ISSN 1066-5307, Mathematical Methods of Statistics, 2017, Vol. 26, No. 3, pp. 226–235. © Allerton Press, Inc., 2017.

Classes of Improved Estimators for Parameters of a Pareto Distribution

L. K. Patra[1] **and S. Kumar**[1*]

[1]*Dept. of Math., Indian Inst. of Technology Kharagpur, Kharagpur, India*

Received October 6, 2016; in final form, January 31, 2017

Abstract—The problem of estimating parameters of a Pareto distribution is investigated under a general scale invariant loss function when the scale parameter is restricted to the interval $(0,1]$. We consider the estimation of shape parameter when the scale parameter is unknown. Techniques for improving equivariant estimators developed by Stein, Brewster–Zidek and Kubokawa are applied to derive improved estimators. In particular improved classes of estimators are obtained for the entropy loss and a symmetric loss. Risk functions of various estimators are compared numerically using simulations. It is also shown that the technique of Kubokawa produces improved estimators for estimating the scale parameter when the shape parameter is known.

Keywords: restricted maximum likelihood estimator, generalized Bayes estimator, scale invariant loss function, integral expression of risk difference.

2000 Mathematics Subject Classification: primary 62F10; secondary 62C15.

DOI: 10.3103/S106653071703005X

1. INTRODUCTION

The problem of point estimation of restricted parameters has been studied extensively by statisticians. The problem arises in several practical situations in agriculture, biological, industrial and economic experiments. Incorporating prior information about restrictions on parameter space leads to more efficient estimators. Decision-theoretic estimation of restricted parameters was first considered by Katz [7]. He showed that the generalized Bayes estimator of restricted mean is minimax and admissible in a normal distribution with known variance. Farrell [4] established the minimaxity and admissibility of estimators of restricted parameters in one-dimensional general location family. One may refer to [13, 11, 10, 18, 6, 21] for detailed review of work on estimation problems under restrictions on the parameter space.

In this paper, we consider the estimation of unknown parameters of a Pareto $P(\alpha,\beta)$ distribution under a restriction on the scale parameter. The density function of a Pareto distribution is given by

$$f_X(x,\alpha,\beta)=\frac{\beta\alpha^{\beta}}{x^{\beta+1}},\qquad \alpha\le x<\infty,\quad \beta>0. \tag{1.1}$$

Here α is the scale parameter and β is the shape parameter. Pareto distribution was originally developed as a model for representing distribution of income. Later it has found applications in modeling the lifetime of systems. See for example [5, 12]. For a review on the problem of estimating the parameters of a Pareto distribution one may refer to [1, 8, 14, 15, 16, 3].

Tripathi et al. [19] considered the estimation of the parameters of a Pareto distribution with respect to a quadratic loss when the scale parameter is constrained. Recently Tripathi et al. [20] considered the estimation of shape parameter of a Pareto distribution with respect to a quadratic loss function when the shape parameter is bounded below.

For an estimator δ of θ assume that the loss function is $L\left(\frac{\delta}{\theta}\right)$, where $L(t)$ is a twice differentiable strictly convex function with $L(1)=0$, that is, the derivative $L'(t)$ is strictly increasing for $t>0$. We also

*E-mail: smsh@maths.iitkgp.ernet.in

assume that integrals involving $L(t)$ are finite and differentiation under the integral sign is permissible. Here we consider the estimation of parameters of a Pareto distribution under the scale invariant loss function $L(t)$ when the scale parameter is constrained.

This paper is organized as follows. In Section 2, estimation of β is considered when the scale parameter α is unknown and $0 < \alpha \leq 1$ under a general scale invariant loss function. Using the techniques of Stein [17] and Kubokawa [9] we derive classes of scale equivariant estimators which improve upon the best affine equivariant estimator. These results are applied to entropy loss and a symmetric loss functions for deriving specific classes of improved estimators. Numerical comparisons of risk functions are carried out using simulations. It is also shown that when the shape parameter is known a priori, the improvement results using the integral expression of risk difference (IERD) technique of Kubokawa [9] can be obtained for estimating the scale parameter.

