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Matrix Variate Distribution Theory under Elliptical Models—V: The Non-Central Wishart and Inverted Wishart Distributions

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Abstract—The non-central Wishart and inverted Wishart distributions are studied in this work under elliptical models; some distributional results are based on some generalizations of the well-known Kummer relations, which leds us to determine that some moments have a polynomial representation. Then the non-central F and "studentized Wishart" distributions are derived in a general setting. After some generalizations, including the so called non-central generalized inverted Wishart distribution, the classical results based on Gaussian models are derived here as corollaries.

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1. INTRODUCTION

The main role of matrix variate distributions via zonal polynomials for statistics based on Gaussian models is well known. Most of the essential results were proposed very early in the 1960's with the works of A.T. James and A.G. Constantine, and others, which were collected and improved in the 1980's by [41]. A number of works and compilations in such context of central Gaussian models via zonal polynomials, can be found in the posterior literature, see for example, [28, 44, 45] and the references therein. Only a few extensions from the old, assuming non Gaussian models, have appeared recently and no further substantial aggregation of the fundamental theory summarized by Muirhead, has been proposed, with the exception of the recent apparition of the efficient computation of zonal polynomials by [36], which opened the possibility of working with the exact densities of a number of results described and passed to asymptotics in chapters 7–9 in the referred classical book. Translation of this theory in the general setting of the elliptical distributions of [26, 27], in the context of zonal polynomials, has been the goal of a series of papers recently studied by the authors. In the previous four parts we have considered the generalized Wishart and its application in the likelihood ratio statistic for the homogeneity of covariance matrices [6] and the sphericity test [7] under elliptical models; moreover [9] studied the test about a specified value of a covariance matrix, and specified values for the mean vector and covariance matrix, also, [8] proposed some aspects of chapter 9 of [41] under the perspective of elliptical models. The exact densities are computable under certain conditions of the kernel function and do not require the use of asymptotic distributions. Lemma 7.2.12 of [41] and Theorem 2.2 of [38] were corrected, then their main consequences around elliptical sample covariance matrix distributions were proposed. In this fifth installment, a number of generalizations of results from chapters 3, 7, and 10 by [41] are considered under the elliptical framework, such as the non-central Wishart (Section 4) and the non-central F and "studentized Wishart" distributions (Section 4).

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2. JACK POLYNOMIALS AND DAVIS INVARIANT POLYNOMIALS OF MATRIX ARGUMENTS

In this section we provide a summary of the polynomials of matrix arguments as the foundation of the distribution theory here derived. We focus the history of their computation and the related problems for a realibable non-central matrix variate distributions.

In the last seventy years, the matrix variate distribution theory has been arisen as one of the most interesting source of challenging problems in polynomials of Hermitian matrix arguments under real normed division algebras.

Jack polynomials $C_\kappa^\alpha(\lambda_1,\ldots,\lambda_m)$, indexed by a real parameter α and an ordered partition κ of a positive integer k are functions of m positive real variables $\lambda_1,\ldots,\lambda_m$ originally introduced in [30]. When the $\lambda's$ are seen as the eigenvalues of a general Hermitian matrix under a specified field, they have recovered attention in distribution theory, because they can provide a unified approach under real normed division algebras over symmetric cones. A collection of the classical results of real ($\alpha=2$) and complex ($\alpha=1$) matrix variate statistics into a general setting involving also the quaternion ($\alpha=1/2$) and the elusive octonion ($\alpha=1/4$) fields. Jack or zonal spherical functions were inspired by the so called zonal polynomials of James, introduced in the sixties in order to compute the integrals for $\alpha=2,1$,

$$\int\limits_{\mathbf{H}\in\mathfrak{U}^\beta(m)} C_\kappa^\beta(\mathbf{X}\mathbf{H}^*\mathbf{Y}\mathbf{H})(d\mathbf{H})=\frac{C_\kappa^\beta(\mathbf{X})C_\kappa^\beta(\mathbf{Y})}{C_\kappa^\beta(\mathbf{I})},$$

where $\mathbf{X},\mathbf{Y}\in\mathfrak{S}_m^\beta$ are positive definite or Hermitian matrices, and $\mathfrak{U}^\beta(m)$ is the orthogonal or the unitary group, respectively. The real and complex zonal polynomials were separated and unconnected by decades. Despite Jack polynomial theory, in terms of functions of the $\lambda's$ and any real α, advanced in numerous works involving combinatorial, algebra and partition theory, much geometrical ligands with the original works of James, in the setting of matrix theory, were also missing.

Zonal polynomials of positive definite matrices were constructed in different ways by James, but the most interesting discover for computation of such polynomials arrived when they were seen as eigenfunctions of the Laplace Beltrami Operator. The seminal paper of [31] for $\alpha=2$ was the inspiration of Jack polynomial subsequent papers to provide the required extensions, most of them verbatim copies of the derivations given by James for real zonal polynomials. In particular, [46] just proved algebraically that the Jack polynomials for general α are eigenfunctions of the Laplace Beltrami Operator, but he could not provide any associated interpretation of the operator from its nature theoretical group. Only the real case was understood by James as spherical functions for the pair $(GL_n(R),O_n(R))$; this lack of comprehension continues through the years in the publish papers of the line of Jack polynomials. However, Stanley's proof was sufficient for translating the second discover of [31] about a recurrent method to construct the polynomials. Then [37] provided a recurrence formula for computing Jack polynomials. Later, [25, 36] used that result to construct numerical and symbolical software for calculating the polynomials. In this sense, the efficient computation for any α, seems to be the end of the problem, but the required interpretation for the Laplace-Beltrami operator when $\alpha\neq2$ remained unsolved in the Stanley's line of research. Curiously, the other line of research of matrix variate distributions, promoted by James, conducted to the explanation of the complex case $\alpha=1$. Few years after the publication of [31], some papers conjectured the Laplace-Beltrami operator for zonal polynomials of positive definite Hermitian matrix arguments, when they are just not seen as simple Jack polynomials of λ's and $\alpha=1$. In fact, those works assume the validity of that conjecture and used it without proof in other results, see [10, Lemma 2.1; 25, Definition 2.10; 47, Eq. (4.16)] with $\alpha=1$. The group theoretical nature of the complex case, claimed by [46] appeared in [23]. The implemented approach required a new derivation of [31], due to [22], in order to connect both real and complex cases with the unified theory of linear structures of [40]. The addressed new method for the real case also derived the semidefinite positive symmetric zonal polynomials, we must note that all the referred works and software of Jack or zonal polynomials and their extensions to several matrices only include the definite positive case.

For completeness, the addressed definition of Jack or zonal polynomials $C_\kappa^\alpha(Y)$ under real normed division algebras by the Laplace–Beltrami operator is given by:

$$\sum_{i=1}^{m} y_i^2 \frac{\partial^2 C_\kappa^{(\alpha)}(Y)}{\partial y_i^2} + \frac{2}{\alpha}\sum_{i=1}^{m} y_i^2 \sum_{j\neq i} \frac{1}{y_i - y_j}\frac{\partial C_\kappa^{(\alpha)}(Y)}{\partial y_i} = \sum_{i=1}^{m} k_i(k_i - 1 + \frac{2}{\alpha}(m-i))C_\kappa^{(\alpha)}(Y).$$

Where $y_1, \ldots, y_m$ are the eigenvalues of the matrix Y, and

$$\sum_\kappa C_\kappa^\alpha(Y) = (\mathrm{tr}(Y))^k.$$

The summation is indexed by all the ordered partitions κ of k, i.e., they are decreasing sequence of nonnegative integers $\kappa = (k_1, k_2, \ldots)$ with only finitely many nonzero terms such as $k = \sum k_i$.

Finally, the matrix variate distribution theory also lead to a generalization which is not attempted in the subsequent works derived from [30]. As we described, zonal polynomials of James appeared in the computation of certain integrals over the orthogonal and unitary groups when the joint density function of the latent roots of a central Wishart matrix was required. But, if the non central Wishart is considered, then a more difficult integral arrives and demanded the introduction of new polynomials of several matrix arguments. If A.T. James was the figure for setting the theory of zonal and Jack polynomials, then we can say that A.W. Davis promoted the more complex theory of symmetric polynomials of two and more matrices. However, Davis polynomials have not the same luck of Jack polynomials, most of the well known facts for zonal polynomials are out of any research in the Davis invariant polynomials or are still open problems. The extensions of the Davis invariant polynomials of several positive definite matrix arguments to another fields took much time, at present only a detailed construction of those polynomials was carried out for hermitian matrices, by using extensions of idempotent theory and James doublets [2], a number of applications and classical integrals were also provided. Finally, a parametric family for invariant polynomials of two matrix arguments under real normed division algebras was defined in equation (5.1) of [21]. The addressed polynomials, parametrized by $\beta = \alpha/2$, where defined via an integral over the general Stiefel manifold of the product of two Jack polynomials. The existence and construction of such polynomials for the quaternion and the octonion cases are an open problem today. Also the conditions of the real and complex matrix arguments for such general integral must be considered; the addressed Eq. (5.1) of [21], under the extension and adjustment of [16] for the real case and [2] for the complex case, takes the form:

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \prod_{i=1}^{r} C_{\kappa(i)}^\beta(\mathbf{A}_i\mathbf{H}^*\mathbf{X}_i\mathbf{H})(d\mathbf{H}) = \sum_{\phi\in\kappa(1)\cdot\ldots\cdot\kappa(r)} \frac{C_\phi^{[\beta]\kappa[r]}(\mathbf{A}_{[r]})C_\phi^{[\beta]\kappa[r]}(\mathbf{X}_{[r]})}{C_\phi^\beta(\mathbf{I})},$$

where $\kappa[r]$ runs through all ordered partitions $\kappa(1), \ldots, \kappa(r)$ of $k(1), \ldots, k(r)$, respectively, into less or equal than m parts, and ϕ through the decomposition of $\otimes_{i=1}^r[2\kappa(i)]$. Here ϕ is an ordered partition of $f = \sum_{i=1}^r k(i)$ into less or equal m parts. The Davis invariant polynomials $C_\phi^{[\beta]\kappa[r]}(\mathbf{X}_{[r]})$ of r general Hermitian matrix arguments $\mathbf{X}_1, \ldots, \mathbf{X}_r$ are invariant under the simultaneous transformations $\mathbf{X}_i \to \mathbf{H}^*\mathbf{X}_i\mathbf{H}$ with $\mathbf{H} \in \mathfrak{U}^\beta(m))$, $i = 1, \ldots, r$.

About the construction of Davis invariant polynomials under the approach of the Laplace Beltrami Operator, an interesting path of the problem can be traced. Thirty nine years after their apparition in [18], the literature counts dozens of theoretical papers involving invariant polynomials, most of them expressing the results in colossal multiple series of those polynomials, which at present, cannot be computed for large degrees (more than the 5th degree). For years [16] was the unique work devoted to the construction of non-trivial invariant polynomials (non-top and non-lower order polynomials, see for example [11] for top order polynomials), and contrary to the constructions of zonal polynomials, a unique basis for invariant polynomials was used, i.e., products of powers of traces. Computation of invariant polynomials in terms of symmetric functions was not available in literature during Davis works; recall that this basis allowed a successful iterative method for calculating the zonal polynomials by using the Laplace–Beltrami operator. In fact, [17] addressed about that problem: "We finally note that the invariant polynomials of this paper are not zonal polynomials, and lack certain properties of the latter. In particular, there is no guarantee that they will possess an analogue of the Laplace–Beltrami operator for the zonal polynomials [31]". In fact, 27 years later, the creator of those polynomials, tried to illustrate the method of the Laplace–Beltrami operator for degree two in the usual basis, and then He wrote in [19]: "However, this approach to constructing the polynomials has not yet implemented". Inspired in these claims, [3] developed the necessary aspects to translate the Laplace–Beltrami operator applied to zonal polynomials by James into the setting of Davis invariant polynomials. The addressed work involves. (1) A definition and study of the computation of the monomial symmetric functions of several matrix arguments, extending the well known symmetric monomial functions. (2) A new basis for the invariant polynomials in terms of the preceding functions. (3) A method of constructing the invariant polynomials by using the Laplace–Beltrami operator in terms of symmetric functions instead of the usual basis proposed by [19]. Finally, [3] illustrates the procedure in the first few cases and arrives to a non-expected result, a simple counterexample showing that the invariant polynomials cannot be computed in a recurrence way as the zonal polynomials. This result ends the search of exact parallelism between zonal and invariant polynomials.

