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On Likelihood Ratio Ordering of Parallel Systems with Exponential Components

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Abstract—Let $T(\lambda_1, \ldots, \lambda_n)$ be the lifetime of a parallel system consisting of exponential components with hazard rates $\lambda_1, \ldots, \lambda_n$, respectively. For systems with only two components, Dykstra *et. al.* (1997) showed that if (λ_1, λ_2) majorizes (γ_1, γ_2), then, $T(\lambda_1, \lambda_2)$ is larger than $T(\gamma_1, \gamma_2)$ in likelihood ratio order. In this paper, we extend this theorem to general parallel systems. We introduce a new partial order, the so-called d-larger order, and show that if $(\lambda_1, \ldots, \lambda_n)$ is d-larger than $(\gamma_1, \ldots, \gamma_n)$, then $T(\lambda_1, \ldots, \lambda_n)$ is larger than $T(\gamma_1, \ldots, \gamma_n)$ in likelihood ratio order.

Keywords: likelihood ratio order, order statistic, parallel system, stochastic comparison.

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1. INTRODUCTION

Order statistics play an important role in statistical inference, reliability theory, operations research, and many other areas. For $i = 1, \ldots, n$, let X_i be the lifetime of ith component in a parallel system, then the largest of $X_1, \ldots, X_n$ is the lifetime of the parallel system.

Since exponential distribution has nice mathematical form and is commonly-used in survival analysis, reliability analysis, and many other fields, so in this paper, we focus on exponential components. An exponential component means its lifetime follows an exponential distribution. Specifically, assume, for $i = 1, \ldots, n$, that X_i follows the exponential distribution with hazard rate λ_i, then we denote by $T(\lambda_1, \ldots, \lambda_n) = \max\{X_1, \ldots, X_n\}$ as the lifetime of the parallel system consisting of components whose lifetimes are $X_1, \ldots, X_n$ respectively. By symmetry, we assume the hazard rate vector $(\lambda_1, \ldots, \lambda_n)$ is in increasing order, that is, $0 < \lambda_1 \leq \cdots \leq \lambda_n$.

So far, there is an extensive literature on stochastic comparison between two parallel systems with lifetimes $T(\lambda_1, \ldots, \lambda_n)$ and $T(\gamma_1, \ldots, \gamma_n)$. For example, Pledger and Proschan (1971) showed that $(\lambda_1, \ldots, \lambda_n) \overset{m}{\succ} (\gamma_1, \ldots, \gamma_n)$ implies $T(\lambda_1, \ldots, \lambda_n) \geq_{st} T(\gamma_1, \ldots, \gamma_n)$. Dykstra *et. al.* (1997) enhanced the above result to reversed hazard rate order. Recently, Misra and Misra (2013) further extended the result to weak majorization. Here and in the sequel, $\overset{m}{\succ}$ stands for majorization order; $\overset{w}{\succ}$ for weak majorization order; $\geq_{st}$ for the usual stochastic order; $\geq_{hr}$ for hazard rate order; $\geq_{rh}$ for reversed hazard rate order; and $\geq_{lr}$ for likelihood ratio order. For more details on these majorization-type orders and various stochastic orders, see Shaked and Shanthikumar (2007) and Marshall *et al.* (2011).

As we know, the likelihood ratio order implies other stochastic orders. Hence the likelihood ratio order is the most interesting order in stochastic comparison. However, due to technical cumbersomeness, the results about likelihood ratio order between $T(\lambda_1, \ldots, \lambda_n)$ and $T(\gamma_1, \ldots, \gamma_n)$ are relatively few.

In the case of $n = 2$, Dykstra *et. al.* (1997) showed that if $(\lambda_1, \lambda_2) \overset{m}{\succ} (\gamma_1, \gamma_2)$, then $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$. We refer to this result as DKR theorem. Wang and Laniado (2015) extended this result to: If $(\gamma_1, \gamma_2) - (\lambda_1, \lambda_2) = a(1, -1) + b(1, \frac{1}{2})$, where $a, b \geq 0$, then $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$.