2. ESTIMATION OF THE SHAPE PARAMETER WITH UNKNOWN SCALE PARAMETER

In this section, we consider estimation of the shape parameter when the scale parameter is unknown and restricted to the interval $(0, 1]$. Let $X_1, X_2, \ldots, X_n$ be a random sample taken from a Pareto $P(\alpha, \beta)$ distribution. We consider the estimation of β when α is unknown and $0 < \alpha \leq 1$. Define $T = \log X_{(1)}$ and $S = \frac{1}{n}\sum_{i=1}^{n} \log \frac{X_i}{X_{(1)}}$, where $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$. Then (T, S) is a complete sufficient statistic with T having exponential $\mathrm{Exp}(\mu, \sigma)$ with $\mu = \log \alpha$, $\mu \leq 0$ and S having $\mathrm{Gamma}(n-1, \sigma)$ distribution with $\sigma = n\beta$. The density function of T and S are given by

$$f_T(t, \mu, \sigma) = \sigma e^{-\sigma(t-\mu)}, \qquad \mu \leq t < \infty, \quad \sigma > 0,$$

and

$$g_S(s, \sigma) = \frac{\sigma^{n-1}}{\Gamma(n-1)} e^{-\sigma s} s^{n-2}, \qquad s > 0, \quad \sigma > 0,$$

respectively.

Now we consider the equivalent problem of estimation of σ under the scale invariant loss function $L\left(\frac{\delta}{\sigma}\right)$ when μ is unknown and satisfies the inequality $\mu \leq 0$. The maximum likelihood estimator (MLE) of σ for the restricted parameter space $\mu \leq 0$ is

$$\delta_{ML} = \frac{n}{S + \max(0, T)}. \tag{2.1}$$

Next we prove a complete class result based on restricted MLE δ_{ML}. Consider a class of estimators based on the restricted MLE as

$$\delta_d = \frac{d}{S + \max(0, T)}, \quad d > 0. \tag{2.2}$$

Let $d_0(\mu, \sigma)$ minimize the risk function $R(\delta_d, \alpha) = EL\left(\frac{\delta_d}{\alpha}\right)$ of δ_d. Denote

$$d_* = \inf_{\mu \leq 0, \sigma > 0} d_0(\mu, \sigma) \qquad \text{and} \qquad d^* = \sup_{\mu \leq 0, \sigma > 0} d_0(\mu, \sigma).$$

An application of the Brewster and Zidek technique [2] gives a complete class of estimators.

Lemma 2.1. *For estimating σ when $\mu(\leq 0)$ is unknown, the class of estimators $\{\delta_d \colon d_* \leq d \leq d^*\}$ forms a complete class among all estimators of the form* (2.2) *with respect to the scale invariant loss function $L(t)$.*

Consider the affine group of transformations $G = \{g_{p,q} \colon g_{p,q}(x) = px + q,\ p > 0,\ q \in \mathbb{R}\}$. Under this group the estimation problem is invariant. The best affine equivariant estimator is

$$\delta_0 = \frac{a_0}{S}, \tag{2.3}$$

where a_0 is the unique solution of

$$\int_0^\infty L'\left(\frac{a_0}{z}\right) z^{n-3} e^{-z}\, dz = 0. \tag{2.4}$$

For improving upon δ_0, we consider a scale equivariant estimator of the form

$$\delta_\phi = \frac{\phi(V)}{S}, \tag{2.5}$$

where $V = \frac{T}{S}$. The joint density of S and V is given by

$$f_{S,V}(s,v) = \frac{\sigma^n}{\Gamma(n-1)} e^{-\sigma(sv+s-\mu)}, \qquad s > \max(0, \mu/v), \quad -\infty < v < \infty.$$

The following theorem proves the inadmissibility of δ_0 by deriving a Stein-type estimator [17].

Theorem 2.2. *The estimator δ_0 is inadmissible and dominated by the estimator*

$$\delta_{ST}(T,S) = \begin{cases} \max\left\{a_0, \frac{b_0}{(1+V)}\right\} S^{-1}, & V > 0, \\ a_0 S^{-1}, & V \le 0, \end{cases}$$

under the scale invariant loss function $L(t)$, where b_0 is the unique solution of the equation

$$\int_0^\infty L'\left(\frac{b_0}{z}\right) z^{n-2} e^{-z}\, dz = 0. \tag{2.6}$$

Proof. The risk function of δ_ϕ given by (2.5) depends on σ and μ only through $\mu\sigma$. So without loss of generality we take $\sigma = 1$ and the risk function of δ_ϕ can be written as

$$R(\delta_\phi, \mu) = E_\mu L\left(\frac{\phi(V)}{S}\right) = E_\mu\left[E_\mu\left\{L\left(\frac{\phi(V)}{S}\right)\middle| V = v\right\}\right].$$

Denote

$$R_*(c,\mu) = E_\mu\left\{L\left(\frac{c}{S}\right)\middle| V = v\right\}.$$

Since $R_*(c,\mu)$ is strictly convex in c, the choice of $c = c(\mu)$ minimizing $R_*(c,\mu)$ satisfies the equation

$$E_\mu\left\{L'\left(\frac{c(\mu)}{S}\right)\frac{1}{S}\middle| V = v\right\} = 0.$$