3. THE GENERALIZED NON-CENTRAL WISHART DISTRIBUTION

It is known that the non-central Wishart distribution generalizes the non-central χ^2 distribution in the same way that the usual or central Wishart distribution generalizes the χ^2 distribution. In the same way a central Wishart distribution, which is based on a central Gaussian distribution, can be generalized under any elliptically countered distribution, which is referred to as the central generalized Wishart distribution [6], and at the same time the last one can be generalized to the non-central generalized Wishart distribution, which is the aim of this section.

Now, the matrix variate elliptically contoured distributions (which have been studied by various authors, such as [26, 27]) are defined by:

Definition 1. $\mathbf{X}$, a matrix of dimensions $N \times K$, has a matrix variate elliptically contoured distribution if its density function is

$$g_{\mathbf{X}}(\mathbf{X}) = \frac{1}{|\mathbf{\Sigma}|^{n/2}|\mathbf{\Theta}|^{m/2}} h\left(\operatorname{tr}\mathbf{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})'\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right),$$

where $\mathbf{\Sigma} : N \times N$, $\mathbf{\Theta} : K \times K$, $\mathbf{\Sigma} > 0$, $\mathbf{\Theta} > 0$ and the function $h : \Re \to [0, \infty)$ is termed the generator function, and is such that $\int_0^\infty u^{NK-1}h(u^2)du < \infty$. This distribution is denoted by $E_{N\times K}(\boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{\Theta}, h)$.

Given that this work tries the generalization for the non-central case of the exposition due to [41], we always follow all the restrictions and notations of that book, in order to guide an easy comparison and derivations of the corresponding corollaries.

From [6], the central elliptical Wishart $EW_m(n, \Sigma, h)$ density is the distribution of the $m \times m$ random matrix $A = Z'Z$, where $Z(n \times m)$ is $E(0, I_n \otimes \Sigma, h)$. When $n < m$, A is singular and the $EW_m(n, \Sigma, h)$ distribution does not have a density function with respect to the Lebesgue measure.

Lemma 2. *If A is positive definite distributed as $EW_m(n, \Sigma, h)$ with $n \geq m$ then the density function of A is*

$$\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{1}{2}n\right)|\Sigma|^{n/2}}|A|^{(n-m-1)/2}h\left(\operatorname{tr}\Sigma^{-1}A\right), \tag{1}$$

where $\Gamma_m(\cdot)$ is the multivariate gamma function

$$\Gamma_m(a) = \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma\left[a - \frac{1}{2}(i-1)\right], \quad \text{with } \Re(a) > \frac{1}{2}(m-1).$$

Two simple properties of the central elliptical Wishart are the following.

Theorem 3. *If $A \sim EW_m(n, \Sigma, h)$ and M is a $t \times m$ matrix of rank t then $MAM' \sim W_t(n, M\Sigma M', h)$.*

Proof. From Lemma 2, the characteristic function of $A \sim EW_m(n, \Sigma, h)$ is given by

$$E\left[\operatorname{etr}\left(\frac{i}{2}A\Gamma\right)\right] = \frac{\pi^{\frac{1}{2}mn}|\Sigma|^{-\frac{1}{2}n}}{\Gamma_m\left(\frac{1}{2}n\right)}\sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}\sum_{\kappa}\int_{A}\operatorname{etr}\left(\frac{i}{2}A\Gamma\right)|A|^{\frac{1}{2}(n-m-1)}C_\kappa\left(\Sigma^{-1}A\right)(dA)$$

$$= \frac{\left|-\frac{i}{2}\Gamma\Sigma\right|^{-\frac{1}{2}n}}{\pi^{-\frac{1}{2}mn}} \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} \sum_{\kappa} \left(\frac{1}{2}n\right)_{\kappa} C_{\kappa}\left(\Sigma^{-1}\left(-\frac{i}{2}\Gamma\right)^{-1}\right), \tag{2}$$

where we have used [41, Theorem 7.2.7] in the last line. Note that (2) is a generalization of [41, Theorem 3.2.3], which can be obtained by taking $h(y) = \frac{1}{(2\pi)^{\frac{1}{2}mn}} \mathrm{e}^{-\frac{1}{2}y}$ and recalling that ${}_1F_0(a; X) = |I - X|^{-a}$.

Now, if M is a $t \times m$ matrix of rank t, then the characteristic function of MAM' follows easily from (2); i.e.,

$$E\left[\mathrm{etr}\left(\frac{i}{2}MAM'\Gamma\right)\right]$$

$$= \frac{\left|-\frac{i}{2}\Gamma M\Sigma M'\right|^{-\frac{1}{2}n}}{\pi^{-\frac{1}{2}tn}} \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} \sum_{\kappa} \left(\frac{1}{2}n\right)_{\kappa} C_{\kappa}\left(\left(M\Sigma M'\right)^{-1}\left(-\frac{i}{2}\Gamma\right)^{-1}\right),$$

which implies that MAM' is $EW_t\left(n, M\Sigma M', h\right)$. □

An important consequence of this result is:

Corollary 4. *If $A \sim EW_m(n, \Sigma, h)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where A_{11} and Σ_{11} are $k \times k$ matrices, then $A_{11} \sim EW_k(n, \Sigma_{11})$ and $A_{22} \sim EW_{m-k}(n, \Sigma_{22})$.*

Proof. The demonstration is the same from the Gaussian case (see [41, Corollary 3.2.6]). The first result follows substituting the $k \times m$ matrix M in Theorem 3 by $[I_k|0]$, then $MAM' = A_{11}$ and $M\Sigma M' = \Sigma_{11}$. The second part is obtained by considering $M = [0|I_{m-k}]$. □

Then we are in position to define a non-central generalized Wishart matrix.

Definition 5. Let Z be $E_{n\times m}(M, I_n \otimes \Sigma), h$, so the $m \times m$ matrix $A = Z'Z$ is said to have the non-central Wishart distribution with n degrees of freedom, covariance matrix Σ and matrix of non-centrality parameters $\Omega = \Sigma^{-1}M'M$. This will be denoted by $A \sim EW_m(n, \Sigma, \Omega, h)$.

As in the central case, when $n < m$, the matrix A is singular and it does not have a density with respect to the Lebesgue Measure. We can now derive the density of a A, if $n \geq m$.

Theorem 6. *Let Z be $E_{n\times m}(M, I_n \otimes \Sigma, h)$, with $n \geq m$. The distribution of the non-central generalized Wishart matrix $A = Z'Z > 0$ is given by*

$$\frac{\pi^{\frac{1}{2}mn}|A|^{(n-m-1)/2}}{\Gamma_m\left[\frac{1}{2}n\right]|\Sigma|^{\frac{n}{2}}} {}_0P_1\left(h^{(2k)}\left(\mathrm{tr}\left(\Sigma^{-1}A + \Omega\right)\right) : \frac{1}{2}n; \Omega\Sigma^{-1}A\right), \tag{3}$$

where $\Omega = \Sigma^{-1}M'M$. This will be denoted by $A \sim EW_m(n, \Sigma, \Omega, h)$.

Proof. Start with the density of Z, i.e., $E_{n\times m}(M, I_n \otimes \Sigma, h)$:

$$|\Sigma|^{-\frac{n}{2}} h\left(\mathrm{tr}\left(\Sigma^{-1}Z'Z + \Omega\right) - 2\,\mathrm{tr}\left(\Sigma^{-1}M'Z\right)\right)(dZ).$$

As in the central case, perform the factorization $Z = H_1T$, where $H_1 \in V_{m,n}$, the Stiefel manifold of $n \times m$ matrices such that $H_1'H_1 = I_m$, and T is an upper triangular matrix; and consider $A = Z'Z = T'T$, where the jacobian follows from [41, Theorem 2.1.14], i.e.,

$$(dZ) = 2^{-m}|A|^{(n-m-1)/2}(dA)(H_1'dH_1).$$

Then, the joint density of A and H_1 has the form

$$\frac{|A|^{(n-m-1)/2}}{2^m|\Sigma|^{\frac{n}{2}}} h\left(\mathrm{tr}\left(\Sigma^{-1}A + \Omega\right) - 2\,\mathrm{tr}\left(T\Sigma^{-1}M'H_1\right)\right)(dA)(H_1'dH_1), \tag{4}$$

where $\Omega = \Sigma^{-1}M'M$ is the non-centrality parameter matrix.

The required density is obtained by integrating over $V_{m,n}$:

$$\frac{|A|^{(n-m-1)/2}}{2^m|\Sigma|^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{h^{(k)}\left(\operatorname{tr}\left(\Sigma^{-1}A+\Omega\right)\right)}{k!}\int\limits_{V_{m,n}}\left[\operatorname{tr}\left(-2T\Sigma^{-1}M'H_1\right)\right]^k(H_1'dH_1);$$

but this integral is zero for odd k, then by using [32, Eq. (22)], and noting that $(1/2)_k 2^{2k}/(2k!) = 1/k!$ we obtain

$$\frac{\pi^{\frac{1}{2}mn}|A|^{(n-m-1)/2}}{\Gamma_m\left[\frac{1}{2}n\right]|\Sigma|^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{h^{(2k)}\left(\operatorname{tr}\left(\Sigma^{-1}A+\Omega\right)\right)}{k!}\sum_{\kappa}\frac{1}{\left(\frac{1}{2}n\right)_\kappa}C_\kappa\left(\Omega\Sigma^{-1}A\right).$$

Finally, by applying the following definition of a generalized hypergeometric function (see [5]),

$${}_0P_1(f(X,k):a;X)=\sum_{k=0}^{\infty}\frac{f(X,k)}{k!}\sum_{\kappa}\frac{1}{(a)_\kappa}C_\kappa(X),$$

we have the result. □

When $M=0$, $\Omega=0$, $C_\kappa(0)=0$ for $k>0$; $(\cdot)_\kappa C_\kappa(0)=1$ for $k=0$ and $h^{(0)}(\cdot)=h(\cdot)$ then (3) becomes the generalized central Wishart distribution (1) as it ought to be.

For the classical Gaussian case given by [41, Theorem 10.3.2], just consider $h(y)=\frac{1}{(2\pi)^{mn/2}}\,\mathrm{e}^{-\frac{1}{2}y}$, so $h^{(2k)}(y)=\frac{1}{(2\pi)^{mn/2}}\left(\frac{1}{2}\right)^{2k}\mathrm{e}^{-\frac{1}{2}y}$, and recall that ${}_0P_1\left(1:a;X\right)={}_0F_1(a;X)$, then the following familiar result is obtained:

$$\frac{2^{-\frac{1}{2}mn}|A|^{(n-m-1)/2}}{|\Sigma|^{\frac{n}{2}}\Gamma_m\left[\frac{1}{2}n\right]}\operatorname{etr}\left(-\frac{1}{2}\left[\Sigma^{-1}A+\Omega\right]\right){}_0F_1\left(\frac{1}{2}n;\frac{1}{4}\Omega\Sigma^{-1}A\right).$$

Now, the density of $A\sim EW(M,\Sigma,\Omega,h)$, given in (3), can be expanded in terms of zonal polynomials and then written in terms of invariant polynomials of [18] as follows:

$$\frac{\pi^{\frac{1}{2}mn}|A|^{(n-m-1)/2}}{|\Sigma|^{\frac{n}{2}}\Gamma_m\left[\frac{1}{2}n\right]}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{t=0}^{\infty}\frac{h^{(2k+t)}\left(\operatorname{tr}\Omega\right)}{t!}\sum_{\kappa}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\kappa}\times\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}C_\phi^{\kappa,\tau}\left(\Omega\Sigma^{-1}A,\Sigma^{-1}A\right),$$

where $\Omega=\Sigma^{-1}M'M$. And applying

$$\int\limits_{R>0}\operatorname{etr}(-RW)|R|^{t-p}C_\phi^{\kappa,\lambda}(XR,YR)(dR)=\frac{\Gamma_m(t)(t)_\phi}{|W|^t}C_\phi^{\kappa,\lambda}(XW^{-1},YW^{-1}),$$

see [18], we get the following result.

