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Jensen's Inequality Connected with a Double Random Good

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Abstract—In this paper, we define a multiple random good of order 2 denoted by X_{12} whose possible values are of a monetary nature. A two-risky asset portfolio is a multiple random good of order 2. It is firstly possible to establish its expected return by using a linear and quadratic metric. We secondly establish the expected return on X_{12} denoted by $\mathbf{P}(X_{12})$ by using a multilinear and quadratic metric. An extension of the notion of mathematical expectation of X_{12} is carried out by using the notion of α-norm of an antisymmetric tensor of order 2. An extension of the notion of variance of X_{12} denoted by $\text{Var}(X_{12})$ is shown by using the notion of α-norm of an antisymmetric tensor of order 2 based on changes of origin. An extension of the notion of expected utility connected with X_{12} is considered. An extension of Jensen's inequality is shown as well. We focus on how the decision-maker maximizes the expected utility connected with multiple random goods of order 2 being chosen by her under conditions of uncertainty and riskiness.

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1. INTRODUCTION

1.1. A Contingent Consumption Plan

In this paper, the investor is modeled as being a consumer. It is not money alone that matters, but it is the average consumption that money can buy that is the ultimate good being chosen by her. A state of the world of a contingent consumption plan is a single event, so it is a well-determined proposition identified with a real number such that, by betting on it, it is possible to establish whether it is true or false (see also [16]).

Let X be a random good. Let $I(X) = \{x^1, x^2, \ldots, x^m\}$ be the set of the possible values for X, where we have $x^1 < x^2 < \ldots < x^m$ without loss of generality because $I(X)$ identifies a finite partition of m mutually exclusive states of the world of a contingent consumption plan. We write $\inf I(X) = x^1$ and $\sup I(X) = x^m$. The elements of $I(X)$ are of a monetary nature. They give rise to an m-dimensional consumption vector denoted by

$$(x^1, x^2, \ldots, x^m).$$

It expresses all possible quantitative states of the world of a contingent consumption plan. It is possible to verify that it is contained in a closed structure which is a linear space over $\mathbb{R}$ (see also [36]). Indeed, its elements may be added together and multiplied by real numbers called scalars to obtain other elements belonging to the same linear structure. Such elements identify other contingent consumption plans. A linear space over $\mathbb{R}$ is a space furnished with a metric measure. We denote it by E^m. It has a Euclidean structure. A located vector at the origin of E^m is entirely determined by its end point. Accordingly, an

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ordered m-tuple of real numbers can indifferently be called either an m-dimensional point of $\mathcal{E}^m$ (affine space) or an m-dimensional vector of E^m, where $\mathcal{E}^m$ and E^m are isomorphic.

Uncertainty about a state of the world of a contingent consumption plan is of a personalistic nature in the sense that uncertainty about an event ceases only when the investor receives sure information about it (see also [9]).

The probability associated with a state of the world of a contingent consumption plan is the degree of belief in the occurrence of it attributed by a given investor at a given instant and with a given set of information and knowledge (see also [10]). We think of probability as being a mass. It is not a measure according to measure theory. Within this context, a mechanical transposition of all the notions, procedures, results of measure theory into the calculus of probability does not take place. The well-known implications of the mechanical meaning of mass make clear all those probabilistic properties meant as knowledge of the barycenter of a nonparametric distribution of mass or of moments of inertia. The concept of probability does not exist independently of the evaluations the investor makes of it mentally or instinctively (see also [23]). Such evaluations can be based on objective elements such as a judgment of equal probability expressing symmetry or a judgment based on statistical frequencies (see also [15]). Nevertheless, they do not exist outside of the investor's judgment whose nature is always subjective.[1)]

A function defined on the set of all possible quantitative states of the world of a contingent consumption plan coincides with X. Its domain expressed by $I(X)$ is a finite collection of possible events, where each of them is generically denoted by E_i, $i = 1, \ldots, m$. We write

$$X = x^1|E_1| + x^2|E_2| + \ldots + x^m|E_m|,$$

where we have

$$|E_i| = \begin{cases} 1, & \text{if } E_i \text{ is true} \\ 0, & \text{if } E_i \text{ is false} \end{cases}$$

for every $i = 1, \ldots, m$.

One and only one of all possible quantitative states of the world of a contingent consumption plan belonging to $I(X)$ will be true at the right time (see also [14]). We establish the following.

Definition 1. Let $id_{\mathbb{R}} \colon \mathbb{R} \to \mathbb{R}$ be the identity function on $\mathbb{R}$, where $\mathbb{R}$ is a linear space over itself. Given m incompatible and exhaustive states of the world of a contingent consumption plan, a random good denoted by X is the restriction of $id_{\mathbb{R}}$ to $I(X) = \{x^1, x^2, \ldots, x^m\} \subset \mathbb{R}$ such that it turns out to be $id_{\mathbb{R}|I(X)} \colon I(X) \to \mathbb{R}$.

We consider the finest possible partition into elementary events. They are not further subdivisible for the purposes of the problem under consideration. We do not consider other events. That alternative which will turn out to be verified "a posteriori" is nothing but a random point contained in $I(X)$. It expresses everything there is to be said whenever uncertainty ceases (see also [22]).

We say that a probability distribution associated with the possible values for a random good can vary from investor to investor. It can vary in accordance with the state of information and knowledge associated with each investor. Each investor is faced with m masses denoted by $p_1, p_2, \ldots, p_m$. They are located on m real numbers denoted by $x^1, x^2, \ldots, x^m$.

Each single state of the world of a contingent consumption plan could uniquely be expressed by infinite real numbers, so we could also write

$$\{x^1 + a,\, x^2 + a,\, \ldots,\, x^m + a\},$$

where $a \in \mathbb{R}$ is an arbitrary constant. We consider infinite changes of origin in this way. It is possible to consider different quantities from a geometric point of view. They are nevertheless the same quantity from a randomness point of view because states of the world and probabilities associated with them do not change.

[1)] Different nonparametric distributions of mass underlie different measures of a metric nature. Nevertheless, if we talk about mass then there always exists the physical feeling of being able to move it in a coherent way. This is because the notion of mass has no intrinsically a special status unlike the one of measure treated by measure theory. To move all the masses under consideration in a coherent way implies that each mass is found between 0 and 1, end points included, and their sum is finitely equal to 1.

1.2. Contravariant and Covariant Indices Associated with a Contingent Consumption Plan

Let E_i, $i = 1, \ldots, m$, be a generic state of the world of a contingent consumption plan. We establish the following.

Definition 2. Let X be a random good. The investor is in doubt between m monetary values for X, so x^1 is the return on X if E_1 occurs with probability denoted by p_1, ..., x^m is the return on X if E_m occurs with probability denoted by p_m. It is also possible to say that x^1 is the wealth that X yields and that can be spent by the investor if E_1 occurs with probability denoted by p_1, ..., x^m is the wealth that X yields and that can be spent by the investor if E_m occurs with probability denoted by p_m.

We write

$$(x^1, p_1), (x^2, p_2), \ldots, (x^m, p_m)$$

in order to identify masses associated with the possible values for X. Masses are expressed by using covariant indices. They are used together with contravariant ones. We wish to distinguish possibility from probability in this way. We use contravariant indices to identify the possible values for X. We use covariant indices to denote the corresponding probabilities that are assigned to them. The conditions of coherence are of an objective nature. They impose no limits on the probabilities that the investor may subjectively assign (see also [5]).