We consider $v > 0$. For any μ the conditional density of S given $V = v$ when $\sigma = 1$ is

$$h_S(s,\mu) \propto s^{n-1} e^{-(1+v)s}, \qquad s > \max\{0, \mu/v\}.$$

Again $R_*(c, 0)$ is a strictly convex function and minimized at $c = c(0)$. For $\mu \le 0$ the conditional density of S given $V = v$ is independent of μ. Hence $c(0)$ is given by

$$E_0\left\{L'\left(\frac{c(0)}{S}\right)\frac{1}{S}\middle| V = v\right\} = 0.$$

Now $c(0)$ is the unique solution of

$$\int_0^\infty L'\left(\frac{c(0)}{s}\right) s^{n-2} e^{-(1+v)s}\, ds = 0 \qquad \text{or} \qquad \int_0^\infty L'\left(\frac{c(0)(1+v)}{z}\right) z^{n-2} e^{-z}\, dz = 0. \tag{2.7}$$

Comparing (2.7) with (2.6) we get $c(0) = \frac{b_0}{(1+v)}$. We define a function $\phi(v)$ as

$$\phi(v) = \begin{cases} \max\left\{a_0, \frac{b_0}{(1+v)}\right\}, & v > 0, \\ a_0, & v \le 0. \end{cases}$$

Then we have $c(\mu) > \phi(v) > a_0$ on a set of positive probability for $\mu \le 0$. Hence from the strict convexity of $R_*(c,\mu)$ we obtain

$$E_\mu\Big\{L\Big(\frac{\phi(v)}{S}\Big)\Big|V=v\Big\} \le E_\mu\Big\{L\Big(\frac{a_0}{S}\Big)\Big|V=v\Big\}.$$

This implies that $R(\delta_{ST},\mu) \le R(\delta_0,\mu)$. This proves the result. □

The following theorem gives another class of improved affine equivariant estimators. We define a class of estimators of the form

$$\delta_K(T,S) = \begin{cases} \frac{\phi(V)}{S}, & V > 0, \\ \frac{a_0}{S}, & V \le 0, \end{cases} \tag{2.8}$$

where ϕ is an absolutely continuous positive function.

Theorem 2.3. *Let the function $\phi(x)$ satisfy the following conditions:*

(i) *$\phi(x)$ is nondecreasing and $\lim_{x\to\infty}\phi(x) = a_0$,*

(ii) $\displaystyle\int_0^\infty\int_0^x L'\Big(\frac{\phi(x)}{y}\Big)y^{n-2}e^{-y(1+v)}\,dv\,dy \le 0.$

Then the estimator δ_K has uniformly smaller risk than δ_0 under the scale invariant loss function $L(t)$, where a_0 is the unique solution of

$$\int_0^\infty L'\Big(\frac{a_0}{y}\Big)y^{n-3}e^{-y}\,dy = 0.$$

Proof. The risk difference of δ_K and δ_0 can be written as

$$\begin{aligned}\Delta(\delta_K,\delta_0) &= E\int_1^\infty L'\Big(\frac{\phi(zv)}{y}\Big)\frac{v}{y}\phi'(zv)I(v>0)\,dz \\ &= \int_0^\infty\int_{\max(0,\frac{\mu\sigma}{y})}^\infty\int_1^\infty L'\Big(\frac{\phi(zv)}{y}\Big)\frac{v}{y}\phi'(zv)f_{Y,V}(y,v)\,dz\,dv\,dy,\end{aligned}$$

where $f_{Y,V}(y,v)$ is the joint density of $Y=\sigma S$ and $V=\frac{T}{S}$. For $\mu \le 0$ we have

$$\begin{aligned}\Delta(\delta_K,\delta_0) &= \int_0^\infty\int_0^\infty\int_1^\infty L'\Big(\frac{\phi(zv)}{y}\Big)\frac{v}{y}\phi'(zv)f_{Y,V}(y,v)\,dz\,dv\,dy \\ &= \int_0^\infty\int_0^\infty\int_v^\infty L'\Big(\frac{\phi(x)}{y}\Big)\frac{\phi'(x)}{y}f_{Y,V}(y,v)\,dx\,dv\,dy \\ &= \int_0^\infty \phi'(x)\Big[\int_0^\infty\int_0^x L'\Big(\frac{\phi(x)}{y}\Big)\frac{1}{y}f_{Y,V}(y,v)\,dv\,dy\Big]\,dx.\end{aligned}$$