Theorem 7. *Let n be a real number with $n>m-1$. Then the characteristic function of a non-central generalized Wishart matrix $A\sim EW(M,\Sigma,\Omega,h)$ is given by*

$$\frac{\pi^{\frac{1}{2}mn}}{|\Sigma|^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{t=0}^{\infty}\frac{h^{(2k+t)}\left(\operatorname{tr}\Omega\right)}{t!}\sum_{\kappa}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\kappa}\times\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}\left(\frac{1}{2}n\right)_\phi(2i)^{\frac{mn}{2}}|\Gamma|^{-\frac{n}{2}}C_\phi^{\kappa,\tau}\left(2i\Omega\Sigma^{-1}\Gamma^{-1},2i\Sigma^{-1}\Gamma^{-1}\right).$$

For the Gaussian case [41, Theorem 10.3.3] consider again $h(y)=\frac{1}{(2\pi)^{mn/2}}\,\mathrm{e}^{-\frac{1}{2}y}$, and by a suitable use of the properties of invariant and zonal polynomials, we have the following chain of transformations which provides the result. Start by using the integral representation of $\left(\frac{1}{2}n\right)_\phi(2i)^{\frac{n}{2}}|\Gamma|^{-\frac{n}{2}}\times$

$C_\phi^{\kappa,\tau}(2i\Omega\Sigma^{-1}\Gamma^{-1}, 2i\Sigma^{-1}\Gamma^{-1})$, see [18]. Then we get:

$$\frac{2^{\frac{1}{2}mn}}{\Gamma_m\left(\frac{1}{2}n\right)|\Sigma|^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{t=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^t \operatorname{etr}\left(-\frac{1}{2}\Omega\right)}{t!}\sum_{\kappa}\sum_{\tau}\frac{1}{\left(\frac{1}{2}n\right)_\kappa}$$
$$\times\int\limits_{R>0}\operatorname{etr}\left(-R\left(-\frac{i}{2}\Gamma\right)\right)|R|^{(n-m-1)/2}\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}C_\phi^{\kappa,\tau}\left(\frac{1}{4}\Omega\Sigma^{-1}R, \Sigma^{-1}R\right)(dR).$$

Now write $\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}C_\phi^{\kappa,\tau}\left(\frac{1}{4}\Omega\Sigma^{-1}R, \Sigma^{-1}R\right)$ as a product of zonal polynomials, then group the series indexed by t and locate its corresponding Taylor function as follows:

$$\frac{2^{\frac{1}{2}mn}}{\Gamma_m\left(\frac{1}{2}n\right)|\Sigma|^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{1}{\left(\frac{1}{2}n\right)_\kappa}$$
$$\times\int\limits_{R>0}\operatorname{etr}\left(-R\left(-\frac{i}{2}\Gamma\right)\right)|R|^{(n-m-1)/2}C_\kappa\left(\frac{1}{4}\Omega\Sigma^{-1}R\right)$$
$$\times\operatorname{etr}\left(-\frac{1}{2}\left(\Omega+\Sigma^{-1}R\right)\right)(dR),$$

now, use [41, Theorem 7.2.7], thus

$$|I_m - i\Gamma\Sigma|^{-\frac{n}{2}}\operatorname{etr}\left(-\frac{1}{2}\Omega\right)\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}C_\kappa\left(\frac{1}{2}\Omega\left(I_m - i\Gamma\Sigma\right)^{-1}\right)$$

and finally invert the series to obtain.

Corollary 8. *The characteristic function of a non-central Wishart distribution based on a Gaussian model is given by*

$$E\left[\operatorname{etr}\left(\frac{i}{2}A\Gamma\right)\right] = |I_m - i\Gamma\Sigma|^{-\frac{n}{2}}\operatorname{etr}\left(-\frac{1}{2}\Omega\right)\operatorname{etr}\left(\frac{1}{2}\Omega\left(I_m - i\Gamma\Sigma\right)^{-1}\right),$$

where $\Omega = \Sigma^{-1}M'M$.

In order to obtain the next result we need the general integral proved by [5, Theorem 1].

Lemma 9. *If* $A \sim EW_m(n, \Sigma, \Omega, h)$*, where* $n \geq m$*, then*

$$E\left[|A|^r\right] = \frac{\Gamma_m\left(\frac{1}{2}n + r\right)}{\pi^{-\frac{1}{2}mn}|\Sigma|^{-r}\Gamma_m\left[\frac{1}{2}n\right]}$$
$$\times\,{}_1P_1\left(\frac{\sum_{t=0}^{\infty}\frac{\int_0^\infty h^{(2k+t)}(w)w^{\frac{1}{2}m(n+2r)+k-1}dw}{t!}\operatorname{tr}^t(\Omega)}{\Gamma\left[\frac{1}{2}m(n+2r)+k\right]} : \frac{1}{2}n + r; \frac{1}{2}n; \Omega\right).$$

Proof. First expand the density (3) of $A \sim EW(M, \Sigma, \Omega, h)$, in the following different way to the proof of Theorem 7:

$$\frac{\pi^{\frac{1}{2}mn}|A|^{(n-m-1)/2}}{|\Sigma|^{\frac{n}{2}}\Gamma_m\left[\frac{1}{2}n\right]}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{t=0}^{\infty}\frac{h^{(2k+t)}\left(\operatorname{tr}\left[\Sigma^{-1}A\right]\right)}{t!}$$
$$\times\sum_{\kappa}\frac{1}{\left(\frac{1}{2}n\right)_\kappa}C_\kappa\left(\Omega\Sigma^{-1}A\right)\sum_{\tau}C_\tau(\Omega).$$

Thus by [5, Theorem 1]

$$E\left[|A|^r\right] = \frac{\pi^{\frac{1}{2}mn}\Gamma_m\left(\frac{1}{2}n + r\right)}{|\Sigma|^{-r}\Gamma_m\left[\frac{1}{2}n\right]}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left[\frac{1}{2}m(n+2r)+k\right]}$$

$$\times \left\{ \sum_{t=0}^{\infty} \frac{\int\limits_0^{\infty} h^{(2k+t)}(w) w^{\frac{1}{2}m(n+2r)+k-1} dw}{t!} \operatorname{tr}^t(\Omega) \right\} \sum_{\kappa} \frac{\left(\frac{1}{2}n + r\right)_{\kappa}}{\left(\frac{1}{2}n\right)_{\kappa}} C_{\kappa}(\Omega),$$

which is the required result given in terms of the generalized hypergeometric ${}_1P_1(\cdot)$. □

The above moments have an important restriction, that is, it is an infinite series of zonal polynomials (their computability depends on some open problems in the area, see [36]). Although it is easy to observe that the series belongs to a general class studied by [24], who proved that a generalized confluent type series becomes a polynomial if the parameters satisfy some conditions (this fact was applied for example in the context of shape theory, which significantly reduces the inference because the densities are polynomials, see [4]).

First we start with some facts of the so called matrix generalized Kummer relation of [24].

Definition 10. Let $\mathbf{X} > 0$ be an $m \times m$ positive definite matrix. The hypergeometric generalised function ${}_1P_1$ of the matrix argument is defined by

$${}_1P_1(f(t, \operatorname{tr}(\mathbf{X})) : a; c; \mathbf{X}) = \sum_{t=0}^{\infty} \frac{f(t, \operatorname{tr}(\mathbf{X}))}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} C_{\tau}(\mathbf{X}), \tag{5}$$

where $\sum_{\tau}$ denotes the summation over all partitions τ, $\tau = (t_1, \cdots, t_m)$, $t_1 \geq t_2 \cdots \geq t_m > 0$, of t, $C_{\tau}(\mathbf{X})$ is the zonal polynomial of $\mathbf{X}$ corresponding to τ, the function $f(t, \operatorname{tr}(\mathbf{X}))$ is independent of τ and the generalised hypergeometric coefficient $(b)_{\tau}$ is given by

$$(\beta)_{\tau} = \prod_{i=1}^{m} \left(\beta - \frac{1}{2}(i-1)\right)_{t_i},$$

where

$$(b)_t = b(b+1)\cdots(b+t-1), \quad (b)_0 = 1.$$

Here, $\mathbf{X}$ is the argument of the function, is a complex symmetric $m \times m$ matrix and the parameters a, c are arbitrary complex numbers. The parameter c cannot be zero or an integer or a half-integer $\leq (m-1)/2$. If the parameter a is a negative integer, say, $a = -l$, then the function (5) is a polynomial of degree ml, because for $t \geq ml + 1$, $(a)_{\tau} = (-l)_{\tau} = 0$, see [41, p. 258]. In particular, note that, ${}_1P_1(1 : a; c; \mathbf{X}) = {}_1F_1(a; c; \mathbf{X})$.

Thus, using this notation we observe that the Kummer relation described by [29], ${}_1F_1(a; c; \mathbf{X}) = \operatorname{etr}(\mathbf{X})\, {}_1F_1(c - a; c; -\mathbf{X})$, is a particular case of a general type of expressions with the following form:

$${}_1P_1\left(f^{(t)}(0) : a; c; \mathbf{X}\right) = {}_1P_1\left(f^{(t)}(\operatorname{tr}(\mathbf{X})) : c - a; c; -\mathbf{X}\right), \tag{6}$$

where $f^{(t)}(y)$ denotes the tth derivative of the function $f(y)$ which allows a Taylor expansion in zonal polynomials.

Lemma 11. *If $f(y)$ allows a Taylor expansion in zonal polynomials, then the generalised Kummer relation is given by*

$${}_1P_1\left(f^{(t)}(0) : a; c; \mathbf{X}\right) = {}_1P_1\left(f^{(t)}(\operatorname{tr}(\mathbf{X})) : c - a; c; -\mathbf{X}\right), \tag{7}$$

where $\mathbf{X} > 0$, $\Re(c) > (m-1)/2$ and a is arbitrary (or at least $\Re(a) > (m-1)/2$, if the integral representation of ${}_1F_0$ is used, see [29, p. 485] or [41, Corollary 7.3.5]).

Proof. It is given in [24]. □

Therefore, we have the necessary elements fot the improvement of Lemma 12.

Theorem 12. *Given $A \sim EW_m(n, \Sigma, \Omega, h)$, where $n \geq m$, and assume that there is $f(\cdot)$ such that it accepts a Taylor expansion in zonal polynomials and*

$$f^{(k)}(0) = \frac{\sum_{t=0}^{\infty} \frac{\int\limits_0^{\infty} h^{(2k+t)}(w) w^{\frac{1}{2}m(n+2r)+k-1} dw}{t!} \operatorname{tr}^t(\Omega)}{\Gamma\left[\frac{1}{2}m(n+2r)+k\right]},$$

so

$$E\left[|A|^r\right] = \frac{\Gamma_m\left(\frac{1}{2}n + r\right)}{\pi^{-\frac{1}{2}mn}|\Sigma|^{-r}\Gamma_m\left[\frac{1}{2}n\right]} {}_1P_1\left(f^{(k)}(\operatorname{tr}\Omega) : -r; \frac{1}{2}n; -\Omega\right).$$

And if r is a positive integer the rth moment is a polynomial of degree mr.