1.3. The Objectives of the Paper

In this paper, we define a multiple random good of order 2 denoted by X_{12} whose possible values are of a monetary nature. A two-risky asset portfolio is a multiple random good of order 2. It is firstly possible to establish its expected return by using a linear and quadratic metric. Given ${}_1X$ and ${}_2X$, where ${}_1X$ and ${}_2X$ are the components of X_{12}, whenever we use a linear metric in order to establish the expected return on a two-risky asset portfolio, we focus on the components of X_{12} only. We focus on ${}_1X$ and ${}_2X$ only. We secondly establish the expected return on X_{12} denoted by $\mathbf{P}(X_{12})$ by using a multilinear and quadratic metric. Whenever we use a multilinear metric in order to establish the expected return on a two-risky asset portfolio, we focus on X_{12}. Whenever we use a multilinear metric, we are not interested in studying separately the components of X_{12} denoted by ${}_1X$ and ${}_2X$, but we are interested in studying X_{12} as a whole. If the decision-maker is risk neutral then $\mathbf{P}(X_{12})$ is a subjective price coinciding with the certainty equivalent to X_{12}. An extension of the notion of mathematical expectation of X_{12} denoted by $\mathbf{P}(X_{12})$ is carried out by using the notion of α-norm of an antisymmetric tensor of order 2. We prove a theorem about this. An extension of the notion of variance of X_{12} denoted by $\mathrm{Var}(X_{12})$ is shown by using the notion of α-norm of an antisymmetric tensor of order 2 based on changes of origin. We prove a theorem about this. An extension of the notion of expected utility connected with X_{12} is considered. An extension of Jensen's inequality is shown as well. Whenever the decision-maker maximizes the expected utility of X_{12}, she maximizes the utility of average quantities of consumption. We focus on how the decision-maker maximizes the expected utility connected with multiple random goods of order 2 being chosen by her under conditions of uncertainty and riskiness. What she actually chooses inside of her budget set underlies all of this.

2. LOGICAL AND PROBABILISTIC ASPECTS CONCERNING AN ORDERED PAIR OF CONTINGENT CONSUMPTION PLANS

Let $\mathcal{B}_m^{\perp} = \{\mathbf{e}_i \,|\, i \in I_m = \{1, \ldots, m\}\}$ be an orthonormal basis of E^m. Two marginal random goods denoted by ${}_1X$ and ${}_2X$ give rise to a joint random good denoted by ${}_1X\,{}_2X$ whenever all its possible monetary values are obtained by considering the Cartesian product of the possible values for ${}_1X$ and ${}_2X$ belonging to $I({}_1X)$ and $I({}_2X)$ respectively. Two random goods are logically independent if and only if there are m^2 possible values for ${}_1X\,{}_2X$. Let $({}_1X, {}_2X)$ be an ordered pair of random goods (see also [25]). We are faced with two different partitions, where each of them is characterized by m incompatible and exhaustive events. After considering $I({}_1X) = \{{}_{(1)}x^1, \ldots, {}_{(1)}x^m\}$ and $I({}_2X) = \{{}_{(2)}x^1, \ldots, {}_{(2)}x^m\}$ we establish the following

Definition 3. All states of the world of an ordered pair of contingent consumption plans are obtained by considering the Cartesian product of the possible values for two logically independent random goods

Table 1

Random good 1 \ Random good 2	0	10	11	Sum
0	0	0	0	0
6	0	0.3	0.1	0.4
7	0	0.1	0.5	0.6
Sum	0	0.4	0.6	1

denoted by ${}_1X$ and ${}_2X$. Such marginal random goods give rise to a joint random good denoted by ${}_1X\,{}_2X$. It is a function written in the form ${}_1X\,{}_2X\colon I({}_1X) \times I({}_2X) \to \mathbb{R}$, where it turns out to be ${}_1X\,{}_2X({}_{(1)}x^i, {}_{(2)}x^j) = {}_{(1)}x^i\,{}_{(2)}x^j$, with $i, j = 1, \ldots, m$.

We are evidently faced with

$$ {}_1X\,{}_2X = {}_{(1)}x^1\,{}_{(2)}x^1 |{}_{(1)}E_1||{}_{(2)}E_1| + \ldots + {}_{(1)}x^m\,{}_{(2)}x^m |{}_{(1)}E_m||{}_{(2)}E_m|, \tag{1} $$

where it is possible to write

$$ |{}_{(1)}E_i||{}_{(2)}E_j| = \begin{cases} 1, & \text{if } {}_{(1)}E_i \text{ and } {}_{(2)}E_j \text{ are both true} \\ 0, & \text{otherwise} \end{cases} \tag{2} $$

for every $i, j = 1, \ldots, m$.

We geometrically consider ${}_{(1)}\mathbf{x} \in E^m$ as well as ${}_{(2)}\mathbf{x} \in E^m$. We write

$$ {}_{(1)}\mathbf{x} = {}_{(1)}x^i \mathbf{e}_i $$

and

$$ {}_{(2)}\mathbf{x} = {}_{(2)}x^i \mathbf{e}_i, $$

where we use the Einstein summation convention. We note that ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$ are uniquely represented with respect to $\mathcal{B}_m^{\perp}$. There exists one and only one m-tuple of real numbers coinciding with the set $\{{}_{(1)}x^i\}$ and satisfying the first linear combination that appears. There also exists one and only one m-tuple of real numbers coinciding with the set $\{{}_{(2)}x^i\}$ and satisfying the second linear combination that appears. We associate the contravariant components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$ with the possible values for ${}_1X\,{}_2X$ expressed in the same unit of measurement (see also [28]).

The covariant components of an affine tensor of order 2 represent the joint probabilities of the joint distribution of ${}_1X$ and ${}_2X$. We associate in an orderly manner the covariant components of an affine tensor of order 2 with the joint probabilities of the joint distribution of ${}_1X$ and ${}_2X$. Their number is overall equal to m^2. We write

$$ p = p_{ij} \tag{3} $$

with $p \in E^m \otimes E^m$. Since it turns out to be

$$ \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} = 1, \tag{4} $$

all probabilistic evaluations being made by the investor are coherent. Conditions of coherence pertain to the meaning of probability. They do not pertain to motives of a mathematical nature (see also [27]).

We note the following.

Remark 1. Given an orthonormal basis of E^m, the contravariant and covariant components of a same vector of E^m coincide. They represent the same numbers. Accordingly, we could indifferently use lower indices instead of upper ones and vice versa. □

Table 2

Random good 1 \ Random good 2	0	6	7	Sum
0	0	0	0	0
6	0	0.4	0	0.4
7	0	0	0.6	0.6
Sum	0	0.4	0.6	1

2.1. Metric Aspects Concerning an Ordered Pair of Contingent Consumption Plans

We say that an ordered pair of random goods denoted by $({}_1X,\ {}_2X)$ is represented by an ordered triple of geometric entities denoted by

$$\left({}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}, p_{ij}\right) \tag{5}$$

with $(i,\ j) \in I_m \times I_m$.