Since $\phi(x)$ is nondecreasing, that is, $\phi'(x) \ge 0$, the risk difference is nonpositive if

$$\int_0^\infty\int_0^x L'\Big(\frac{\phi(x)}{y}\Big)\frac{1}{y}f_{Y,V}(y,v)\,dv\,dy \le 0$$

which is equivalent to

$$\int_0^\infty\int_0^x L'\Big(\frac{\phi(x)}{y}\Big)y^{n-2}e^{-y(1+v)}\,dv\,dy \le 0.$$

This proves the result. □

Example 2.1 (Entropy Loss). Let $L(t) = t - \log t - 1$. For this loss function the best affine equivariant estimator is $\delta_0 = \frac{n-2}{S}$. In this case $a_0 = n - 2$.

Consider an estimator of the form (2.2). By Lemma 2.1 we get the following result which gives a class of improved estimators based on the restricted MLE δ_{ML} given by (2.1).

Lemma 2.4. *The class of estimators* $\{\delta_c\colon (n-2) \le c \le (n-1)\}$ *forms an admissible class of estimators among all estimators of the form* (2.2).

Remark 2.1. The restricted MLE $\delta_{ML} = \delta_n$ is inadmissible and improved by the estimator δ_{n-1}.

The following result gives a Stein type improved estimator.

Theorem 2.5. *The best affine equivariant estimator* δ_0 *is inadmissible for estimating* σ *when* $\mu(\le 0)$ *is unknown under the entropy loss function and is improved by the estimator*

$$\delta_{ST}(V) = \begin{cases} \max\left\{(n-2), \frac{n-1}{(V+1)}\right\} S^{-1}, & V > 0, \\ \frac{n-2}{S}, & V \le 0. \end{cases}$$

An application of Theorem 2.3 yields the following result.

Theorem 2.6. *Assume that the function* $\phi(x)$ *satisfies the following conditions:*

(i) $\phi(x)$ *is nonincreasing and* $\lim_{x\to\infty} \phi(x) = a_0$,

(ii) $\phi(x) \le \phi_0(x)$ *for all* $x > 1$ *with* $\phi_0(x) = (n-2)\left(\dfrac{1-(1+x)^{1-n}}{1-(1+x)^{2-n}}\right)$.

Then the estimator δ_K *given by* (2.8) *improves upon the best affine equivariant estimator* δ_0 *under the entropy loss function.*

The Brewster–Zidek type improved estimator is

$$\delta_{BZ}(V) = \begin{cases} \frac{(n-2)}{S}\left(\frac{1-(1+V)^{1-n}}{1-(1+V)^{2-n}}\right), & V > 0, \\ \frac{n-2}{S}, & V \le 0. \end{cases}$$

Remark 2.2. Consider the noninformative prior on the restricted parameter space of μ and σ as

$$g(\mu, \sigma) = \frac{1}{\sigma}, \qquad 0 < \sigma < \infty, \quad \infty < \mu < 0. \tag{2.9}$$

Under this prior the generalized Bayes estimator of σ with respect to the entropy loss function is

$$\delta_{GB} = \begin{cases} \frac{n-2}{T+S}, & T > 0, \\ \frac{n-2}{S}, & T \le 0. \end{cases}$$

From Lemma 2.4 it is seen that the generalized Bayes estimator δ_{GB} belongs to the class δ_d given by (2.2) and improves upon the restricted maximum likelihood estimator.

Example 2.2 (Symmetric Loss). Let $L(t) = t + \frac{1}{t} - 2$. For this loss function the best affine equivariant estimator is $\delta_0 = \frac{\sqrt{(n-1)(n-2)}}{S}$ for $n > 2$. For this case $a_0 = \sqrt{(n-1)(n-2)}$.

Consider an estimator of the form (2.2). The minimizing choice of d is obtained as

$$d(\mu,\sigma) = \left(\sigma^2 \frac{E[S+\max(0,T)]}{E\left[\frac{1}{S+\max(0,T)}\right]}\right)^{1/2}.$$

It is easily seen that

$$E\left[\frac{1}{S+\max(0,T)}\right] = \frac{\sigma e^{\mu\sigma}}{n-2}\left[e^{-\mu\sigma} - \frac{1}{n-1}\right] \tag{2.10}$$

and

$$E[S+\max(0,T)] = \frac{(n-1)e^{\mu\sigma}}{\sigma}\left[e^{-\sigma\mu} + \frac{1}{n-1}\right]. \tag{2.11}$$

Using (2.10) and (2.11) we have

$$d(\mu,\sigma) = \sqrt{(n-1)(n-2)}\left(\frac{e^{-\mu\sigma} + \frac{1}{n-1}}{e^{-\mu\sigma} - \frac{1}{n-1}}\right)^{1/2}, \qquad \mu\sigma \le 0. \tag{2.12}$$

Now we have

$$\sup_{\mu\sigma\le 0} d(\mu,\sigma) = \sqrt{n(n-1)} \qquad \text{and} \qquad \inf_{\mu\sigma\le 0} d(\mu,\sigma) = \sqrt{(n-1)(n-2)}. \tag{2.13}$$

Since the risk function is convex in d, we get the following result.