The Gaussian case was derived in its entirety by [41, Theorem 10.3.7], here it follows straightforward: Take $h(y) = \frac{1}{(2\pi)^{mn/2}} \mathrm{e}^{-\frac{1}{2}y}$, which implies that

$$\int_0^\infty h^{(2k+t)}(w) w^{\frac{1}{2}m(n+2r)+k-1} dw = (-1)^t \pi^{-\frac{mn}{2}} 2^{mr-k-t} \Gamma\left[\frac{mn}{2} + mr + k\right].$$

Then

$$f^{(k)}(0) = \pi^{-\frac{mn}{2}} 2^{mr-k} \operatorname{etr}\left(-\frac{1}{2}\Omega\right).$$

Thus

$$f(y) = \pi^{-\frac{mn}{2}} 2^{mr} \operatorname{etr}\left(-\frac{1}{2}\Omega\right) \mathrm{e}^{\frac{1}{2}y}$$

and

$$f^{(k)}(\operatorname{tr}\Omega) = \pi^{-\frac{mn}{2}} 2^{mr-k} \operatorname{etr}\left(-\frac{1}{2}\Omega\right) \operatorname{etr}\left(\frac{1}{2}\Omega\right) = \pi^{-\frac{mn}{2}} 2^{mr-k}.$$

Therefore

Corollary 13. *If A is $W_m(n, \Sigma, \Omega)$ with $n \geq m$, then*

$$E\left[|A|^r\right] = \frac{2^{mr}\Gamma_m\left(\frac{1}{2}n + r\right)}{\Gamma_m\left(\frac{1}{2}n\right)} |\Sigma|^r {}_1F_1\left(-r; \frac{1}{2}n; -\frac{1}{2}\Omega\right),$$

which is a polynomial of degree mr if r is a positive integer.

Now, we generalize the so called Bartlett decomposition [1], which states the distributions of the elements from the triangular matrix involved in such transformation.

First we start with the central case.

Lemma 14. *Consider the Bartlett decomposition $A = T'T$ of a central generalized Wishart matrix $A \sim EW_m(n, I_m, h)$, $n \geq m$; where T is an upper triangular $m \times m$ matrix with positive diagonal elements. Then the distribution of T is given by*

$$\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{1}{2}n\right)} \prod_{i=1}^m (t_{ii}^2)^{(n-i-1)/2} \sum_{k=0}^\infty \frac{h^{(k)}(0)}{k!} \sum_\kappa C_\kappa(T'T),$$

Proof. From Lemma 2, the density function of $A \sim EW_m(n, \Sigma, h)$, with $n \geq m$ is given by

$$\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{1}{2}n\right)} |A|^{(n-m-1)/2} h\left(\operatorname{tr} A\right).$$

Replacing $A = T'T$, then $(dA) = 2^m \prod_{i=1}^m t_{ii}^{m+1-i}(dT)$, $|A| = |T'T| = |T|^2 = \prod_{i=1}^m t_{ii}^2$, and $\operatorname{tr} A = \operatorname{tr} T'T = \sum_{i \leq j}^m t_{ij}^2$.

Hence the density function of T is given by

$$\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{1}{2}n\right)} \prod_{i=1}^m (t_{ii}^2)^{(n-i-1)/2} h\left(\sum_{i \leq j}^m t_{ij}^2\right)(dT).$$

Finally, the distribution of T, takes the required form, which depends on the model and needs to reverse the series in order to see the particular non independent distributions of the elements of T. □

Following a kind comment of a Referee, if the generator function satisfies the property, $h(x+y) = h(x)h(y)$, then according to Hamel–Cauchy equation, only the Gaussian family case is obtained. The associated corollary was full derived by [1], see also [41, Theorem 3.2.14], which follows straightforwardly by taking $h(y) = (2\pi)^{-mn/2}\,\mathrm{e}^{-y/2}$.

4. THE GENERALIZED NON-CENTRAL F AND THE STUDENTIZED WISHART DISTRIBUTIONS

In this section we introduce the generalization of the classical non-central F distributions under elliptical models.

But first, we need a generalization of [5, Theorem 1], see [18] for notations and restrictions.

Theorem 15. *Let Z be a complex symmetric $m\times m$ matrix with $\mathrm{Re}(Z) > 0$ and let Y and W be symmetric $m\times m$ matrices. If $f = k + l$ is a nonnegative integer and ϕ is a partition of f, in the Kronecker product of partitions κ and λ of respective nonnegative integers k and l, then, for $\mathrm{Re}(a) > (m-1)/2$ and usual kernel elliptical generator $h(\cdot)$:*

$$\int\limits_{X>0} h(\mathrm{tr}\,XZ)|X|^{a-(m+1)/2}C_{\phi}^{\kappa,\lambda}(YX,WX)(dX)$$

$$= \frac{\Gamma_m(a)\int\limits_0^{\infty} h(w)w^{ma+f-1}dw}{\Gamma(ma+f)}|Z|^{-a}(a)_{\phi}C_{\phi}^{\kappa,\lambda}(YZ^{-1},WZ^{-1}). \tag{8}$$

Proof. Consider as usual $V = Z^{\frac{1}{2}}XZ^{\frac{1}{2}}$, which implies that $(dV) = |Z|^{(m+1)/2}(dX)$, then

$$\int\limits_{X>0} h(\mathrm{tr}\,XZ)|X|^{a-(m+1)/2}C_{\phi}^{\kappa,\lambda}(YX,WX)(dX)$$

$$= |Z|^{-a}\int\limits_{V>0} h(\mathrm{tr}\,V)|V|^{a-(m+1)/2}C_{\phi}^{\kappa,\lambda}(Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}}V, Z^{-\frac{1}{2}}WZ^{-\frac{1}{2}}V)(dV),$$

then, from [18, Eq. (2.4)], the last integral becomes

$$|Z|^{-a}\int\limits_{V>0} h(\mathrm{tr}\,V)|V|^{a-(m+1)/2}$$

$$\times \int\limits_{O(m)} C_{\phi}^{\kappa,\lambda}(H'Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}}HV, H'Z^{-\frac{1}{2}}WZ^{-\frac{1}{2}}HV)(dH)(dV)$$

$$= |Z|^{-a}\frac{C_{\phi}^{\kappa,\lambda}(YZ^{-1},WZ^{-1})}{C_{\phi}(I)}\int\limits_{V>0} h(\mathrm{tr}\,V)|V|^{a-(m+1)/2}C_{\phi}(V)(dV),$$

and applying [5, Theorem 1] we obtain the desired result. □

4.1. Case 1: $r \geq m$

Now, we can generalize the classical central F distribution (see for example [41, Theorem 10.4.1]) by considering $A \sim EW_m(r, I, \Omega, h)$ and $B \sim EW_m(n-p, I, g)$ independent, with $r \geq m$, $n-p \geq m$, this will be useful in the so called generalized MANOVA under elliptical models. The joint density function of A and B comes from Theorem 6 and Lemma 2, respectively

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]}|A|^{(r-m-1)/2}|B|^{(n-p-m-1)/2}$$

$$\times\, g\,(\operatorname{tr} B)\,{}_0P_1\left(h^{(2k)}\,(\operatorname{tr}(A+\Omega)) : \frac{1}{2}r; \Omega A\right)(dA)(dB).$$

The distribution of $F = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ is required, then consider this transformation and $U = A$, which implies that $(dA)(dB) = |U|^{\frac{1}{2}(m+1)}|F|^{-(m+1)}(dU)(dF)$; then the joint density function of F and U is given by

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]}|F|^{-\frac{1}{2}(n-p+m+1)}|U|^{\frac{1}{2}(n+r-p-m-1)}$$
$$\times\, g\left(\operatorname{tr} F^{-1}U\right){}_0P_1\left(h^{(2k)}\,(\operatorname{tr}(U+\Omega)) : \frac{1}{2}r; \Omega U\right)(dU)(dF).$$

Expanding $h^{(2k)}\,(\operatorname{tr}(U+\Omega))$ in terms of zonal polynomials, and expressing the resulting product in terms of invariant polynomials, see [18, Eq. (2.8)], we have

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]}|F|^{-\frac{1}{2}(n-p+m+1)}|U|^{\frac{1}{2}(n+r-p-m-1)}g\left(\operatorname{tr} F^{-1}U\right)$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{t=0}^{\infty}\frac{h^{(2k+t)}\,(\operatorname{tr}\Omega)}{t!}\sum_{\tau}\sum_{\kappa}\frac{1}{\left(\frac{1}{2}r\right)_{\kappa}}\sum_{\phi\in\kappa\cdot\tau}\theta_{\phi}^{\kappa,\tau}C_{\phi}^{\kappa,\tau}\left(\Omega U, U\right)(dU)(dF).$$

Then the marginal density of F has the form

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]}|F|^{-\frac{1}{2}(n-p+m+1)}$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{t=0}^{\infty}\frac{h^{(2k+t)}\,(\operatorname{tr}\Omega)}{t!}\sum_{\tau}\sum_{\kappa}\frac{1}{\left(\frac{1}{2}r\right)_{\kappa}}\sum_{\phi\in\kappa\cdot\tau}\theta_{\phi}^{\kappa,\tau}$$
$$\times\int\limits_{U>0} g\left(\operatorname{tr} F^{-1}U\right)|U|^{\frac{1}{2}(n+r-p-m-1)}C_{\phi}^{\kappa,\tau}\left(\Omega U, U\right)(dU)(dF).$$

Finally, the required density of the non-central generalized F distribution can be obtained via Theorem 15.

Theorem 16. *Let $A \sim EW_m(r, I, \Omega, h)$ and $B \sim EW_m(n-p, I, g)$ independent generalized non-central and central Wishart matrices, respectively; where $r \geq m$ and $n-p \geq m$. Then the density function of the generalized non-central matrix $F = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ has the form*

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}\Gamma_m\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]}|F|^{\frac{1}{2}(r-m-1)}$$
$$\times\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{h^{(2k+t)}\,(\operatorname{tr}\Omega)\int_0^{\infty} g(w)w^{\frac{1}{2}m(n+r-p)+f-1}dw}{k!t!\Gamma\left[\frac{1}{2}m(n+r-p)+f\right]}$$
$$\times\sum_{\kappa}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\frac{\theta_{\phi}^{\kappa,\tau}\left(\frac{1}{2}(n+r-p)\right)_{\phi}}{\left(\frac{1}{2}r\right)_{\kappa}}C_{\phi}^{\kappa,\tau}(\Omega F, F),$$

where $f = k+t$ and $\Omega = M'M$.

As a simple consequence, we get the classical non-central F distribution under a Gaussian model, fully derived by [41, Theorem 10.4.1].

Corollary 17. *Let A and B be independent, where $A \sim W_m(r, I, \Omega)$ and $B \sim W_m(n-p, I)$, where $r \geq m$, $n-p \geq m$. So the density function of $F = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ is*

$$\frac{\Gamma_m\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]}\frac{|F|^{\frac{1}{2}(r-m-1)}}{|I+F|^{\frac{1}{2}(n+r-p)}}\operatorname{etr}\left(-\frac{1}{2}\Omega\right){}_1F_1\left(\frac{1}{2}(n+r-p); \frac{1}{2}r; \frac{1}{2}\Omega F(I+F)^{-1}\right).$$

Proof. Here the proof is straightforward.

Consider $h(y) = \frac{1}{(2\pi)^{mr/2}} \mathrm{e}^{-\frac{1}{2}y}$ and $g(y) = \frac{1}{(2\pi)^{m(n-p)/2}} \mathrm{e}^{-\frac{1}{2}y}$ so that $h^{(2k+t)}(y) = \frac{1}{(2\pi)^{mr/2}} \left(-\frac{1}{2}\right)^{2k+t} \mathrm{e}^{-\frac{1}{2}y}$ and $\int_0^\infty g(w) w^{\frac{1}{2}m(n+r-p)+f-1} dw = 2^{\frac{1}{2}mr+f} \pi^{\frac{1}{2}m(p-n)} \Gamma\left[\frac{1}{2}m(n+r-p)+f\right]$.