We consider the notion of α-product between ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$ in order to establish a quadratic metric on E^m. It is a scalar or inner product obtained by using the joint probabilities of the joint distribution of ${}_1X$ and ${}_2X$ together with the contravariant components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. We write

$$\langle {}_{(1)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha = {}_{(1)}x^i\,{}_{(2)}x^j p_{ij} = {}_{(1)}x^i\,{}_{(2)}x_i, \tag{6}$$

where

$${}_{(2)}x^j p_{ij} = {}_{(2)}x_i \tag{7}$$

is a vector homography by means of which we pass from ${}_{(2)}x^j$ to ${}_{(2)}x_i$ by using p_{ij}. For instance, from Table 1 it follows that it turns out to be $\mathbf{P}({}_1X\,{}_2X) = 70.1$. Given the contravariant components of ${}_{(2)}\mathbf{x}$ identifying the following column vector

$$\begin{pmatrix} 0 \\ 10 \\ 11 \end{pmatrix},$$

its covariant components are expressed by

$$0 \cdot 0 + 10 \cdot 0 + 11 \cdot 0 = 0,$$

$$0 \cdot 0 + 10 \cdot 0.3 + 11 \cdot 0.1 = 4.1,$$

and

$$0 \cdot 0 + 10 \cdot 0.1 + 11 \cdot 0.5 = 6.5,$$

so it is possible to write the following result

$$\left\langle \begin{pmatrix} 0 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 4.1 \\ 6.5 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha = \mathbf{P}({}_1X\,{}_2X) = 70.1.$$

On the other hand, after calculating the covariant components of ${}_{(1)}\mathbf{x}$ in a similar way, we write

$$\left\langle \begin{pmatrix} 0 \\ 2.5 \\ 4.1 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 11 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha = \mathbf{P}({}_1X\,{}_2X) = 70.1.$$

From the notion of α-product it follows the one of α-norm of an m-dimensional vector. We write

$$||_{(1)}\mathbf{x}||_\alpha^2 = \langle {}_{(1)}\mathbf{x},\ {}_{(1)}\mathbf{x}\rangle_\alpha = {}_{(1)}x^i\,{}_{(1)}x^i p_{ii} = {}_{(1)}x^i\,{}_{(1)}x_i \tag{8}$$

as well as

$$||_{(2)}\mathbf{x}||_\alpha^2 = \langle {}_{(2)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha = {}_{(2)}x^i\,{}_{(2)}x^i p_{ii} = {}_{(2)}x^i\,{}_{(2)}x_i \tag{9}$$

because the joint probabilities of the particular joint distributions under consideration whose covariant indices are not equal coincide with 0. For instance, from Table 2 it follows that it turns out to be

$$\left\langle \begin{pmatrix} 0 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.4 \\ 4.2 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x},\ {}_{(1)}\mathbf{x}\rangle_\alpha = ||_{(1)}\mathbf{x}||_\alpha^2 = 43.8.$$

Also, it is possible to show two metric inequalities. The Schwarz's α-generalized inequality is given by

$$\left|\langle {}_{(1)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha\right| \leq ||_{(1)}\mathbf{x}||_\alpha ||_{(2)}\mathbf{x}||_\alpha, \tag{10}$$

whereas the α-triangle inequality is expressed by

$$||_{(1)}\mathbf{x} + {}_{(2)}\mathbf{x}||_\alpha \leq ||_{(1)}\mathbf{x}||_\alpha + ||_{(2)}\mathbf{x}||_\alpha. \tag{11}$$

From (10) it follows the notion of α-cosine, so it is possible to write

$$\cos({}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x})_\alpha = \frac{\langle {}_{(1)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha}{||_{(1)}\mathbf{x}||_\alpha ||_{(2)}\mathbf{x}||_\alpha}. \tag{12}$$

2.2. The Relative and Subjective Nature of the Joint Probabilities Associated with an Ordered Pair of Contingent Consumption Plans

The covariant components of an affine tensor of order 2 belonging to $E^m \otimes E^m$ are joint probabilities whose nature is relative (see also [6]). They depend on the variable group of circumstances supposed to be of interest to the occurrence of a specific state of the world characterizing ${}_1X\,{}_2X$. Such circumstances are known at the time. They generally vary from instant to instant. It follows that probabilities vary according to the state of information and knowledge associated with a given investor which can be enriched by the flow of information and results that are learned or observed with respect to more or less similar cases. We note that each new piece of information is able to modify the evaluations of probability being made by the investor according to Bayes' rule (see also [8]). The nature of the joint probabilities is also subjective in the sense that a probability concerning a state of the world of a contingent consumption plan and depending on the variable state of information and knowledge associated with a given investor is intrinsically personalized. It follows that different investors having the same state of information and knowledge could give a greater attention to certain circumstances than to others. The state of information and knowledge associated with a given investor can also modify the set of all possible quantitative states of the world of a contingent consumption plan, where each of them is a real number (see also [13]). Accordingly, the absolute value of each real number can change.

Since it turns out to be

$$\dim(E^m \otimes E^m) = m^2,$$

there exists an isomorphism between $E^m \otimes E^m$ and E^{m^2}. We can think of locating m^2 masses on m^2 points, where each point of them is a real number denoted by ${}_{(1)}x^i\,{}_{(2)}x^j$, $i, j = 1, \ldots, m$. We write

$$(\mathbf{x},\ \mathbf{p}) \subset E^{m^2},$$

where $\mathbf{x}$ and $\mathbf{p}$ are two m^2-dimensional vectors.

If we write

$$\langle {}_{(1)}\mathbf{x},\ {}_{(2)}\mathbf{x}\rangle_\alpha = {}_{(1)}x^i\,{}_{(2)}x^j p_{ij} = \mathbf{P}({}_1X\,{}_2X),$$

then we observe a "reduction of dimension" because we pass from m^2 points to 1 point, where the latter is always studied together with its Cartesian coordinates. If we consider all coherent previsions of ${}_1X\,{}_2X$ then $\mathbf{P}({}_1X\,{}_2X) = (\mathbf{P}({}_1X),\,\mathbf{P}({}_2X))$ is a point of a two-dimensional convex set coinciding with the budget set of the investor (see also [7]). Such a convex set is a continuous subset of $\mathbb{R}\times\mathbb{R}$. All coherent previsions of ${}_1X\,{}_2X$ are obtained by taking all the values between 0 and 1, end points included, into account for each mass of m^2 masses. The number of such values is infinite. $\mathbf{P}({}_1X\,{}_2X)$ is always decomposed into two linear measures, $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$, respectively. Each of them shows a "reduction of dimension" because we pass from m one-dimensional points which are found on a one-dimensional straight line to 1 one-dimensional point which is found on the same line. All coherent previsions of ${}_1X$ and ${}_2X$ are obtained by taking all the values between 0 and 1, end points included, into account for each mass of m masses. The number of such values is infinite.

3. TWO CONTINGENT CONSUMPTION PLANS JOINTLY CONSIDERED THAT ARE INDEPENDENT OF THE NOTION OF ORDERED PAIR

We note the following.