Lemma 2.7. *The class of estimators* $\{\delta_c\colon \sqrt{(n-1)(n-2)} \le c \le \sqrt{n(n-1)}\}$ *forms an admissible class of estimators among all estimators of the form* (2.2).

Remark 2.3. The restricted MLE δ_n is inadmissible and is improved by the estimator

$$\delta_{IML} = \frac{\sqrt{n(n-1)}}{S+\max(0,T)}.$$

The following result gives Stein-type improved estimator for σ when $\mu(\le 0)$ is unknown.

Theorem 2.8. *The best affine equivariant estimator δ_0 is inadmissible for estimating σ with unknown $\mu(\le 0)$ under the symmetric loss function and is improved by the estimator*

$$\delta_{ST}(V) = \begin{cases} \max\left\{\sqrt{(n-1)(n-2)}, \frac{\sqrt{n(n-1)}}{(V+1)}\right\}S^{-1}, & V > 0, \\ \frac{\sqrt{(n-1)(n-2)}}{S}, & V \le 0. \end{cases}$$

An application of Theorem 2.3 yields the following result.

Theorem 2.9. *Assume that the function $\phi(x)$ satisfies the following conditions:*

(i) $\phi(x)$ *is nonincreasing and* $\lim_{x\to\infty}\phi(x) = a_0$,

(ii) $\phi(x) \le \phi_0(x)$ *for all* $x > 1$ *with* $\phi_0(x) = \left[(n-1)(n-2)\frac{1-(1+x)^{-n}}{1-(1+x)^{2-n}}\right]^{1/2}$.

Then the estimator δ_K given by (2.8) *improves upon the best affine equivariant estimator δ_0 under the symmetric loss function.*

The Brewster–Zidek type improved estimator is

$$\delta_{BZ}(V) = \begin{cases} \frac{\sqrt{(n-1)(n-2)}}{S}\left[\frac{1-(1+V)^{-n}}{1-(1+V)^{2-n}}\right]^{1/2}, & V > 0, \\ \frac{\sqrt{(n-1)(n-2)}}{S}, & V \leq 0. \end{cases}$$

Remark 2.4. Under the noninformative prior (2.9) a generalized Bayes estimator of σ with respect to this symmetric loss function is

$$\delta_{\mathrm{GB}} = \begin{cases} \frac{\sqrt{(n-1)(n-2)}}{T+S}, & T > 0, \\ \frac{\sqrt{(n-1)(n-2)}}{S}, & T \leq 0. \end{cases}$$

From Lemma 2.7 it is seen that the generalized Bayes estimator δ_{GB} belongs to the class δ_d given by (2.2) and improves upon the restricted maximum likelihood estimator.

2.1. Numerical Comparisons

In this section, we numerically compare the risk performance of various estimators. The risk values of estimators are calculated using simulations based on 10,000 samples of sizes $n = 10$ and $n = 20$ with $\beta = 1$. In Figs. 1 and 2, we plot the risk functions of the estimators δ_{ML}, δ_{n-1}, δ_0, δ_{GB}, δ_{ST} and δ_{BZ} for entropy loss function, and in the Figs. 3 and 4, we plot the risk functions of the estimators δ_{ML}, δ_{IML} δ_0, δ_{GB}, δ_{ST} and δ_{BZ} for symmetric loss function.

For entropy loss function from the Figs. 1 and 2 we made the following observations.

(i) From the Figs. 1 (a) and 2 (a), it is seen that

- (a) δ_{GB} and δ_{n-1} improve upon the restricted MLE δ_{ML}.
- (b) δ_{GB} improves upon δ_{n-1} for all α except near $\alpha = 1$.

(ii) From Figs. (b) and 2 (b) we can see that δ_{ST} and δ_{BZ} improve upon the best affine equivariant estimator δ_0.

(iii) Comparing risk plots in Figs. 1 (a), 1 (b), 2 (a), 2 (b), we see that

- (a) δ_0, δ_{BZ} and δ_{ST} improve upon δ_{ML}.
- (b) δ_0 and δ_{BZ} improve δ_{n-1} for all α except in a neighborhood of 1.
- (c) δ_{ST} uniformly improves upon δ_{n-1} and δ_{GB}.