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It is well known that the DKR theorem cannot be extended directly to the cases of $n \geq 3$ (see Boland *et al.* (1994)). Under what condition $T(\lambda_1, \ldots, \lambda_n) \geq_{lr} T(\gamma_1, \ldots, \gamma_n)$ can hold is a problem that has not been well solved yet. Up to now, only a few results for special situations are available. Among them, Kochar and Xu (2015) showed that $T(\lambda_1, \ldots, \lambda_n) \geq_{lr} T(\bar{\lambda}, \ldots, \bar{\lambda})$, where $\bar{\lambda}$ is the average of $\lambda_1, \ldots, \lambda_n$, and Torrado and Kochar (2015) showed that if $(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2) \overset{m}{\succ} (\gamma_1, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_2)$, then $T(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2) \geq_{lr} T(\gamma_1, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_2)$.

The DKR theorem indicates that the reliability of a parallel system (with two components) is stochastically increasing (in terms of likelihood ratio order) by unbalancing its components. So, $T(\lambda_1, \lambda_2) \geq_{lr} T(\lambda_1 + \delta, \lambda_2 - \delta)$ (with symmetry condition $\lambda_1 + \delta \leq \lambda_2 - \delta$). We thus say that the reliability of a parallel system is decreasing in the direction $(1, -1)$. As we can verify, when the best components in two parallel systems are identical, there is no likelihood ratio order between the systems. This fact indicates that the reliability of a parallel system is not decreasing in the direction $(0, 1)$, or $(1, \infty)$. However, by intuition, if the quality of the best component along with other components becomes better, the reliability of the system should improve. Hence the reliability of a parallel system should be decreasing in the direction $(1, a)$, $a < \infty$. Wang and Laniado (2015) proved this is true for $a = 1/2$. Thus the reliability of a parallel system (with two components) is decreasing in the directions $(1, -1)$, $(1, \frac{1}{2})$, and their combinations.

In this paper, we extend such a result to general parallel systems. We introduce a new partial order, the so-called d-larger order among the hazard rate vectors (denote as $\overset{d}{\succeq}$), and show that

$$(\lambda_1, \ldots, \lambda_n) \overset{d}{\succeq} (\gamma_1, \ldots, \gamma_n) \Longrightarrow T(\lambda_1, \ldots, \lambda_n) \geq_{lr} T(\gamma_1, \ldots, \gamma_n).$$

The paper is organized as follows. In Section 2, we give some notation, definitions, and lemmas. Section 3 provides the proofs of the main results. The paper ends with a short discussion in Section 4. The proofs of the lemmas are moved to the Appendix.

2. NOTATION, DEFINITIONS, AND LEMMAS

Let X be a nonnegative continuous random variable with distribution function F_X, survival function $\bar{F}_X = 1 - F_X$, and density function f_X. The hazard function and the reversed hazard function of X are defined as $\lambda_X = f_X/\bar{F}_X$ and $r_X = f_X/F_X$, respectively. For two nonnegative continuous random variables X and Y, we say that X is larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$; X is larger than Y in hazard rate order (denoted by $X \geq_{hr} Y$) if $\lambda_X(t) \leq \lambda_Y(t)$; X is larger than Y in reversed hazard rate order (denoted by $X \geq_{rh} Y$) if $r_X(t) \geq r_Y(t)$; X is larger than Y in likelihood ratio order (denoted by $X \geq_{lr} Y$) if the ratio $f_X(t)/f_Y(t)$ is increasing in t. It is well known that the likelihood ratio order implies both the hazard rate order and reversed hazard rate order, while these two orders imply the usual stochastic order.

Given two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ with elements in increasing order, the vector $\boldsymbol{a}$ is said to majorize the vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{m}{\succ} \boldsymbol{b}$) if and only if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, \ldots, n-1$. If $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for all $k = 1, \ldots, n$, then the vector $\boldsymbol{a}$ is said to weakly majorize the vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b}$).