Remark 2. Let ${}_1X$ and ${}_2X$ be two marginal random goods, where each of them is characterized by m possible values. The two m-dimensional vectors, whose contravariant components represent the possible values for two random goods which are separately considered, are assumed to be linearly independent. The possible values for two logically independent random goods which are jointly considered have to be represented by the contravariant components of a tensor of order 2. It is an antisymmetric tensor of order 2 whenever we are interested in handling a multiple random good of order 2 denoted by X_{12}, where its components are expressed by ${}_1X$ and ${}_2X$. □

We pass from an ordered pair of contingent consumption plans to two contingent consumption plans which are jointly considered regardless of the notion of ordered pair. We have to consider a multiple random good of order 2 (double random good) denoted by

$$X_{12} = \{{}_1X,\,{}_2X\}, \tag{13}$$

whose possible values coincide with the contravariant components of an antisymmetric tensor of order 2. Given the marginal probabilities of ${}_1X$ and ${}_2X$, after choosing m^2 joint probabilities connected with ${}_1X\,{}_2X$, it is necessary to consider four joint distributions characterizing ${}_1X\,{}_1X$, ${}_1X\,{}_2X$, ${}_2X\,{}_1X$, and ${}_2X\,{}_2X$, with

$${}_1X\,{}_1X\colon I({}_1X)\times I({}_1X)\to\mathbb{R}, \tag{14}$$

$${}_2X\,{}_2X\colon I({}_2X)\times I({}_2X)\to\mathbb{R}, \tag{15}$$

and

$${}_2X\,{}_1X\colon I({}_2X)\times I({}_1X)\to\mathbb{R}, \tag{16}$$

in order to release X_{12} from the notion of ordered pair of contingent consumption plans. We note that ${}_1X$ and ${}_2X$ are not put near unlike what happens when we jointly consider ${}_iX$ and ${}_jX$, where we have $i,j=1,2$. We can think of putting the m^2 joint probabilities into a two-way table having m rows and m columns. Each probability distribution of a marginal random good is viewed to be as a particular joint distribution. This is because all off-diagonal joint probabilities of the two-way table under consideration coincide with 0. It is possible to show that the mathematical expectation of ${}_iX\,{}_jX$, with $i,j=1,2$, is always bilinear (see also [37]). This means that it is separately linear in each marginal random good. We prove the following.

Theorem 1. *The mathematical expectation of $X_{12}=\{{}_1X,\,{}_2X\}$ denoted by $\mathbf{P}(X_{12})$ coincides with the determinant of a square matrix of order 2. Each element of such a determinant is a real number coinciding with the mathematical expectation of ${}_iX\,{}_jX$, where we have $i,j=1,2$.*

Proof. An affine tensor of order 2 representing the possible values for ${}_1X\,{}_2X$, where ${}_1X\,{}_2X$ corresponds to $({}_1X,\,{}_2X)$, is written in the form

$$T = {}_{(1)}\mathbf{x}\otimes{}_{(2)}\mathbf{x} = {}_{(1)}x^i\,{}_{(2)}x^j\,\mathbf{e}_i\otimes\mathbf{e}_j. \tag{17}$$

An affine tensor of order 2 representing the possible values for ${}_2X\,{}_1X$, where ${}_2X\,{}_1X$ corresponds to $({}_2X,\,{}_1X)$, is conversely written in the form

$$T = {}_{(2)}\mathbf{x} \otimes {}_{(1)}\mathbf{x} = {}_{(2)}x^j\,{}_{(1)}x^i \mathbf{e}_j \otimes \mathbf{e}_i. \tag{18}$$

We wrote a same affine tensor of order 2 denoted by T whose m^2 contravariant components are not the same. If we pass from (17) to (18) then we note that the contravariant components whose upper indices are equal do not change. If we pass from (17) to (18) then we note that the contravariant components whose upper indices are not equal change. It follows that we write an antisymmetric tensor of order 2 in the form

$$T = \sum_{i<j} \left({}_{(1)}x^i\,{}_{(2)}x^j - {}_{(1)}x^j\,{}_{(2)}x^i\right) \mathbf{e}_i \otimes \mathbf{e}_j \tag{19}$$

because we have to consider (17) and (18) together. We wrote $i < j$ under the summation symbol because if it turns out to be $i = j$ then every contravariant component inside parentheses is equal to 0. Hence, we denote by ${}_{12}x$ an antisymmetric tensor of order 2 identifying X_{12}. We write

$${}_{12}x^{(ij)} = \begin{vmatrix} {}_{(1)}x^i & {}_{(1)}x^j \\ {}_{(2)}x^i & {}_{(2)}x^j \end{vmatrix} = {}_{(1)}x^i\,{}_{(2)}x^j - {}_{(1)}x^j\,{}_{(2)}x^i \tag{20}$$

in order to identify the strict contravariant components of it. We have $i < j$. The number of such components is overall equal to

$$\binom{m}{2}.$$

The corresponding strict covariant components of ${}_{12}x$ are given by

$${}_{12}x_{(ij)} = \begin{vmatrix} {}_{(1)}x_i & {}_{(1)}x_j \\ {}_{(2)}x_i & {}_{(2)}x_j \end{vmatrix} = \begin{vmatrix} {}_{(1)}x^j p_{ji} & {}_{(1)}x^i p_{ij} \\ {}_{(2)}x^j p_{ji} & {}_{(2)}x^i p_{ij} \end{vmatrix}, \tag{21}$$

where we have $i < j$. We do not compute the scalar value of (21). The number of the strict contravariant and covariant components of ${}_{12}x$ is absolutely unimportant. We always obtain the same outcome independently of such a number. We put together (20) and (21), where (20) and (21) contain all strict contravariant and covariant components of ${}_{12}x$ at the same time. We always put together (20) and (21) in the same way. We always associate ${}_{(1)}x^i$ with ${}_{(1)}x_i$, ${}_{(1)}x^j$ with ${}_{(2)}x_j$, ${}_{(2)}x^i$ with ${}_{(1)}x_i$, and ${}_{(2)}x^j$ with ${}_{(2)}x_j$. After putting together (20) and (21), whose structure is evidently the one of two determinants because we are considering multilinear matters, we obtain different single terms (monomials). It follows that a variable index appearing twice in a monomial implies summation of it over all values of the index (hence, every time it is possible to obtain a polynomial by using the Einstein notation). On the other hand, all strict contravariant and covariant components of ${}_{12}x$ are simultaneously identified with two determinants because, in general, the determinant of a square matrix is the most exemplary multilinear relationship as well as a linear combination of basis vectors is the most exemplary linear relationship. We obtain the mathematical expectation of X_{12} given by

$$\begin{aligned} ||{}_{12}x||^2_\alpha &= \begin{vmatrix} ||{}_{(1)}\mathbf{x}||^2_\alpha & \langle {}_{(1)}\mathbf{x},\,{}_{(2)}\mathbf{x}\rangle_\alpha \\ \langle {}_{(2)}\mathbf{x},\,{}_{(1)}\mathbf{x}\rangle_\alpha & ||{}_{(2)}\mathbf{x}||^2_\alpha \end{vmatrix} \\ &= ||{}_{(1)}\mathbf{x}||^2_\alpha ||{}_{(2)}\mathbf{x}||^2_\alpha - \left(\langle {}_{(1)}\mathbf{x},\,{}_{(2)}\mathbf{x}\rangle_\alpha\right)^2, \end{aligned} \tag{22}$$

where we evidently observe

$$\langle {}_{(1)}\mathbf{x},\,{}_{(2)}\mathbf{x}\rangle_\alpha = \langle {}_{(2)}\mathbf{x},\,{}_{(1)}\mathbf{x}\rangle_\alpha. \tag{23}$$

By putting together (20) and (21) we are always faced with four joint distributions characterizing ${}_1X\,{}_1X$, ${}_1X\,{}_2X$, ${}_2X\,{}_1X$, and ${}_2X\,{}_2X$ that are all summarized. We write

$$||{}_{12}x||^2_\alpha = \mathbf{P}(X_{12}) > 0, \tag{24}$$

where it turns out to be

$$\mathbf{P}(X_{12}) = \begin{vmatrix} ||_{(1)}\mathbf{x}||^2_\alpha & \langle_{(1)}\mathbf{x},\ _{(2)}\mathbf{x}\rangle_\alpha \\ \langle_{(2)}\mathbf{x},\ _{(1)}\mathbf{x}\rangle_\alpha & ||_{(2)}\mathbf{x}||^2_\alpha \end{vmatrix}$$

$$= \begin{vmatrix} _{(1)}x^i\ _{(1)}x^i p^{(11)}_{ii} = {_{(1)}x^i}\ _{(1)}x_i & _{(1)}x^j\ _{(2)}x^i p^{(12)}_{ij} = {_{(1)}x^j}\ _{(2)}x_j \\ _{(2)}x^i\ _{(1)}x^j p^{(21)}_{ji} = {_{(2)}x^i}\ _{(1)}x_i & _{(2)}x^j\ _{(2)}x^j p^{(22)}_{jj} = {_{(2)}x^j}\ _{(2)}x_j \end{vmatrix}. \tag{25}$$