(iv) The risk values of all estimators decrease as n increases.

Similar observations are made from risk plots in Figs. 3 and 4 for symmetric loss function.

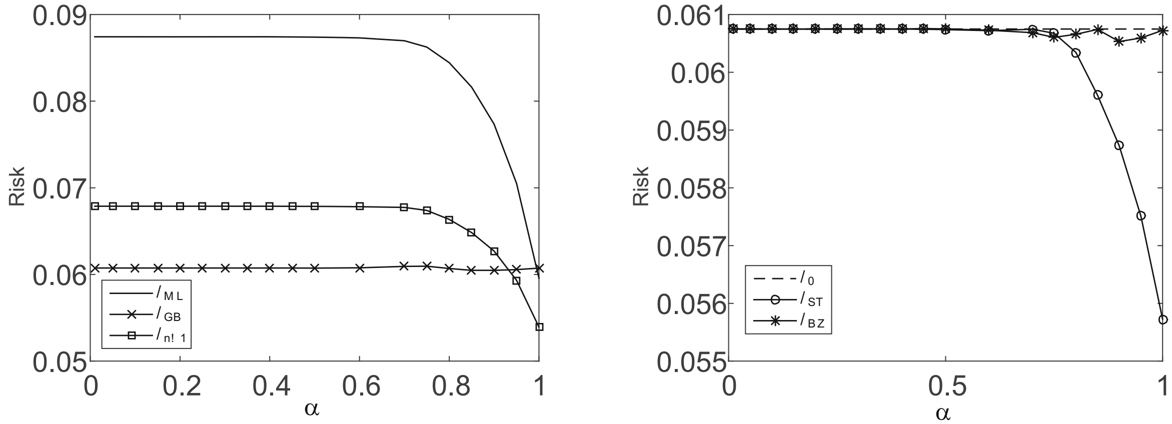
Thus, it may be recommended that estimate δ_{ST} be used for this estimation problem.

Remark 2.5. Consider the estimation of α $(0 < \alpha \leq 1)$ when β is known under a general scale invariant loss function $L(t)$. For this problem $Y = X_{(1)}$ is a complete sufficient statistic, where $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$. The distribution of Y is $P(\alpha, n\beta)$. For existence of moments, it is also assumed that $n\beta > 1$. The best scale equivariant estimator of α is given by $\delta_{c_0} = c_0 Y$, where c_0 is the unique solution of the equation $\int_1^\infty L'(c_0 y)\frac{n\beta}{y^{n\beta}}\,dy = 0$.

Consider an estimator of the form

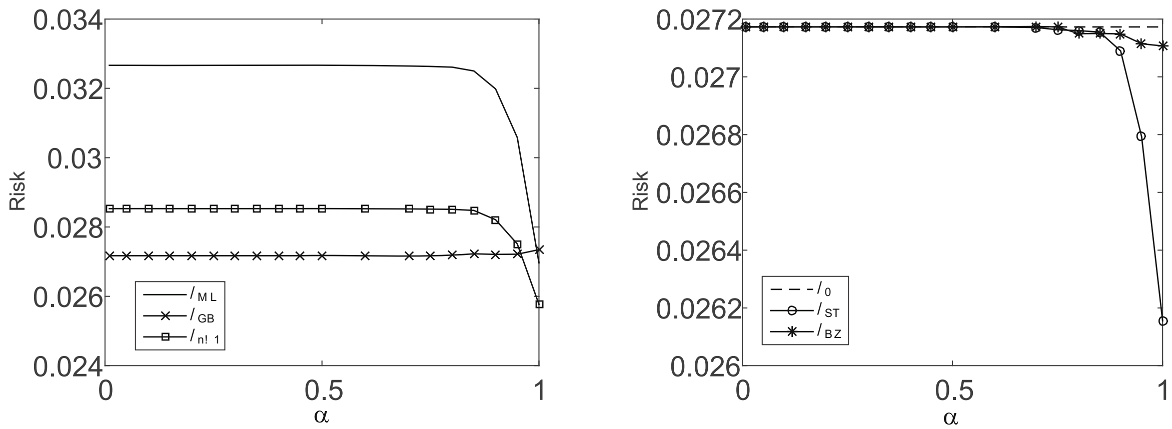
$$\delta_\phi(Y) = \begin{cases} \phi(Y)Y & \text{if} \quad Y > 1, \\ c_0 Y & \text{if} \quad Y \leq 1, \end{cases} \tag{2.14}$$

where ϕ is an absolutely continuous positive function. Using the IERD approach of Kubokawa as in Theorem 2.3, we can establish the following result.