Then the density of F takes the form

$$\frac{\Gamma_m\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]} |F|^{\frac{1}{2}(r-m-1)} \operatorname{etr}\left(-\frac{1}{2}\Omega\right)$$
$$\times \sum_{k=0}^\infty \sum_{t=0}^\infty \frac{(-1)^t 2^{-2k-t}}{k!t!} \sum_\kappa \sum_\tau \sum_{\phi\in\kappa\cdot\tau} \frac{\theta_\phi^{\kappa,\tau} 2^f \left(\frac{1}{2}(n+r-p)\right)_\phi}{\left(\frac{1}{2}r\right)_\kappa} C_\phi^{\kappa,\tau}(\Omega F, F).$$

Now, by [18, Eq. (2.6)], if $W \sim W_m(n,\Sigma)$ then $E_W[C_\kappa(XW)C_\lambda(YW)] = \sum_{\phi\in\kappa\cdot\lambda} 2^f (n/2)_\phi \theta_\phi^{\kappa,\lambda} \times C_\phi^{\kappa,\lambda}(X\Sigma, Y\Sigma)$, so we have that

$$\frac{\Gamma_m\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]} |F|^{\frac{1}{2}(r-m-1)} \operatorname{etr}\left(-\frac{1}{2}\Omega\right)$$
$$\times \sum_{k=0}^\infty \sum_{t=0}^\infty \frac{(-1)^t 2^{-2k-t}}{k!t!} \sum_\kappa \sum_\tau \frac{1}{\left(\frac{1}{2}r\right)_\kappa}$$
$$\times \int\limits_{W>0} \frac{\operatorname{etr}\left(-\frac{1}{2}F^{-1}W\right) |W|^{\frac{1}{2}((n+r-p)-m-1)} C_\kappa(\Omega W) C_\tau(W)}{2^{m(n+r-p)/2}\Gamma_m\left[\frac{1}{2}(n+r-p)\right] |F|^{\frac{1}{2}(n+r-p)}} (dW).$$

Noting that $\sum_{t=0}^\infty \frac{(-2)^{-t}}{t!} \sum_\tau C_\tau(W) = \operatorname{etr}\left(-\frac{1}{2}W\right)$, we get

$$\frac{2^{-m(n+r-p)/2}}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]} |F|^{\frac{1}{2}(p-n-m-1)} \operatorname{etr}\left(-\frac{1}{2}\Omega\right) \sum_{k=0}^\infty \frac{1}{k!} \sum_\kappa \frac{1}{\left(\frac{1}{2}r\right)_\kappa}$$
$$\times \int\limits_{W>0} \operatorname{etr}\left(-\frac{1}{2}\left(I+F^{-1}\right)W\right) |W|^{\frac{1}{2}((n+r-p)-m-1)} C_\kappa\left(\frac{1}{4}\Omega W\right) (dW)$$

and using [41, Theorem 7.2.7] we obtain the required result. □

It is important to note, that only some densities avoid the invariant polynomials and allows the reversal of the series as in the Gaussian case, so in general the non-central generalized F distribution is difficult to compute given that at present there are no algorithms for calculating invariant polynomials of higher degrees, see [3].

We can proceed as in Gaussian law [13], identifying in this case the distribution of the latent roots of a generalized non-central F matrix. It is based on a known result, see for example, [41, Theorem 3.2.17]. Applying that result in Theorem 16, we have that.

Theorem 18. *Let $A \sim EW_m(r, I, \Omega, h)$ and $B \sim EW_m(n-p, I, g)$ independent generalized non-central and central Wishart matrices, respectively; where $r \geq m$ and $n-p \geq m$. Then the joint density function of the latent roots $f_1 > \ldots > f_m > 0$ of $F = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ is*

$$\frac{\pi^{\frac{1}{2}m(m+n+r-p)}\Gamma_m\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]\Gamma_m\left(\frac{1}{2}m\right)} \prod_{i=1}^m f_i^{\frac{1}{2}(r-m-1)} \prod_{i<j}^m (f_i - f_j)$$
$$\times \sum_{k=0}^\infty \sum_{t=0}^\infty \frac{h^{(2k+t)}(\operatorname{tr}\Omega) \int\limits_0^\infty g(w) w^{\frac{1}{2}m(n+r-p)+f-1} dw}{k!t!\Gamma\left[\frac{1}{2}m(n+r-p)+f\right]}$$
$$\times \sum_\kappa \sum_\tau \sum_{\phi\in\kappa\cdot\tau} \frac{\left(\theta_\phi^{\kappa,\tau}\right)^2 \left(\frac{1}{2}(n+r-p)\right)_\phi}{\left(\frac{1}{2}r\right)_\kappa} \frac{C_\kappa(\Omega) C_\phi(L)}{C_\kappa(I)},$$

where $L = \operatorname{diag}(f_1, \cdots, f_m)$.

Proof. The density follows from the expression

$$\frac{\pi^{m^2/2}}{\Gamma_m\left(\frac{1}{2}m\right)} \prod_{i<j}^{m} (f_i - f_j) \frac{\pi^{\frac{1}{2}m(n+r-p)} \Gamma_m\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_m\left[\frac{1}{2}r\right]\Gamma_m\left[\frac{1}{2}(n-p)\right]} |L|^{\frac{1}{2}(r-m-1)}$$
$$\times \sum_{k=0}^{\infty}\sum_{t=0}^{\infty} \frac{h^{(2k+t)}\left(\operatorname{tr}\Omega\right) \int_0^{\infty} g(w) w^{\frac{1}{2}m(n+r-p)+f-1} dw}{k!t!\Gamma\left[\frac{1}{2}m(n+r-p)+f\right]}$$
$$\times \sum_{\kappa}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau} \frac{\theta_{\phi}^{\kappa,\tau}\left(\frac{1}{2}(n+r-p)\right)_{\phi}}{\left(\frac{1}{2}r\right)_{\kappa}} C_{\phi}^{\kappa,\tau}(\Omega, I) C_{\phi}(L)/C_{\phi}(I),$$

where we have used [18, Eq. (2.4)]. And recalling that $C_{\phi}^{\kappa,\tau}(\Omega, I) = [\theta_{\phi}^{\kappa,\tau} C_{\phi}(I)/C_{\kappa}(I)]C_{\kappa}(\Omega)$, we obtain the desired joint density. □

The Gaussian case of [41, Theorem 10.4.2] can be derived by understanding that the series of invariant polynomials can be reversed as it was done in the proof of Corollary 8 then the hypergeometric series of two arguments appears by applying [41, Theorems 3.2.17 and 7.3.3].

4.2. Case 2: $r < m$

The natural way of obtaining this result comes from considering a connection with the so called generalized inverted Wishart. In order to set the open problem under this consideration, we propose the following alternative proof of [32, Eq. (72)]:

Lemma 19. *Let* $X \sim N_{r\times m}(M, I_r \otimes \Sigma)$ *and* $Y \sim N_{(n-p)\times m}(0, I_{n-p} \otimes \Sigma)$ *independent, with* $n - p \geq m \geq r$, *then the distribution of* $F = X(Y'Y)^{-1}X'$ *is*

$$\frac{\Gamma\left[\frac{1}{2}(n+r-p)\right] \operatorname{etr}\left(-\frac{1}{2}M'M\Sigma^{-1}\right)}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n+r-p-m)\right]} |I+F|^{-\frac{1}{2}(n+r-p)} |F|^{-\frac{1}{2}(r-m+1)}$$
$$\times {}_1F_1\left(\frac{1}{2}(n+r-p); \frac{1}{2}m; \frac{1}{2}M\Sigma^{-1}M'\left(I+F^{-1}\right)^{-1}\right). \tag{9}$$

Proof. The first part of the proof is essentially the same of [41, Theorem 10.4.4]. Transform the distributions in order to obtain a Kronecker product of identities as the covariance matrices; i.e., consider $U = X\Sigma^{-\frac{1}{2}}$ and $V = Y\Sigma^{-\frac{1}{2}}$, then $F = X(Y'Y)^{-1}X' = U(V'V)^{-1}U'$, where $U \sim N_{r\times m}(M\Sigma^{-\frac{1}{2}}, I_r \otimes I_m)$ and $V \sim N_{(n-p)\times m}(0, I_{n-p} \otimes I_m)$ are independent. Performing the QR decomposition of U', we have

$$U' = H_1 T = [H_1|H_2]\begin{bmatrix} T \\ 0 \end{bmatrix} = H\begin{bmatrix} T \\ 0 \end{bmatrix}, \tag{10}$$

where $H_1 : m \times r$ with orthogonal columns, i.e., $H_1'H_1 = I_r$; H_2 is any $m \times (m-r)$ such that $H = [H_1|H_2]$ is orthogonal; and $T : r \times r$ is upper triangular. So, if $Z = VH$, which implies that $Z'Z$ is $W_m(n-p, I_m)$, then matrix F can be expressed in terms of T and Z as follows:

$$F = U(V'V)^{-1}U' = [T'|0]H'(V'V)^{-1}H\begin{bmatrix} T \\ 0 \end{bmatrix} = [T'|0](Z'Z)^{-1}\begin{bmatrix} T \\ 0 \end{bmatrix} = T'B^{-1}T, \tag{11}$$

where $B^{-1} : r \times r$ is the matrix constituted by the first r rows and columns of $(Z'Z)^{-1}$.

Now [41, p. 452], considers that $Z'Z$ is $W_m(n-p, I_m)$ so that B is $W_m(n+r-p-m, I_r)$ and then the distribution of F is obtained.

We propose the following alternative approach based on the density of $(Z'Z)^{-1}$, but this deserves some details which we explain shortly.

- (i) Recall that the distribution of an inverted Wishart $m \times m$ matrix $A > 0$, with $n > 2m$ degrees of freedom and parameter matrix $V > 0$ (which is denoted by $A \sim W_m^{-1}(n, V)$) is given by

$$\frac{2^{-\frac{m}{2}(n-m-1)}|V|^{\frac{1}{2}(n-m-1)}}{\Gamma\left[\frac{1}{2}(n-m-1)\right]|A|^{\frac{n}{2}}} \operatorname{etr}\left(-\frac{1}{2}A^{-1}V\right),$$

so it is easy to prove that if $A \sim W_m(n, \Sigma)$ then $A^{-1} \sim W_m^{-1}\left(n+m+1, \Sigma^{-1}\right)$.

- (ii) Now, from [41, Theorem 3.2.10], if $A \sim W_m(n, \Sigma)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, then $A_{11.2} \sim W_k(n-m+k, \Sigma_{11.2})$, where A_{11} and Σ_{11} are $k \times k$ matrices.

- (iii) Thus, we can use the above result and the fact that $B_{11} = A_{11.2}^{-1}$ in order to prove that: if $B \sim W_m(n, V)$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, then $B_{11} \sim W_k^{-1}(n-2m+2k, V_{11})$, where B_{11} and V_{11} are $k \times k$ matrices.

Assembling these facts in our problem, we have that $Z'Z \sim W_m(n-p, I_m)$ implies that $(Z'Z)^{-1} \sim W_m^{-1}(n-p+m+1, I_m)$, and noting that B^{-1} in (11) is formed with the first r rows and columns of $(Z'Z)^{-1}$, then

$$B^{-1} \sim W_r^{-1}(n-p-m+2r+1, I_r).$$

So the joint density function of $B^{-1} = C$ and U is

$$\frac{2^{-\frac{1}{2}r(n-p-m+r)}(2\pi)^{-\frac{1}{2}mr}}{\Gamma_r\left[\frac{1}{2}(n+r-p-m)\right]}|C|^{-\frac{1}{2}(n-p-m+2r+1)} \operatorname{etr}\left[-\frac{1}{2}C^{-1}\right]$$
$$\times \operatorname{etr}\left[-\frac{1}{2}(U-M\Sigma^{-\frac{1}{2}})'(U-M\Sigma^{-\frac{1}{2}})\right](dU)(dC).$$

From (10), $U' = H_1T$ then $(dU) = 2^{-r}|T'T|^{\frac{1}{2}(m-r-1)}(d(T'T))(H_1'dH_1)$, thus the joint density function becomes

$$\frac{2^{-\frac{1}{2}r(n-p-m+r)-r}\operatorname{etr}\left(-\frac{1}{2}M'M\Sigma^{-1}\right)}{(2\pi)^{\frac{1}{2}mr}\Gamma_r\left[\frac{1}{2}(n+r-p-m)\right]}|C|^{-\frac{1}{2}(n-p-m+2r+1)} \operatorname{etr}\left(-\frac{1}{2}C^{-1}\right)$$
$$\times \operatorname{etr}\left(-\frac{1}{2}T'T\right)\operatorname{etr}\left(H_1TM\Sigma^{-\frac{1}{2}}\right)|T'T|^{\frac{1}{2}(m-r-1)}(dC)(d(T'T))(H_1'dH_1).$$