We note that $p^{(11)}$ is the tensor of all joint probabilities associated with $(_{(1)}\mathbf{x},\ _{(1)}\mathbf{x})$. The same is true for all others contained in (25). It is possible to observe that in general it turns out to be

$$\mathbf{P}(_1X\ _2X) \neq \mathbf{P}(X_{12}). \tag{26}$$

We finally write

$$\mathbf{P}(X_{12}) = \begin{vmatrix} \mathbf{P}(_1X\ _1X) & \mathbf{P}(_1X\ _2X) \\ \mathbf{P}(_2X\ _1X) & \mathbf{P}(_2X\ _2X) \end{vmatrix}, \tag{27}$$

where the determinant of the square matrix of order 2 under consideration is a bilinear function of the columns of it. □

Given $_1X$ and $_2X$ and their coherent previsions denoted by

$$\mathbf{P}(_1X) = {_{(1)}x^1}\ _{(1)}p_1 + \ldots + {_{(1)}x^m}\ _{(1)}p_m$$

and

$$\mathbf{P}(_2X) = {_{(2)}x^1}\ _{(2)}p_1 + \ldots + {_{(2)}x^m}\ _{(2)}p_m,$$

where it turns out to be

$$_{(1)}p_1 + \ldots + {_{(1)}p_m} = 1$$

as well as

$$_{(2)}p_1 + \ldots + {_{(2)}p_m} = 1,$$

with $0 \leq {_{(1)}p_i} \leq 1$, $0 \leq {_{(2)}p_j} \leq 1$, $i, j = 1, \ldots, m$, it is possible to consider all deviations from $\mathbf{P}(_1X)$ and $\mathbf{P}(_2X)$ of the possible values for $_1X$ and $_2X$. We are evidently faced with the marginal distributions of the joint distribution of $_1X$ and $_2X$ (see also [29]). We prove the following

Theorem 2. *The variance of $X_{12} = \{_1X,\ _2X\}$ denoted by Var(X_{12}) coincides with the determinant of a square matrix of order 2. Each element of such a determinant is a real number coinciding with the variance of $_1X$ and $_2X$ and with their covariance.*

Proof. All deviations from $\mathbf{P}(_1X)$ and $\mathbf{P}(_2X)$ of the possible values for $_1X$ and $_2X$ are translations. They are changes of origin. It is possible to write

$$||_{12}d||^2_\alpha = \begin{vmatrix} ||_{(1)}\mathbf{d}||^2_\alpha & \langle_{(1)}\mathbf{d},\ _{(2)}\mathbf{d}\rangle_\alpha \\ \langle_{(2)}\mathbf{d},\ _{(1)}\mathbf{d}\rangle_\alpha & ||_{(2)}\mathbf{d}||^2_\alpha \end{vmatrix}$$

$$= ||_{(1)}\mathbf{d}||^2_\alpha ||_{(2)}\mathbf{d}||^2_\alpha - \left(\langle_{(1)}\mathbf{d},\ _{(2)}\mathbf{d}\rangle_\alpha\right)^2, \tag{28}$$

where $_{12}d$ is an antisymmetric tensor of order 2 representing X_{12} from a logical point of view. We are faced with changes of origin of the possible values for $_1X$ and $_2X$. We write

$$||_{12}d||^2_\alpha = \text{Var}(X_{12}) = \sigma^2_{X_{12}}. \tag{29}$$

We note that it turns out to be

$$\langle_{(1)}\mathbf{d},\ _{(2)}\mathbf{d}\rangle_\alpha = \langle_{(2)}\mathbf{d},\ _{(1)}\mathbf{d}\rangle_\alpha = \text{Cov}(_1X,\ _2X) = \text{Cov}(_2X,\ _1X), \tag{30}$$

so it is possible to write

$$\mathrm{Var}(X_{12}) = \begin{vmatrix} \mathrm{Var}({}_1X) & \mathrm{Cov}({}_1X,\, {}_2X) \\ \mathrm{Cov}({}_2X,\, {}_1X) & \mathrm{Var}({}_2X) \end{vmatrix}. \tag{31}$$

If we are faced with the variance of X_{12} then ${}_1X$ and ${}_2X$ are fused together. In general, if we compute only the covariance of ${}_1X$ and ${}_2X$ (in addition to the variance of each of them) then they are simply put near. □

We note the following.

Remark 3. Given X_{12}, $\mathbf{P}(X_{12})$ is coherent in the same way as $\mathbf{P}({}_1X)$, $\mathbf{P}({}_2X)$ as well as $\mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X)$, where $\mathbf{P}(X_{12})$ and $\mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X)$ are both of them bilinear indices. $\mathbf{P}(X_{12})$ is an aggregate index, whereas $\mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X)$ is a disaggregate index. □

Remark 4. The origin of the variability of X_{12} is not standardized, but it depends on the variable state of information and knowledge associated with a given investor. All deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$ depend on her variable state of information and knowledge. □

4. THE BUDGET SET OF THE INVESTOR

Given the two-good assumption, the objects of investor choice are of a bilinear nature (see also [3]). We consider two mutually orthogonal axes of a two-dimensional Cartesian coordinate system on which an origin, a same unit of length and an orientation are established (see also [12]). All the m^2 possible states of the world of two contingent consumption plans which are jointly considered belong to a finite subset of a two-dimensional Cartesian coordinate system, where each axis of it contains m possible states of the world of a contingent and marginal consumption plan. It is possible to consider two half-lines, where each of them extends indefinitely in a positive direction from zero before being restricted (see also [19]). Only a joint distribution is considered inside of the budget set of the investor. Whenever we summarize it, we obtain a bilinear measure. It coincides with a possible object of investor choice. It is a synthesized element of the Frèchet class. We consider all coherent previsions of a joint random good (see also [26]). All coherent previsions of a joint random good denoted by ${}_1X\,{}_2X$ are expressed by $\mathbf{P}({}_1X\,{}_2X)$. They are disaggregate and bilinear measures, so $\mathbf{P}({}_1X\,{}_2X)$ is always decomposed into two coherent previsions of two marginal random goods denoted by $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$, respectively. All coherent previsions of a joint random good identify a two-dimensional convex set denoted by $\mathcal{P} \subset \mathbb{R} \times \mathbb{R}$. It is a right triangle whose catheti belong to the first quadrant of a two-dimensional Cartesian coordinate system. They meet at the point denoted by $(0, 0)$. Its hypotenuse is the budget line identifying the budget set of the investor. It is a hyperplane embedded in a two-dimensional Cartesian coordinate system. It does not separate any point $\mathbf{P}$ of $\mathcal{P}$ from the set $\mathcal{Q} = I({}_1X) \times I({}_2X)$ of all possible points for ${}_1X$ and ${}_2X$ belonging to $I({}_1X)$ and $I({}_2X)$, respectively. It makes sense to consider possible values for a joint random entity as well as for two marginal random entities. Given the marginal previsions of ${}_1X$ and ${}_2X$ denoted by $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$, the budget constraint of the investor is a linear inequality expressed by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) \leq c.$$

It is characterized by three strictly positive real numbers, c_1, c_2, and c (see also [11]). They are the two objective prices, c_1 and c_2, of the two goods under consideration besides the amount of money she has to spend (see also [34]). The two prices of the two goods under consideration identify the negative slope of the budget line (see also [35]). We deal with two continuous goods because what the investor actually chooses inside of her budget set is an average quantity of consumption associated with each of them. The number of possible and coherent average quantities of consumption associated with each marginal good is infinite. Two one-dimensional convex sets are identified because the two-dimensional convex set coinciding with the budget set of the investor contains infinite coherent previsions of a bilinear nature, where each of them is always decomposed into two previsions of a linear nature. Two one-dimensional convex sets coincide with two line segments belonging to the two axes of a two-dimensional Cartesian coordinate system. It is clear that each average quantity of consumption associated with random good 1 and random good 2 does not depend on objective elements only, but it depends on subjective elements as well (see also [2]).