(a) (b)

Fig. 1. Risk plots for the Entropy loss function ($n = 10$).



(a) (b)

Fig. 2. Risk plots for the Entropy loss function ($n = 20$).

Theorem 2.10. *Assume that the function $\phi(w)$ satisfies the following conditions:*

(i) $\phi(w)$ *is nonincreasing and* $\lim_{w\to 1}\phi(w) = c_0$,

(ii) $\displaystyle\int_w^\infty L'(\phi(w)z)\frac{n\beta}{z^{n\beta}}\,dz \geq 0.$

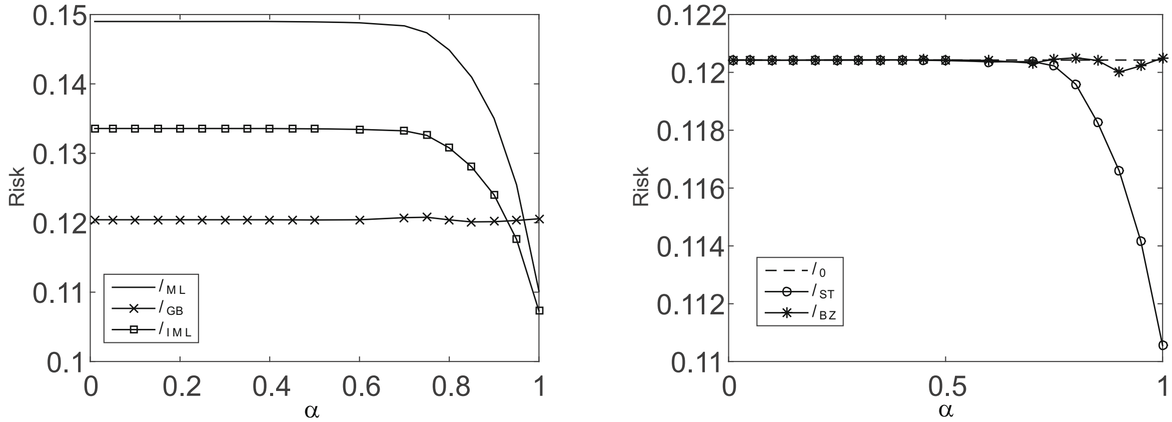
Then the estimator δ_ϕ have uniformly smaller risk than the best scale equivariant estimator δ_{c_0} under the scale equivariant loss function $L(t)$.

Remark 2.6. When the shape parameter β is known and $0 < \alpha \leq 1$, the MLE of α is $\delta_{ML}(Y) = \min\{Y, 1\}$. Consider a class of estimators based on the restricted MLE as

$$\delta_d = d\delta_{ML}(Y), \qquad d > 0. \tag{2.15}$$

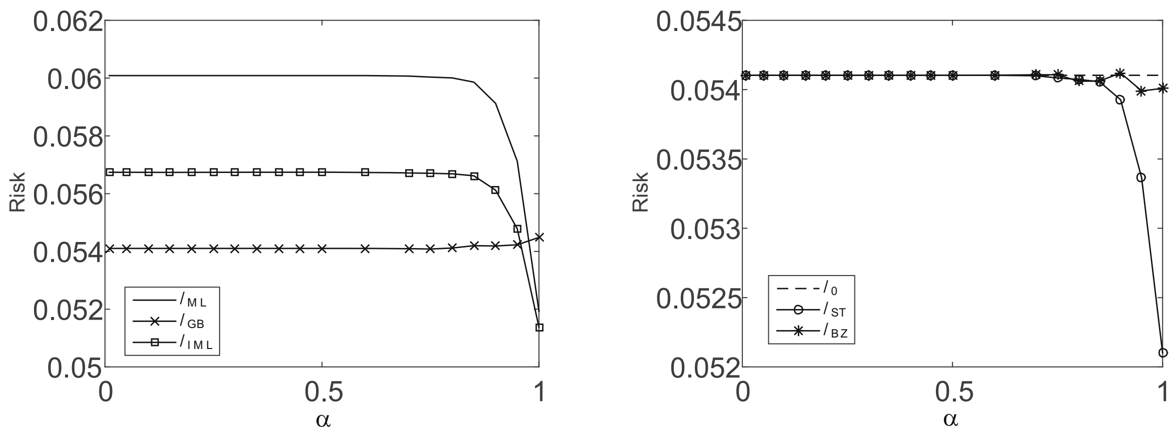
Let $d_0(\alpha)$ be the choice of d minimizing the risk function $R(\delta_d, \alpha) = EL\left(\frac{\delta_d}{\alpha}\right)$ of δ_d. Denote

$$d_* = \inf_{0<\alpha\leq 1} d_0(\alpha) \qquad \text{and} \qquad d^* = \sup_{0<\alpha\leq 1} d_0(\alpha).$$



(a) (b)

Fig. 3. Risk plots for the Symmetric loss function ($n = 10$).



