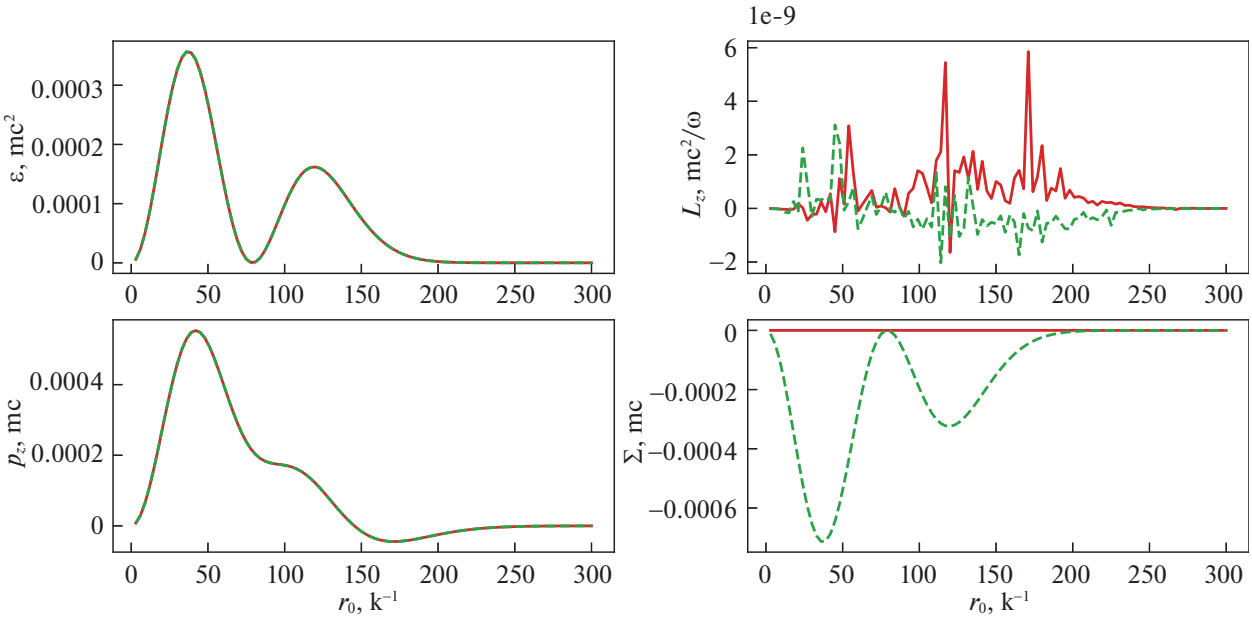

Fig. 4. Dependence of the transferred kinetic energy ε, longitudinal momentum p_z, angular momentum L_z, and Σ on the initial distance r_0 of the electron on the beam axis during the motion in the Laguerre–Gaussian quasi-monochromatic mode with $w_0 = 100$ and $n = 12$, with regard to the derivative of the envelope and field amplitude with respect to z coordinate, obtained by numerical simulation for left-hand (green broken curve) and right-hand (red solid curve) polarized modes.

4. CONCLUSIONS

The discussed approximations play an important role in describing the interaction of the structured light wave with charged particles. In order that the angular momentum transferred to a particle was larger than errors caused by disregarding the envelope variability and the paraxial approximation accuracy, two inequalities $L_z \gg a_0$ and $L_z \gg \frac{a_0}{w_0^2}\tau$, respectively, should be satisfied. These conditions lead to limiting the dimensionless pulse duration $\tau \lesssim w_0^2$, which does not allow an infinite increase in the pulse duration in the numerical calculation without improving the approximations. In the case of Gaussian beams, it was noted [10] that corrections arising due to the envelope should be considered for short pulses $\tau \lesssim 2w_0$. Thus, the possibility of disregarding the envelope shape is also controlled by an even more stringent condition $w_0 \ll \tau \lesssim w_0^2$.

Since an addition of terms necessary to conserve the integrals of motion, but exceeding the determination accuracy of fields in (7), caused mostly a change in Σ, rather than the energy, we can conclude that these additions define angular momentum absorption. Thus, to correctly describe the angular momentum transfer to a particle in a structured electromagnetic field, it is necessary to take into account corrections to both the paraxial approximation and to slowly varying envelope both in solving Maxwell's equations and in considering equations of motion.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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