We multiply c_1, c_2, and c by a positive number. The investor divides her relative monetary wealth given by

$$\frac{c_1}{c_1 + c_2} \tag{32}$$

and

$$\frac{c_2}{c_1 + c_2} \tag{33}$$

between the two random goods denoted by ${}_1X$ and ${}_2X$. It follows that it turns out to be

$$\frac{c_1}{c_1 + c_2} + \frac{c_2}{c_1 + c_2} = 1. \tag{34}$$

The budget set of the investor does not change (see also [30]). She always chooses one and only one of the points of $\mathcal{P}$ from her budget set. All points of $\mathcal{P}$ are admissible in terms of coherence of $\mathbf{P}$. We write

$$\frac{c_1}{c_1 + c_2}\,\mathbf{P}({}_1X) + \frac{c_2}{c_1 + c_2}\,\mathbf{P}({}_2X) \leq \frac{c}{c_1 + c_2} \tag{35}$$

whenever we use a bilinear measure that is decomposed into two linear measures. The left-hand side of (35) is a weighted average of the two expected returns on ${}_1X$ and ${}_2X$.

4.1. To Go Away from the Budget Set of the Investor: Changes of Origin

Let ${}_1X$ and ${}_2X$ be two random goods coinciding with two risky assets. Given

$$\mathbf{y} = \mu_1\,{}_{(1)}\mathbf{d} + \mu_2\,{}_{(2)}\mathbf{d}, \tag{36}$$

with $\mu_1 = \frac{c_1}{c_1+c_2} \in \mathbb{R}$ and $\mu_2 = \frac{c_2}{c_1+c_2} \in \mathbb{R}$, it is possible to obtain, outside of the budget set of the investor, the following expression given by

$$\begin{aligned} ||\mathbf{y}||_\alpha^2 &= ||\mu_1\,{}_{(1)}\mathbf{d} + \mu_2\,{}_{(2)}\mathbf{d}||_\alpha^2 \\ &= (\mu_1)^2\; ||{}_{(1)}\mathbf{d}||_\alpha^2 + 2\mu_1\,\mu_2\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha + (\mu_2)^2\; ||{}_{(2)}\mathbf{d}||_\alpha^2. \end{aligned} \tag{37}$$

We focus on the riskiness of the components of X_{12} only, where ${}_1X$ and ${}_2X$ are the components of X_{12}. This is because we use a linear metric (see also [4]).

We establish the following.

Definition 4. We call linear metric the expression given by (37). Since it is also possible to write $||{}_{(1)}\mathbf{d} - {}_{(2)}\mathbf{d}||_\alpha^2 = ||{}_{(1)}\mathbf{d}||_\alpha^2 + ||{}_{(2)}\mathbf{d}||_\alpha^2 - 2\langle{}_{(1)}\mathbf{d},\,{}_{(2)}\mathbf{d}\rangle_\alpha$, it derives from the notion of α-distance between the two components of X_{12} denoted by ${}_1X$ and ${}_2X$ whose possible values are subjected to two changes of origin.

We note the following.

Remark 5. Whenever we consider a linear and quadratic metric, we are faced with a joint distribution only. It depends on the notion of ordered pair of random goods. □

We establish the following.

Definition 5. We call non-linear (multilinear) metric the expression given by (28). It is the area of a 2-parallelepiped whose edges are two marginal random goods having their possible values that are subjected to two changes of origin. The strict components of ${}_{12}d$ are the coordinates of such edges denoted by ${}_{(1)}\mathbf{d}$ and ${}_{(2)}\mathbf{d}$.

By using (28), where (28) is an aggregate measure of a bilinear nature, it is possible to obtain the Bravais-Pearson correlation coefficient. We firstly write

$$||{}_{12}\hat{d}||_\alpha^2 = \begin{vmatrix} ||{}_{(1)}\mathbf{d}||_\alpha^2 & 0 \\ 0 & ||{}_{(2)}\mathbf{d}||_\alpha^2 \end{vmatrix}. \tag{38}$$

After some mathematical steps, we obtain

$$-1 \leq \left(1 - \frac{||_{12}d||_{\alpha}^{2}}{||_{12}\hat{d}||_{\alpha}^{2}}\right)^{1/2} \leq 1, \tag{39}$$

where it is possible to realize that the expression within the parentheses coincides with the Bravais-Pearson correlation coefficient intrinsically referred to X_{12}. We write it in the following form expressed by

$$r_{12} = \frac{\langle {}_{(1)}\mathbf{d},\ {}_{(2)}\mathbf{d}\rangle_{\alpha}}{||_{(1)}\mathbf{d}||_{\alpha}\, ||_{(2)}\mathbf{d}||_{\alpha}}. \tag{40}$$

5. UNCERTAINTY AND RISKINESS: PROBABILITY AND UTILITY CONNECTED WITH MULTIPLE RANDOM GOODS OF ORDER 2

Since $E^m \otimes E^m$ is isomorphic to E^{m^2}, it is possible to transfer all the m^2 possible states of the world of two contingent consumption plans jointly considered identifying a joint random good on a one-dimensional straight line on which an origin, a unit of length and an orientation are established. We deal with four joint distributions, so we go away from the budget set of the investor. We transfer all them on a one-dimensional straight line on which an origin, a unit of length and an orientation are chosen.