(a) (b)

Fig. 4. Risk plots for the Symmetric loss function ($n = 20$).

It can be shown that the class of estimators $\{\delta_d\colon d_* \le d \le d^*\}$ forms a complete class among all the estimators of the form (2.15) with respect to the scale invariant loss function $L(t)$.

Results similar those in Examples 2.1 and 2.2 can be obtained for the entropy loss function and a symmetric loss function.

ACKNOWLEDGMENTS

The authors are grateful to the Editor-in-Chief and a reviewer for their suggestions which have improved the presentation of the paper.

REFERENCES

1. B. R. Asrabadi, "Estimation in the Pareto distribution", Metrika **37** (1), 199–205 (1990).
2. J. F. Brewster and J. Zidek, "Improving on Equivariant Estimators", Ann. Statist. **2** (1), 21–38 (1974).
3. U. Dixit and M. J. Nooghabi, "Efficient Estimation in the Pareto Distribution with the Presence of Outliers", Statist. Methodology **8** (4), 340–355 (2011).
4. R. H. Farrell, "Estimators of a Location Parameter in the Absolutely Continuous Case", Ann. Math. Statist., No. 3, 949–998 (1964).

5. C. M. Harris, “The Pareto Distribution as a Queue Service Discipline”, Operat. Res. **16** (2), 307–313 (1968).
6. Kanika and S. Kumar, “Methods for Improving Estimators of Truncated Circular Parameters”, Bernoulli **22** (4), 2521–2547 (2016).
7. M. W. Katz, “Admissible and Minimax Estimates of Parameters in Truncated Spaces”, Ann. Math. Statist. **32** (1), 136–142 (1961).
8. D. Kern, “Minimum Variance Unbiased Estimation in the Pareto Distribution”, Metrika **30** (1), 15–19 (1983).
9. T. Kubokawa, “A Unified Approach to Improving Equivariant Estimators”, Ann. Statist. **22** (1), 290–299 (1994).
10. T. Kubokawa, “Minimaxity in Estimation of Restricted Parameters”, J. Japan Statist. Soc. **34** (2), 229–253 (2004).
11. S. Kumar and D. Sharma, “An Inadmissibility Result for Affine Equivariant Estimators”, Statist. and Decis. **10** (1–2), 87–98 (1992).
12. H. J. Malik, “Estimation of the Parameters of the Pareto Distribution”, Metrika **15** (1), 126–132 (1970).
13. J. Moors, “Inadmissibility of Linearly Invariant Estimators in Truncated Parameter Spaces”, J. Amer. Statist. Assoc. **76** (376), 910–915 (1981).
14. M. Rytgaard, “Estimation in the Pareto Distribution”, Astin Bull. **20** (2), 201–216 (1990).
15. A. P. S. Saksena and A. Johnson, “Best Unbiased Estimators for the Parameters of a Two-Parameter Pareto Distribution”, Metrika **31** (1), 77–83 (1984).
16. D. Singh, G. Prakash, and P. Singh, “Shrinkage Estimators for the Shape Parameter of Pareto Distribution Using Linex Loss Function”, Commun. in Statist. – Theory Methods **36** (4), 741–753 (2007).
17. C. Stein, “Inadmissibility of the Usual Estimator for the Variance of a Normal Distribution with Unknown Mean”, Ann. Inst. Statist. Math. **16** (1), 155–160 (1964).
18. Y. Tripathi, S. Kumar, and C. Petropoulos, “Estimation of the Parameters of an Exponential Distribution under Constrained Location”, Math. Methods Statist. **23** (1), 66–79 (2014).
19. Y. Tripathi, S. Kumar, and C. Petropoulos, “Improved Estimators for Parameters of a Pareto Distribution with a Restricted Scale”, Statist. Methodology **18**, 1–13 (2014).
20. Y. Tripathi, S. Kumar, and C. Petropoulos, “Estimating the Shape Parameter of a Pareto Distribution under Restrictions,” Metrika **79** (1), 91–111 (2016).
21. C. Van Eeden, *Restricted Parameter Space Estimation Problem: Admissibility and Minimaxity Properties*, in *Lecture Notes in Statist.* (Springer, New York, 2006), Vol. 188.