Integration over the Stiefel manifold $V_{r,m}$ (see [41, p. 443]) is given by

$$\int_{V_{r,m}} \operatorname{etr}\left(H_1TM\Sigma^{-\frac{1}{2}}\right)(H_1'dH_1) = \frac{2^r\pi^{\frac{1}{2}mr}}{\Gamma_r\left(\frac{1}{2}m\right)}{}_0F_1\left(\frac{1}{2}m; \frac{1}{4}M\Sigma^{-1}M'T'T\right),$$

and replacing $G = T'T$, we have that the joint density function of C and G is

$$\frac{2^{-\frac{1}{2}r(n+r-p)}\operatorname{etr}\left(-\frac{1}{2}M'M\Sigma^{-1}\right)}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n+r-p-m)\right]}|C|^{-\frac{1}{2}(n-p-m+2r+1)} \operatorname{etr}\left(-\frac{1}{2}C^{-1}\right)$$
$$\times \operatorname{etr}\left(-\frac{1}{2}G\right)|G|^{\frac{1}{2}(m-r-1)}{}_0F_1\left(\frac{1}{2}m; \frac{1}{4}M\Sigma^{-1}M'T'T\right)(dC)(dG).$$

And taking $F = T'CT$, see (11), then $(dC) = |T'T|^{-\frac{1}{2}(r+1)}(dF) = |G|^{-\frac{1}{2}(r+1)}(dF)$ we have that the joint density of F and G is

$$\frac{2^{-\frac{1}{2}r(n+r-p)}}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n+r-p-m)\right]}\operatorname{etr}\left(-\frac{1}{2}M'M\Sigma^{-1}\right)|F|^{-\frac{1}{2}(n-p-m+2r+1)}$$
$$\times \operatorname{etr}\left[-\frac{1}{2}(I+F^{-1})G\right]|G|^{\frac{1}{2}(n-p-1)}{}_0F_1\left(\frac{1}{2}m;\frac{1}{4}M\Sigma^{-1}M'G\right)(dG)(dF).$$

And using [41, Theorem 7.2.7] for the integration over $G > 0$ we get the required result. □

The generalization of the above lemma demands the solution of items (i), (ii) and (iii) in the context of the so called generalized inverted Wishart distribution.

- [i] Clearly, the item (i) is easily extended, i.e., from Lemma 2, if A is $EW_m(n,\Sigma,h)$ with $n \geq m$ then the density function of A is

$$\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{1}{2}n\right)|\Sigma|^{n/2}}|A|^{(n-m-1)/2}h\left(\operatorname{tr}\Sigma^{-1}A\right)(dA).$$

  Let $B = A^{-1}$ which implies that $(dA) = |B|^{-(m+1)}(dB)$, so

$$\frac{\pi^{mn/2}}{\Gamma_m\left(\frac{1}{2}n\right)|\Sigma|^{n/2}}|B|^{-(n+m+1)/2}h\left(\operatorname{tr}\Sigma^{-1}B^{-1}\right)(dB). \tag{12}$$

  And by replacing the parameter we say that the $m\times m$ positive definite matrix $B > 0$ has the generalized inverted Wishart distribution with n degrees of freedom, $n > 2m$, and $m \times m$ parameter matrix $V > 0$, if its density function is given by

$$\frac{\pi^{-\frac{m}{2}(n-m-1)}|V|^{\frac{1}{2}(n-m-1)}}{\Gamma\left[\frac{1}{2}(n-m-1)\right]|B|^{\frac{n}{2}}}h\left(\operatorname{tr}A^{-1}V\right). \tag{13}$$

  This will be denoted by $B \sim W_m^{-1}(n,V)$. And by (12) we have the required generalization of

- (i): if $A \sim EW_m(n,\Sigma,h)$ then $A^{-1} \sim W_m^{-1}(n+m+1,\Sigma^{-1})$.

- [iii] Now, the main obstacle here is the generalization of item (iii) (and then to obtain the density of F when $r < m$) because it requires the computation of the characteristic function of a $A^{-1} \sim W_m^{-1}(n,V)$, but at present, even in the simplest case, the Gaussian model, the characteristic function of an inverted Wishart is not available in the literature.

- [ii] One way of avoiding that problem consists in the generalization of item (ii), which we will study in Lemma 20.

Lemma 20. *If* $A \sim EW_m(n,\Sigma,h)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$, $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, *and* $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, *where* A_{11} *and* Σ_{11} *are* $k \times k$ *matrices; then the density function of* $A_{11.2} > 0$ *is given by*

$$\frac{\pi^{\frac{1}{2}mn}|A_{11.2}|^{\frac{1}{2}(n-m-1)}}{\Gamma_k\left[\frac{1}{2}(n-m+k)\right]|\Sigma_{11.2}|^{\frac{1}{2}(n-m+k)}}{}_0P_0\left(\frac{\int_0^\infty f^{(t)}(y)y^{\frac{1}{2}n(m-k)-1}dy}{\Gamma\left[\frac{1}{2}n(m-k)\right]}:\Sigma_{11.2}^{-1}A_{11.2}\right),$$

where ${}_0P_0 = \sum_{t=0}^{\infty}\frac{g(t,X)}{t!}\sum_\tau C_\tau(X)$ *and* $f(y)$ *is some analytic function depending on the generator function* $h(y)$ *and* t*, such that*

$$f^{(t)}(0)$$

$$= \int\limits_{A_{12}\in\Re^{k(m-k)}} \frac{h^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12})$$

Proof. By [41, Eqs. (9) and (10), p. 94] we have that the joint density function of $A_{11.2}$, A_{12}, and A_{22} is given by

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m-1)}|A_{11.2}|^{\frac{1}{2}(n-m+k-k-1)}}{\pi^{\frac{1}{2}k(m-k)}\Gamma_k\left[\frac{1}{2}(n-m+k)\right]\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}|\Sigma_{11.2}|^{\frac{1}{2}(n-m+k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}}$$
$$\times\, h\left(\operatorname{tr}\left[\Sigma_{11.2}^{-1}A_{11.2}+A_{22}\Sigma_{22}^{-1}+\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right]\right).$$

This general density does not factor in functions of $A_{11.2}$, A_{12}, and A_{22}, as in the Gaussian case, but we can obtain the required marginal from the joint density function of $A_{22}>0$ and $A_{11.2}>0$, which is calculated by the following integration over $A_{12}\in\Re^{k(m-k)}$ given $A_{11.2}$ and A_{22}.

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m+k-1)}|A_{11.2}|^{\frac{1}{2}(n-m-1)}}{\Gamma_k\left[\frac{1}{2}(n-m+k)\right]\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}|\Sigma_{11.2}|^{\frac{1}{2}(n-m+k)}}$$
$$\times\ {}_0P_0\left(H(t,h):\Sigma_{11.2}^{-1}A_{11.2}+A_{22}\Sigma_{22}^{-1}\right), \tag{14}$$

where we have used the notation for the generalized hypergeometric series ${}_0P_0=\sum_{t=0}^{\infty}\frac{g(t,X)}{t!}\times\sum_{\tau}C_{\tau}(X)$, and

$$H(t,h)=\int\limits_{A_{12}\in\Re^{k(m-k)}} \frac{h^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12}). \tag{15}$$

Given $A_{11.2}$ and A_{22}, the integrating over $A_{12}\in\Re^{t(m-t)}$ could be possible once the elliptical model $h(y)$ is set. Moreover, the function $h^{(t)}(y)$ has the same kernel of the generator function $h(y)$, this can be inferred by applying the so called Faà di Bruno's formula (also derived by using the termed C-arrays, [15]). Let h and g have derivatives of all orders, if $w^{(k)}$ denotes $\frac{d^k w(t)}{dt^k}$ then

$$[h(g(t))]^{(k)}=\sum_{\kappa=(k^{\nu_k},(k-1)^{\nu_{k-1}},\cdots,3^{\nu_3},2^{\nu_2},1^{\nu_1})}\frac{k!}{\prod_{i=1}^{k}\nu_i!(i!)^{\nu_i}}h^{(\sum_{i=1}^{k}\nu_i)}\prod_{i=1}^{k}(g^{(i)})^{\nu_i}, \tag{16}$$

where the sum runs over all the partitions κ of k.

Note that the functions $h(y)$ considered in the elliptically contoured distributions admit Taylor expansions then the above expressions always exist for all positive integer t.

Thus, in (15), $\frac{h^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}$ has the same kernel of the elliptical generator function $h(y)$, and for any t, integration over A_{12} is a constant, say $H(k,h)$, (depending on t and $h(y)$, but independent of $A_{11.2}$ and $A_{22}>0$), compare with Definition (1).

Now, given $H(k,f)$, we can study the existence of analytic function $f(y)$ such that

$$f^{(t)}(0)=H(k,f); \tag{17}$$

then the joint density function of A_{22} and $A_{11.2}$, given in (14) can be written as

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m+k-1)}|A_{11.2}|^{\frac{1}{2}(n-m-1)}}{\Gamma_k\left[\frac{1}{2}(n-m+k)\right]\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}|\Sigma_{11.2}|^{\frac{1}{2}(n-m+k)}}$$
$$\times\ {}_0P_0\left(f^{(t)}\left(\operatorname{tr}A_{22}\Sigma_{22}^{-1}\right):\Sigma_{11.2}^{-1}A_{11.2}\right). \tag{18}$$

And for the integration over $A_{22}>0$ we use the following result of [5, Corollary 2]:

$$\int\limits_{X>0} h(\operatorname{tr}XZ)|X|^{a-(m+1)/2}(dX)=\frac{|Z|^{-a}\Gamma_m(a)}{\Gamma(ma)}\int\limits_0^{\infty}h(y)y^{ma-1}dy, \tag{19}$$

thus we get the density function of $A_{11.2}$ as :

$$\frac{\pi^{\frac{1}{2}mn}|A_{11.2}|^{\frac{1}{2}(n-m-1)}}{\Gamma_k\left[\frac{1}{2}(n-m+k)\right]|\Sigma_{11.2}|^{\frac{1}{2}(n-m+k)}} \times \sum_{t=0}^{\infty}\frac{\int\limits_0^{\infty} f^{(t)}(y)y^{\frac{1}{2}n(m-k)-1}dy}{t!\Gamma\left[\frac{1}{2}n(m-k)\right]}\sum_{\tau}C_{\tau}\left(\Sigma_{11.2}^{-1}A_{11.2}\right).$$

□

At this point, we insist that the apparent simplified version of the above series indexed by $H(k,h)$ depends on the solution of (15) and (17), which can be problematic, according to the selected elliptical law $h(y)$.

However in some models the simplification is straightforward, as is the case of Gaussian law, see [41, Theorem 3.2.10(i)].

Corollary 21. *In the context of Lemma* 20, *if the generator function takes the form* $h(y) = \frac{1}{(2\pi)^{\frac{1}{2}mn}}\,\mathrm{e}^{-\frac{1}{2}y}$, *then*

$$A_{11.2} \sim W_k(n-m+k, \Sigma_{11.2}). \tag{20}$$

Proof. Gaussian law implies that $h^{(t)}(y) = \left(-\frac{1}{2}\right)^t \frac{1}{(2\pi)^{\frac{1}{2}mn}}\,\mathrm{e}^{-\frac{1}{2}y}$, then

$$f^{(t)}(0) = H(t,h) = \frac{\pi^{-\frac{1}{2}mn}}{2^{\frac{1}{2}[mn-k(m-k)]}}\left(-\frac{1}{2}\right)^t \times \int\limits_{\Re^{k(m-k)}} \frac{\mathrm{etr}\left(-\frac{1}{2}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{(2\pi)^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12}).$$

So $f^{(t)}(0) = \pi^{-\frac{1}{2}mn}2^{-\frac{1}{2}[mn-k(m-k)]}\left(-\frac{1}{2}\right)^t$ and $f^{(t)}(y) = \pi^{-\frac{1}{2}mn}2^{-\frac{1}{2}[mn-k(m-k)]}\left(-\frac{1}{2}\right)^t \mathrm{e}^{-\frac{1}{2}y}$, thus $\int_0^{\infty} f^{(t)}(y)y^{\frac{1}{2}n(m-k)-1}dy = \pi^{-\frac{1}{2}mn}2^{-\frac{1}{2}k(n-m+k)}\left(-\frac{1}{2}\right)^t\Gamma\left[\frac{1}{2}n(m-k)\right]$. Finally the density of $A_{11.2}$ is given by:

$$\frac{2^{-\frac{1}{2}k(n-m+k)}|A_{11.2}|^{\frac{1}{2}(n-m-1)}}{\Gamma_k\left[\frac{1}{2}(n-m+k)\right]|\Sigma_{11.2}|^{\frac{1}{2}(n-m+k)}}\,\mathrm{etr}\left(-\frac{1}{2}\Sigma_{11.2}^{-1}A_{11.2}\right),$$

which is the desired density, see [41, Theorem 3.2.10(i)]. □

Remark 22. The density function of A_{22} can be obtained in the same way.