Any distribution of mass is completely characterized by its mathematical expectation and variance, where the latter is a measure of the riskiness of the wealth distribution under consideration (see also [18]). Both mathematical expectation and variance of X_{12} have been obtained by means of the notion of α-norm of an antisymmetric tensor of order 2. Accordingly, in general, they are both of them greater than zero (see also [33]). If the investor estimates all joint probabilities of ${}_1X\,{}_2X$ in such a way that there exists an inverse linear relationship between random good 1 and random good 2 then a higher mathematical expectation of X_{12} is good in her opinion, other things being equal, and a higher variance or standard deviation is bad. She is averse to risk. Her continuous utility function denoted by $u(x)$ is a strictly increasing and concave function, where its slope gets flatter as wealth increases (see also [17]). The form and extent of the aversion to risk which is caught by the utility function under consideration will depend on her temperament, her current mood and some other circumstance. This function is graphically represented outside of the budget set of the investor (see also [1]). We use two mutually orthogonal axes of a two-dimensional Cartesian coordinate system on which an origin, a same unit of length and an orientation are established (see also [38]). It follows that we have ${}_{(1)}x^i\,{}_{(2)}x^j$, $i, j = 1, \ldots, m$, together with their masses on the horizontal axis. We have consequently $u({}_{(1)}x^i\,{}_{(2)}x^j)$ together with their masses on the vertical one. We are evidently faced with m^2 masses located on $u(x)$ as well as on two mutually orthogonal axes. We have also to consider ${}_{(1)}x^i\,{}_{(1)}x^i$ and $u({}_{(1)}x^i\,{}_{(1)}x^i)$ together with m^2 masses as well as ${}_{(2)}x^i\,{}_{(2)}x^i$ and $u({}_{(2)}x^i\,{}_{(2)}x^i)$ together with m^2 masses. It is clear that ${}_{(2)}x^j\,{}_{(1)}x^i$, $j, i = 1, \ldots, m$, together with their masses on the horizontal axis as well as $u({}_{(2)}x^j\,{}_{(1)}x^i)$ together with their masses on the vertical one give rise to the same values as ${}_{(1)}x^i\,{}_{(2)}x^j$ and $u({}_{(1)}x^i\,{}_{(2)}x^j)$. Each possible value that is considered on the horizontal axis of a two-dimensional Cartesian coordinate system is expressed by using the arithmetic product of two values associated with two contingent consumption plans which are separately considered. Accordingly, all coherent arithmetic means are considered. They transfer on a one-dimensional straight line all coherent α-products.

We note the following.

Remark 6. Let $u(x)$ be the cardinal utility function identifying a risk-averse investor. It is considered outside of the budget set of the investor. This function lives inside of a two-dimensional Cartesian coordinate system. All masses characterizing each joint distribution which is considered in order to release X_{12} from the notion of ordered pair of contingent consumption plans are located on some points of its diagram. They identify different one-dimensional convex sets as joint probabilities of every joint distribution vary in the interval from 0 to 1 by taking all the values between 0 and 1, end points included, into account. The number of these values is infinite. There are different one-dimensional convex sets on the horizontal axis as well as different one-dimensional convex sets on the vertical one. All marginal probabilities of every marginal distribution under consideration vary in the interval from 0 to 1 by taking all the values between 0 and 1, end points included, into account. The number of these values is infinite. We note that

$$(\mathbf{P}(X_{12}),\, \mathbf{P}[u(X_{12})])$$

is a point of a two-dimensional Cartesian coordinate system belonging to the union of different one-dimensional convex sets. Such one-dimensional convex sets are found on the horizontal axis to which $\mathbf{P}(X_{12})$ belongs as well as on the vertical one to which $\mathbf{P}[u(X_{12})]$ belongs. If $u(x)$ identifies a risk-loving investor or a risk-neutral decision-maker then all of this continues to be valid. If X is a random good having m possible values then $(\mathbf{P}(X),\, \mathbf{P}[u(X)])$ is a two-dimensional point expressing two barycenters of two nonparametric distributions of mass. □

We observe that X_{12} has been constructed in such a way that the marginal distributions of ${}_1X$ and ${}_2X$ never change with respect to the starting ones. The marginal distributions of the joint distribution connected with ${}_1X\,{}_1X$ coincide with the probability distribution of ${}_1X$. The marginal distributions of the joint distribution connected with ${}_2X\,{}_2X$ coincide with the probability distribution of ${}_2X$. The marginal distributions of the joint distribution connected with ${}_1X\,{}_2X$ coincide with the probability distribution of ${}_1X$ and ${}_2X$ respectively. The marginal distributions of the joint distribution connected with ${}_2X\,{}_1X$ coincide with the probability distribution of ${}_2X$ and ${}_1X$, respectively.

The investor estimates all joint probabilities of ${}_1X\,{}_2X$ inside of her budget set in such a way that there exists an inverse linear relationship between ${}_1X$ and ${}_2X$. She is risk averse. For a risk-averse investor, the utility of the mathematical expectation of X_{12} is greater than the expected utility of X_{12} given by

$$\mathbf{P}[u(X_{12})] = \begin{vmatrix} u({}_{(1)}x^i\,{}_{(1)}x^i)p_{ii} & u({}_{(1)}x^i\,{}_{(2)}x^j)p_{ij} \\ u({}_{(2)}x^j\,{}_{(1)}x^i)p_{ji} & u({}_{(2)}x^i\,{}_{(2)}x^i)p_{ii} \end{vmatrix}$$

$$= \Big[u({}_{(1)}x^i\,{}_{(1)}x^i)p_{ii}\, u({}_{(2)}x^i\,{}_{(2)}x^i)p_{ii} - u({}_{(1)}x^i\,{}_{(2)}x^j)p_{ij}\, u({}_{(2)}x^j\,{}_{(1)}x^i)p_{ji}\Big] > 0, \tag{41}$$

where (4) holds with regard to each factor characterizing the minuend and the subtrahend of (41). We consider an extension of Jensen's inequality connected with a discrete probability distribution. It is a nonparametric probability distribution. We denote by

$$x_{12} = \mathbf{P}_u(X_{12}), \tag{42}$$

the certainty equivalent to X_{12} given by

$$\mathbf{P}_u(X_{12}) = u^{-1}\{\mathbf{P}[u(X_{12})]\}. \tag{43}$$

We note that (43) represents an associative mean. It is an increasing transform of the arithmetic mean considered by means of u and obtained by using a bilinear function of the columns of a square matrix of order 2 (see Theorem 1). Since it turns out to be $x_{12} < \mathbf{P}(X_{12})$ on the horizontal axis, it is possible to say that X_{12} is not preferred to x_{12} in opinion of a risk-averse investor. In all cases she will prefer the certain alternative to the uncertain one. She would content herself with receiving with certainty x_{12} which is less than $\mathbf{P}(X_{12})$ in exchange for the hypothetical gain given by $2\mathbf{P}(X_{12})$ whose probability is judged to be equal to 1/2 by her. In the scale of utility in which her judgments of indifference are based it is possible to observe equal levels on the vertical axis in passing from 0 to x_{12} and from x_{12} to $2\mathbf{P}(X_{12})$ on the horizontal axis, where 0 and $2\mathbf{P}(X_{12})$ express two equiprobable events of a partition of two incompatible and exhaustive events. The possibility of inserting the degree of preferability of X_{12} into the scale of the certain amounts is a necessary condition of all rational decision-making criteria that can be followed (see also [32]).

The investor subjectively estimates all joint probabilities of ${}_1X\,{}_2X$ inside of her budget set in such a way that there exists a direct linear relationship between ${}_1X$ and ${}_2X$. She is risk lover. For a risk-loving investor, the expected utility of X_{12} is greater than the utility of the mathematical expectation of X_{12}. Her continuous and cardinal utility function is a strictly increasing and convex function, where its slope gets steeper as wealth increases. The form and extent of this attitude towards risk which is caught by the utility function under consideration will depend on her temperament, her current mood and some other circumstance.

The investor subjectively estimates all joint probabilities of ${}_1X\,{}_2X$ inside of her budget set in such a way that ${}_1X$ and ${}_2X$ are stochastically independent. She is risk neutral. Accordingly, it is possible to observe that among those decisions leading to different joint contingent consumption plans her best choice under conditions of uncertainty and riskiness must be the one leading to the plan with the highest mathematical expectation denoted by $\mathbf{P}(X_{12})$. Her continuous utility function is an increasing linear function. It is the 45° line. Its graphical form is always the same unlike the previous cases.

All of this is compatible with the fact that the notion of risk is intrinsically of a subjective nature.