Lemma 23. *If* $A \sim EW_m(n,\Sigma,h)$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$, $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ *and* $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, *where* A_{11}, *and* Σ_{11} *are* $k \times k$ *matrices; then the density function of* $A_{22} > 0$ *is given by*

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m+k-1)}}{\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}}\,{}_0P_0\left(F(t,f) : A_{22}\Sigma_{22}^{-1}\right),$$

where ${}_0P_0 = \sum_{t=0}^{\infty}\frac{g(t,X)}{t!}\sum_{\tau}C_{\tau}(X)$ *and* $f(y)$ *is some analytic function depending on the generator function* $h(y)$ *and* t, *such that*

$$F(t,f) = \int\limits_{A_{12}\in\Re^{k(m-k)}} \frac{f^{(t)}\left(\mathrm{tr}\,\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12}).$$

Proof. First we integrate over $A_{11.2} > 0$ by using (19), thus we have:

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m-1)}}{\pi^{\frac{1}{2}k(m-k)}\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}}$$

$$\times \sum_{t=0}^{\infty} \frac{\int\limits_0^{\infty} h^{(t)}(y) y^{\frac{1}{2}k(n-m+k)-1} dy}{t!\Gamma\left[\frac{1}{2}k(n-m+k)\right]}$$

$$\times \sum_{\tau} C_{\tau}\left(A_{22}\Sigma_{22}^{-1} + \Sigma_{11.2}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)$$

or in the notation of the generalized hypergeometric series of [5] we can write it as

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m-1)}}{\pi^{\frac{1}{2}k(m-k)}\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}} {}_0P_0\left(\frac{\int\limits_0^{\infty} h^{(t)}(y) y^{\frac{1}{2}k(n-m+k)-1} dy}{\Gamma\left[\frac{1}{2}k(n-m+k)\right]} : X\right);$$

where $X = A_{22}\Sigma_{22}^{-1} + \Sigma_{11.2}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})'$. Recalling that $h(\cdot)$ is a generator function, which implies that $\int_0^{\infty} h^{(t)}(y) y^{\frac{1}{2}mn-1} dy < \infty$, then for a given elliptical model $h(y)$ we can study the existence of an analytic function $f(z)$ such that

$$f^{(t)}(0) = \frac{\int\limits_0^{\infty} h^{(t)}(y) y^{\frac{1}{2}k(n-m+k)-1} dy}{\Gamma\left[\frac{1}{2}k(n-m+k)\right]}, \tag{21}$$

then the joint density of A_{12} and A_{22} becomes

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m+k-1)}}{\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}}$$

$$\times {}_0P_0\left(\frac{f^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}} : A_{22}\Sigma_{22}^{-1}\right).$$

Given A_{22}, the integration over $A_{12} \in \Re^{t(m-t)}$ could be possible once the elliptical model is set.

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m+k-1)}}{\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}}$$

$$\times {}_0P_0\left(\int\limits_{A_{12}\in\Re^{k(m-k)}} \frac{f^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12} - \Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12}) : A_{22}\Sigma_{22}^{-1}\right).$$

Moreover, according to the definition of $f^{(t)}(0) = \frac{\int_0^{\infty} h^{(t)}(y) y^{\frac{1}{2}k(n-m+k)-1} dy}{\Gamma\left[\frac{1}{2}k(n-m+k)\right]}$, the function $f(y)$ has the same kernel of the generator function $h(y)$, see (16), and providing that the generator function $h(y)$ admits a Taylor expansion.

Thus $\frac{f^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}$ has the same kernel of the elliptical generator function $h(y)$, and for any t, integration over A_{12} is a constant, say $F(t, f)$, (depending on t and $f(y)$, but independent of $A_{22} > 0$), compare with Definition (1).

Finally, we get that the density function of A_{22} is

$$\frac{\pi^{\frac{1}{2}mn}|A_{22}|^{\frac{1}{2}(n-m+k-1)}}{\Gamma_{m-k}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{\frac{1}{2}n}} {}_0P_0\left(F(t,f) : A_{22}\Sigma_{22}^{-1}\right),$$

where

$$F(t,f)=\int\limits_{A_{12}\in\Re^{k(m-k)}}\frac{f^{(t)}\left(\operatorname{tr}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{\pi^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12}). \tag{22}$$

□

Note again, that the above density has a general form indexed by $F(k,f)$ which depends on the particular model, and perhaps the solution of (21) and (22) could be difficult to calculate. However, the density based on Gaussian law, which seems to be the only one studied in the literature, is obtained in a trivial way:

Corollary 24. *In the Gaussian case.*

$$A_{22}\sim W_{m-k}(n,\Sigma_{22})$$

Proof. As usual, take $h(y)=\frac{1}{(2\pi)^{\frac{1}{2}mn}}\mathrm{e}^{-\frac{1}{2}y}$, so $h^{(t)}(y)=\left(-\frac{1}{2}\right)^t\frac{1}{(2\pi)^{\frac{1}{2}mn}}\mathrm{e}^{-\frac{1}{2}y}$, then $f^{(t)}(0)=\pi^{-\frac{1}{2}mn}2^{\frac{1}{2}k(n-m+k)-\frac{1}{2}mn}\left(-\frac{1}{2}\right)^t$ and $f^{(t)}(y)=\pi^{-\frac{1}{2}mn}2^{\frac{1}{2}k(n-m+k)-\frac{1}{2}mn}\left(-\frac{1}{2}\right)^t\mathrm{e}^{-\frac{1}{2}y}$, thus

$$F(t,f)=\frac{\pi^{-\frac{1}{2}mn}}{2^{\frac{1}{2}n(m-k)}}\left(-\frac{1}{2}\right)^t$$

$$\times\int\limits_{\Re^{k(m-k)}}\frac{\operatorname{etr}\left(-\frac{1}{2}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{(2\pi)^{\frac{1}{2}k(m-k)}|\Sigma_{11.2}|^{\frac{1}{2}(m-k)}|A_{22}|^{\frac{1}{2}k}}(dA_{12}).$$

Finally the density of A_{22} is

$$\frac{|A_{22}|^{\frac{1}{2}(n-m+k-1)}}{2^{\frac{1}{2}n(m-k)}\Gamma_{m-k}\left(\frac{1}{2}n\right)|A_{22}|^{\frac{1}{2}n}}{}_0P_0\left(\left(-\frac{1}{2}\right)^t:A_{22}\Sigma_{22}^{-1}\right).$$

As it was required, see [41, Theorem 3.2.10(iii)]. □

Finally, with Lemma 20 we can obtain the desired distribution of F when $r<m$. The Gaussian version is sometimes termed the "studentized Wishart" distribution.

Theorem 25. *Let $X\sim E_{r\times m}(M,I_r\otimes\Sigma,h)$ and $Y\sim E_{(n-p)\times m}(0,I_{n-p}\otimes\Sigma,h)$ independent, with $n-p\geq m\geq r$, then the distribution of $F=X(Y'Y)^{-1}X'$ is*

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}\Gamma_r\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}|F|^{-\frac{1}{2}(n-p-m+2r+1)}$$

$$\times\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\tau}\sum_{\kappa}\sum_{\sigma}\sum_{\phi\in\tau\cdot\kappa\cdot\sigma}\frac{\theta_\phi^{\tau,\kappa,\sigma}\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{t!k!s!\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]\left(\frac{1}{2}m\right)_\kappa}$$

$$\times\frac{\int\limits_0^\infty h^{(2k+s)}(w)w^{\frac{1}{2}r(n+r-p)+f-1}dw}{\Gamma\left[\frac{1}{2}r(n+r-p)+f\right]}\left(\frac{1}{2}(n+r-p)\right)_\phi$$

$$\times C_\phi^{\tau,\kappa,\sigma}\left(F^{-1},M'M\Sigma^{-1},M'M\Sigma^{-1}\right);$$

where $f^{(t)}(y)$ (see Lemma 20) and $h^{(2k+s)}(w)$ are defined under $m\times(n-p)$ and $m\times r$ elliptical models, respectively.

Proof. The first steps of the proof are basically the same of those given from the beginning of the proof of Lemma 19 until Eq. (11), but written in terms of elliptical models indexed by the generator function $h(y)$.

Start with $X\sim E_{r\times m}(M,I_r\otimes\Sigma,h)$ and $Y\sim E_{(n-p)\times m}(0,I_{n-p}\otimes\Sigma,h)$ independent, with $n-p\geq m\geq r$, and take $U=X\Sigma^{-\frac{1}{2}}$ and $V=Y\Sigma^{-\frac{1}{2}}$, then the density of $F=X(Y'Y)^{-1}X'=U(V'V)^{-1}U'$

is required. Here $U \sim E_{r\times m}(M\Sigma^{-\frac{1}{2}}, I_r \otimes I_m, h)$ and $V \sim E_{(n-p)\times m}(0, I_{n-p} \otimes I_m, h)$ (independent). Then

$$U' = H_1 T = [H_1|H_2]\begin{bmatrix} T \\ 0 \end{bmatrix} = H\begin{bmatrix} T \\ 0 \end{bmatrix}, \tag{23}$$

where $H_1 : m \times r$ with orthogonal columns, i.e., $H_1'H_1 = I_r$; H_2 is any $m \times (m-r)$ such that $H = [H_1|H_2]$ is orthogonal; and $T : r \times r$ is upper triangular. Replacing $Z = VH$, then $Z'Z$ is $EW_m(n-p, I_m, h)$, and F takes the form:

$$F = U(V'V)^{-1}U' = [T'|0]H'(V'V)^{-1}H\begin{bmatrix} T \\ 0 \end{bmatrix} = [T'|0](Z'Z)^{-1}\begin{bmatrix} T \\ 0 \end{bmatrix} = T'B^{-1}T, \tag{24}$$

where B^{-1} is formed with the first r rows and columns of the $m \times m$ matrix $(Z'Z)^{-1}$.

Then, by Lemma 20, the joint distribution of B and U

$$\frac{\pi^{\frac{1}{2}m(n-p)}|B|^{\frac{1}{2}(n-p-m-1)}}{\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]} : B\right)$$
$$\times\, h\left[(U - M\Sigma^{-\frac{1}{2}})'(U - M\Sigma^{-\frac{1}{2}})\right].$$

As in the Gaussian case, from (23), $U' = H_1T$ so that $(dU) = 2^{-r}|T'T|^{\frac{1}{2}(m-r-1)}(d(T'T))(H_1'dH_1)$, then the joint density function turns after expanding $h(\cdot)$ in Taylor series, as

$$\frac{\pi^{\frac{1}{2}m(n-p)}|B|^{\frac{1}{2}(n-p-m-1)}}{\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]} : B\right)$$
$$\times \sum_{k=0}^{\infty}\frac{h^{(k)}\left(\operatorname{tr}\left(T'T + M'M\Sigma^{-1}\right)\right)}{k!}\left[\operatorname{tr}\left(-2H_1TM\Sigma^{-\frac{1}{2}}\right)\right]^k,$$

and integration over $V_{m,n}$ by the same procedure exposed in the proof of Theorem 6, we obtain:

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}|B|^{\frac{1}{2}(n-p-m-1)}|T'T|^{\frac{1}{2}(m-r-1)}}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]} : B\right)$$
$$\times \sum_{k=0}^{\infty}\frac{h^{(2k)}\left(\operatorname{tr}\left(T'T + M'M\Sigma^{-1}\right)\right)}{k!}\sum_{\kappa}\frac{C_\kappa\left(M\Sigma^{-1}M'T'T\right)}{\left(\frac{1}{2}m\right)_\kappa}(d(T'T))(dB),$$

and replacing $G = T'T$, we have that the joint density function of B and G is

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}|B|^{\frac{1}{2}(n-p-m-1)}|G|^{\frac{1}{2}(m-r-1)}}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]} : B\right)$$
$$\times\, {}_0P_1\left(h^{(2k)}\left(\operatorname{tr}\left(G + M'M\Sigma^{-1}\right)\right) : \frac{1}{2}m; M\Sigma^{-1}M'G\right)(dG)(dB).$$

And taking $F = T'B^{-1}T$, see (24), then $(dB) = |G|^{\frac{1}{2}(r+1)}|F|^{-(r+1)}(dF)$ we have that the joint density of F and G is

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}|F|^{-\frac{1}{2}(n-p-m+2r+1)}|G|^{\frac{1}{2}(n-p-1)}}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}$$

$$\times\,{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]}:F^{-1}G\right)\sum_{k=0}^\infty\frac{1}{k!}\sum_{s=0}^\infty\frac{h^{(2k+s)}\left(\operatorname{tr}G\right)}{s!}$$
$$\times\sum_\kappa\frac{1}{\left(\frac{1}{2}m\right)_\kappa}C_\kappa(M\Sigma^{-1}M'G)\sum_\sigma C_\sigma\left(M'M\Sigma^{-1}\right)(dG)(dF);$$

which can be written in terms of invariant polynomials by noting that $C_\kappa(X)C_\lambda(Y)C_\tau(Z) = \sum_{\phi\in\kappa\cdot\lambda\cdot\tau}\theta_\phi^{\kappa,\lambda,\tau}C_\phi^{\kappa,\lambda,\tau}(X,Y,Z)$ (see [12]), i.e.,

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}|F|^{-\frac{1}{2}(n-p-m+2r+1)}|G|^{\frac{1}{2}(n+r-p-r-1)}}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}$$
$$\times\sum_{t=0}^\infty\sum_{k=0}^\infty\sum_{s=0}^\infty\sum_\tau\sum_\kappa\sum_\sigma\sum_{\phi\in\tau\cdot\kappa\cdot\sigma}\frac{\theta_\phi^{\tau,\kappa,\sigma}\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{t!k!s!\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]\left(\frac{1}{2}m\right)_\kappa}$$
$$\times\,h^{(2k+s)}\left(\operatorname{tr}G\right)C_\phi^{\tau,\kappa,\sigma}\left(F^{-1}G, M\Sigma^{-1}M'G, M'M\Sigma^{-1}\right)(dG)(dF).$$

Finally, integration over $G>0$ gives the required density function of F by a suitable modification of Theorem 15 via [12]. □

Considering the Gaussian law, reversing the series of invariant polynomials and using the computations in the proof of Corollary 21, we have the classical version of the non-central F distribution of [32, Eq. (72)] (see also [41, Theorem 10.4.4]).

Corollary 26. *Let $X\sim N_{r\times m}(M, I_r\otimes\Sigma)$ and $Y\sim N_{(n-p)\times m}(0, I_{n-p}\otimes\Sigma)$ independent, with $n-p\ge m\ge r$, then the distribution of $F=X(Y'Y)^{-1}X'$ is*

$$\frac{\Gamma\left[\frac{1}{2}(n+r-p)\right]\operatorname{etr}\left(-\frac{1}{2}M'M\Sigma^{-1}\right)}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n+r-p-m)\right]}|I+F|^{-\frac{1}{2}(n+r-p)}|F|^{-\frac{1}{2}(r-m+1)}$$
$$\times\,{}_1F_1\left(\frac{1}{2}(n+r-p);\frac{1}{2}m;\frac{1}{2}M\Sigma^{-1}M'\left(I+F^{-1}\right)^{-1}\right). \tag{25}$$

Proof. This case is trivial and we do not need the series in terms of invariant polynomials, so we just reverse the series until the substitution $F=T'B^{-1}T$ is performed, i.e., the joint density of F and G is

$$\frac{\pi^{\frac{1}{2}m(n+r-p)}|F|^{-\frac{1}{2}(n-p-m+2r+1)}|G|^{\frac{1}{2}(n-p-1)}}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}$$
$$\times\,{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]}:F^{-1}G\right)$$
$$\times\,{}_0P_1\left(h^{(2k)}\left(\operatorname{tr}\left(G+M'M\Sigma^{-1}\right)\right):\frac{1}{2}m;M\Sigma^{-1}M'G\right)(dG)(dF). \tag{26}$$

From the proof of Corollary 21 the Gaussian law implies that $h^{(t)}(y)=\left(-\frac{1}{2}\right)^t\frac{1}{(2\pi)^{\frac{1}{2}pq}}\,\mathrm{e}^{-\frac{1}{2}y}$, then

$$f^{(t)}(0)=H(t,h)=\frac{\pi^{-\frac{1}{2}m(n-p)}}{2^{\frac{1}{2}[m(n-p)-r(m-r)]}}\left(-\frac{1}{2}\right)^t$$
$$\times\int\limits_{\Re^{r(m-r)}}\frac{\operatorname{etr}\left(-\frac{1}{2}\Sigma_{11.2}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})A_{22}^{-1}(A_{12}-\Sigma_{12}\Sigma_{22}^{-1}A_{22})'\right)}{(2\pi)^{\frac{1}{2}r(m-r)}|\Sigma_{11.2}|^{\frac{1}{2}(m-r)}|A_{22}|^{\frac{1}{2}r}}(dA_{12}).$$

So $f^{(t)}(0)=\pi^{-\frac{1}{2}m(n-p)}2^{-\frac{1}{2}[m(n-p)-r(m-r)]}\left(-\frac{1}{2}\right)^t$ and $f^{(t)}(y)=\pi^{-\frac{1}{2}m(n-p)}2^{-\frac{1}{2}[m(n-p)-r(m-r)]}\times\left(-\frac{1}{2}\right)^t \mathrm{e}^{-\frac{1}{2}y}$, thus $\frac{\int_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]}=\pi^{-\frac{1}{2}m(n-p)}2^{\frac{1}{2}r(m-n+p-r)}\left(-\frac{1}{2}\right)^t$. This implies that

$$
{}_0P_0\left(\frac{\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]}:F^{-1}G\right)=\pi^{-\frac{1}{2}m(n-p)}2^{\frac{1}{2}r(m-n+p-r)}\operatorname{etr}\left(-\frac{1}{2}F^{-1}G\right).
$$

In the same way,

$$
\begin{aligned}
&{}_0P_1\left(h^{(2k)}\left(\operatorname{tr}\left(G+M'M\Sigma^{-1}\right)\right):\frac{1}{2}m;M\Sigma^{-1}M'G\right)\\
&=\frac{\operatorname{etr}\left[-\frac{1}{2}\left(G+M'M\Sigma^{-1}\right)\right]}{(2\pi)^{\frac{1}{2}mr}}{}_0F_1\left(\frac{1}{2}m;\frac{1}{4}M\Sigma^{-1}M'G\right),
\end{aligned}
$$

then (26) takes the form

$$
\begin{aligned}
&\frac{2^{\frac{1}{2}r(-n+p-r)}|F|^{-\frac{1}{2}(n-p-m+2r+1)}|G|^{\frac{1}{2}(n-p-1)}\operatorname{etr}\left[-\frac{1}{2}\left(I+F^{-1}\right)G\right]}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]\operatorname{etr}\left[\frac{1}{2}M'M\Sigma^{-1}\right]}\\
&\times{}_0F_1\left(\frac{1}{2}m;\frac{1}{4}M\Sigma^{-1}M'G\right)(dG)(dF),
\end{aligned}
$$

which is the last expression of [41, p. 453] in the corresponding proof of the Gaussian case. □

Note again that not all the elliptical models accept the above procedure of reversing the series in such a way that the invariant polynomials can be avoided.

As in the case $r\geq m$, we can find the joint distribution of the latent roots of a generalized F.

Theorem 27. *Let $X\sim E_{r\times m}(M,I_r\otimes\Sigma,h)$ and $Y\sim E_{(n-p)\times m}(0,I_{n-p}\otimes\Sigma,h)$ independent, with $n-p\geq m\geq r$, then joint density function of the latent roots $f_1>\cdots>f_r>0$ of $F=X(Y'Y)^{-1}X'$, with $L=\operatorname{diag}(f_1,\dots,f_r)$ is*

$$
\begin{aligned}
&\frac{\pi^{\frac{1}{2}[m(n+r-p)+r^2]}\Gamma_r\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_r\left(\frac{1}{2}r\right)\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}\prod_{i<j}^r(f_i-f_j)\prod_{i=1}^r f_i^{-\frac{1}{2}(n-p-m+2r+1)}\\
&\times\sum_{t=0}^\infty\sum_{k=0}^\infty\sum_{s=0}^\infty\sum_\tau\sum_\kappa\sum_\sigma\sum_{\phi\in\tau\cdot\kappa\cdot\sigma}\frac{\theta_\phi^{\tau,\kappa,\sigma}\int\limits_0^\infty f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{t!k!s!\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]\left(\frac{1}{2}m\right)_\kappa}\\
&\times\frac{\int\limits_0^\infty h^{(2k+s)}(w)w^{\frac{1}{2}r(n+r-p)+f-1}dw}{\Gamma\left[\frac{1}{2}r(n+r-p)+f\right]}\left(\frac{1}{2}(n+r-p)\right)_\phi\\
&\times C_\phi^{\tau,\kappa,\sigma}\left(L^{-1},M'M\Sigma^{-1},M'M\Sigma^{-1}\right),
\end{aligned}
$$

where $f^{(t)}(y)$ (see Lemma 20) and $h^{(2k+s)}(w)$ are defined under $m\times(n-p)$ and $m\times r$ elliptical models, respectively.

Proof. Applying [41, Theorem 3.2.17] with the density of F given in Theorem 25, we obtain that the required joint density function of the latent roots $f_1>\cdots>f_r>0$ of $F=HLH'$, with $L=\operatorname{diag}(f_1,\dots,f_r)$, is

$$
\begin{aligned}
&\frac{\pi^{r^2/2}}{\Gamma_r\left(\frac{1}{2}r\right)}\prod_{i<j}^r(f_i-f_j)\prod_{i=1}^r f_i^{-\frac{1}{2}(n-p-m+2r+1)}\\
&\times\frac{\pi^{\frac{1}{2}m(n+r-p)}\Gamma_r\left[\frac{1}{2}(n+r-p)\right]}{\Gamma_r\left(\frac{1}{2}m\right)\Gamma_r\left[\frac{1}{2}(n-p-m+r)\right]}
\end{aligned}
$$

$$\times \sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\tau}\sum_{\kappa}\sum_{\sigma}\sum_{\phi\in\tau\cdot\kappa\cdot\sigma}\frac{\theta_{\phi}^{\tau,\kappa,\sigma}\int\limits_{0}^{\infty} f^{(t)}(y)y^{\frac{1}{2}(n-p)(m-r)-1}dy}{t!k!s!\Gamma\left[\frac{1}{2}(n-p)(m-r)\right]\left(\frac{1}{2}m\right)_{\kappa}}$$

$$\times\frac{\int\limits_{0}^{\infty} h^{(2k+s)}(w)w^{\frac{1}{2}r(n+r-p)+f-1}dw}{\Gamma\left[\frac{1}{2}r(n+r-p)+f\right]}\left(\frac{1}{2}(n+r-p)\right)_{\phi}$$

$$\times C_{\phi}^{\tau,\kappa,\sigma}\left(L^{-1}, M'M\Sigma^{-1}, M'M\Sigma^{-1}\right).$$

□

The Gaussian version of Theorem 27 now can be easily derived via Corollary 26, which corresponds to [41, Theorem 10.4.5].

Now, at this point we have all the required tools for studying the so called generalized MANOVA under any elliptical models and the new associated distributions of the classical statistics of the Gaussian MANOVA. This aspect will be provided in a future work. Moreover, [44, 45], specially suggested by a Referee, for real and complex cases can promote in future some interesting studies for the non central case involving Davis invariant polynomials under the general setting of real normed division algebras.

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