5.1. The Criteria of Rational Choices Being Made by the Investor: Multiple Random Goods of Order 2

We establish the following.

Definition 6. The certain amount that the investor subjectively judges to be equivalent to a double random good denoted by X_{12} is expressed by $\mathbf{P}(X_{12})$ whenever she is only interested in the mathematical expectation of X_{12}. It is the price of X_{12} for her whenever her utility function coincides with the 45-degree line. It coincides with its coherent prevision given by

$$\mathbf{P}(X_{12}) = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) & \mathbf{P}({}_1X\,{}_2X) \\ \mathbf{P}({}_2X\,{}_1X) & \mathbf{P}({}_2X\,{}_2X) \end{vmatrix}$$
$$= \mathbf{P}({}_1X\,{}_1X)\,\mathbf{P}({}_2X\,{}_2X) - \mathbf{P}({}_1X\,{}_2X)\,\mathbf{P}({}_2X\,{}_1X) > 0. \tag{44}$$

It represents the price that the investor is willing to pay in order to purchase the right to participate in a gamble identified with X_{12}.

The slope of the budget line is equal to -1 whenever two marginal random goods are the same. The two catheti of the right triangle under consideration are equal.

We note the following.

Remark 7. A choice being made by the investor under conditions of uncertainty and riskiness is rational if and only if she chooses any coherent evaluation of the marginal probabilities together with the joint ones characterizing m^2 possible quantitative states of the world of two contingent consumption plans that are jointly considered. She chooses a continuous and strictly increasing utility function in accordance with her subjective attitude towards risk. She fixes as her goal the maximization of the expected value of her utility, where the nature of such an expected value is firstly bilinear. □

Given a concave utility function denoted by $u(x)$, where x coincides with the monetary wealth of a risk-averse investor, it is possible to say that X_{12} is preferred to another double random good denoted by X_{34} if and only if it turns out to be

$$\mathbf{P}[u(X_{12})] > \mathbf{P}[u(X_{34})] \tag{45}$$

on the vertical axis of a two-dimensional Cartesian coordinate system. It follows that it turns out to be

$$\mathbf{P}_u(X_{12}) > \mathbf{P}_u(X_{34}) \tag{46}$$

on the horizontal axis, where $\mathbf{P}_u(X_{12})$ is less than $\mathbf{P}(X_{12})$, whereas $\mathbf{P}_u(X_{34})$ is less than $\mathbf{P}(X_{34})$. If a risk-averse investor is firstly faced with X_{34} then to pass from X_{34} to X_{12} is an advantageous transaction to her because $\mathbf{P}_u$ increases. Given a convex utility function denoted by $u(x)$, it is possible to say that X_{12} is preferred to X_{34} by a risk-loving investor if and only if (45) and (46) hold. We observe that $\mathbf{P}_u(X_{12})$ is greater than $\mathbf{P}(X_{12})$ as well as $\mathbf{P}_u(X_{34})$ is greater than $\mathbf{P}(X_{34})$ on the horizontal axis. Given the 45-degree line denoted by $u(x)$ identifying the identity of monetary value and utility, it is possible to say that X_{12} is preferred to X_{34} by a risk-neutral investor if and only if (45) and (46) continue to be valid. We note that it turns out to be $\mathbf{P}_u(X_{12}) = \mathbf{P}(X_{12})$ as well as $\mathbf{P}_u(X_{34}) = \mathbf{P}(X_{34})$ on the horizontal axis.

It is also possible to compare more than two joint contingent consumption plans. If the investor prefers X_{12} to X_{34} then it turns out to be $\mathbf{P}[u(X_{12})] > \mathbf{P}[u(X_{34})]$. If she prefers X_{34} to X_{56}, where X_{56} is different from X_{12} and X_{34}, then we observe $\mathbf{P}[u(X_{34})] > \mathbf{P}[u(X_{56})]$. It follows that she rationally prefers X_{12} to X_{56}, so we note $\mathbf{P}[u(X_{12})] > \mathbf{P}[u(X_{56})]$.

A choice is optimal if and only if there exists a utility function whose maximum expected value is firstly bilinear (see also [31]). An extension of Daniel Bernoulli's approach to the notion of expected utility is carried out. All optimal choices under conditions of uncertainty and riskiness can be ranked by the investor (see also [21]). They can be ranked inside of a linear space over $\mathbb{R}$. Accordingly, she is always able to establish which is her best choice (see also [24]).

Even if we jointly study three or more than three contingent consumption plans, it is never practically possible to consider more than two contingent consumption plans at a time from a metric point of view (see also [20]). This is because we use a quadratic metric. It can be a linear or multilinear metric.

6. CONCLUSIONS AND FUTURE PERSPECTIVES

This paper is connected with revealed preference theory. Two marginal risky assets are two marginal random goods that can be studied inside of the budget set of the investor. She is modeled as being a consumer. In this paper, the budget set of the investor is innovatively studied. It makes sense to consider possible values for a joint random good as well as for two marginal random goods. This is because the budget line identifying the budget set of the investor formally coincides with a hyperplane embedded in a two-dimensional Cartesian coordinate system. By definition, a hyperplane never separates any point $\mathbf{P}$ of $\mathcal{P}$ from the set $\mathcal{Q} = I({}_1X) \times I({}_2X)$ of all possible points for ${}_1X$ and ${}_2X$ belonging to $I({}_1X)$ and $I({}_2X)$ respectively. The same is true with regard to two marginal random goods in the sense that the same hyperplane never separates any point $\mathbf{P}$ of $\mathcal{P}$ from the sets $I({}_1X)$ and $I({}_2X)$ of all possible points for ${}_1X$ and ${}_2X$ belonging to $I({}_1X)$ and $I({}_2X)$ respectively. Possible values for a joint random good as well as for two marginal random goods are of an objective nature. They conceptually coincide with what can objectively be observed. Since $\mathbf{P}({}_1X\,{}_2X) = (\mathbf{P}({}_1X),\, \mathbf{P}({}_2X))$ is a synthesized element of the Frèchet class, the notion of risk is intrinsically of a subjective nature.

There is a complete and reversible equivalence between the assumption of a coherent preference as a basis for decisions and the choice of a coherent evaluation for probability and utility. In this paper, all of this is referred to multiple random goods of order 2 whose possible values are of a monetary nature. A multilinear approach to the criteria of investor choice under conditions of uncertainty and riskiness consists in establishing disaggregate and aggregate measures based on what the investor actually chooses inside of her budget set. Aggregate measures obtained by using a multilinear metric allow to identify multiple random goods of order 2. It is also possible to identify multiple random goods whose order is greater than 2. A multilinear approach to the criteria of investor choice under conditions of uncertainty and riskiness shows that an evolution towards a synthesis is intentionally tried. It avoids mistaken ideas such as the attempts to study decisions under conditions of uncertainty forcing everything to intervene except the evaluation of probability being made by the investor. It is a fundamental element. It is a basic and unavoidable result associated with specific conditions identifying decisions where sure elements are absent. Bound choices being made by the investor under conditions of uncertainty and riskiness have to be studied by considering all elements characterizing them. They are not of an objective nature only, but they are also of a subjective nature. Accordingly, an evolution towards a unitary vision in which it appears that a place and a link are found for theories previously viewed as unconnected parts must intentionally be tried. It is possible to recompose these parts in an organized way by using the properties of the notion of prevision of a random good. Such properties have been treated in this paper.

We are able to study multilinear relationships between variables. We are able to propose a multilinear regression model based on the main elements characterizing this study. Multilinear relationships between variables are not dealt with in the literature.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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