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we say $\boldsymbol{a} = \boldsymbol{b}$ if there is a positive number k such that $\boldsymbol{a} = k\boldsymbol{b}$. For given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$, we say that vector $\boldsymbol{u}$ is a combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ if $\boldsymbol{u} = a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m$, with $a_i \geq 0$ for $i = 1, \ldots, m$. For $k = 2, \ldots, n$, let $\boldsymbol{\epsilon}_k$ be the vector whose first element is 1 and the kth element is -1, while all others are 0; and let $\boldsymbol{\delta}_k$ be the vector whose first element is 1 and kth element is $1/2$, while all others are 0. For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, if $\boldsymbol{b} - \boldsymbol{a}$ is a combination of $\boldsymbol{\epsilon}_k, \boldsymbol{\delta}_j$ $(k, j = 2, \ldots, n)$, then we say $\boldsymbol{a}$ is d-larger than $\boldsymbol{b}$ and denote it by $\boldsymbol{a} \overset{d}{\succeq} \boldsymbol{b}$.

Lemma 2.1. *For vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$, if $\boldsymbol{\lambda} \overset{d}{\succ} \boldsymbol{\gamma}$, then $\boldsymbol{\lambda} \overset{w}{\succ} \boldsymbol{\gamma}$.*

Lemma 2.2. *Let $b(x) = x/(1 - e^{-x})$, $c(x) = xe^{-x}/(1 - e^{-x})$. Then, if $x > 0$, $b(x)$ is increasing, $c(x)$ is decreasing, $c'(x)$ is increasing, and $b'(x)c(x)$ is decreasing.*

Lemma 2.3. *Let $b(x) = x/(1 - e^{-x})$, $c(x) = xe^{-x}/(1 - e^{-x})$. Then, for any $x_1, x_2 > 0$,*

$$2b'(x_1)c(x_1) + c'(x_2)[b(x_2) - b(x_1)] > 0.$$

3. MAIN RESULTS AND PROOFS

Theorem 3.1. *Consider two parallel systems with components whose lifetimes are $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, respectively. Assume, for $i = 1, \ldots, n$, that X_i follows an exponential distribution with hazard rate λ_i and Y_i follows an exponential distribution with hazard rate γ_i. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{d}{\succeq} (\gamma_1, \ldots, \gamma_n) \Longrightarrow T(\lambda_1, \ldots, \lambda_n) \geq_{lr} T(\gamma_1, \ldots, \gamma_n).$$

Proof. Let $U = T(\lambda_1, \ldots, \lambda_n)$ and $V = T(\gamma_1, \ldots, \gamma_n)$. By Lemma 2.2, $\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\gamma}$. By the result of Misra and Misra (2013), $U \geq_{rh} V$. From Theorem 1.C.4 (b) of Shaked and Shanthikumar (2007), it is enough to show that the ratio of the reversed hazard rate function of U over that of V is increasing in $t > 0$. Denote the reversed hazard rate function of $U = T(\lambda_1, \ldots, \lambda_n)$ by $r_{\boldsymbol{\lambda}}(t)$, and that of V by $r_{\boldsymbol{\gamma}}(t)$. We have,

$$\psi(t) = \frac{r_{\boldsymbol{\lambda}}(t)}{r_{\boldsymbol{\gamma}}(t)} = \frac{\sum_{i=1}^n \dfrac{\lambda_i e^{-\lambda_i t}}{1 - e^{-\lambda_i t}}}{\sum_{i=1}^n \dfrac{\gamma_i e^{-\gamma_i t}}{1 - e^{-\gamma_i t}}} \overset{\text{def}}{=} \frac{\varphi(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\gamma}; t)}.$$

For convenience, we denote $A \overset{\text{sgn}}{=} B$ if A and B are of the same sign. So,

$$\psi'(t) \overset{\text{sgn}}{=} \varphi'_t(\boldsymbol{\lambda}; t)\varphi(\boldsymbol{\gamma}; t) - \varphi(\boldsymbol{\lambda}; t)\varphi'_t(\boldsymbol{\gamma}; t) \overset{\text{sgn}}{=} \frac{\varphi'_t(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\lambda}; t)} - \frac{\varphi'_t(\boldsymbol{\gamma}; t)}{\varphi(\boldsymbol{\gamma}; t)},$$

where

$$\frac{\varphi'_t(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\lambda}; t)} = -\frac{\sum_{i=1}^n \dfrac{\lambda_i^2 e^{-\lambda_i t}}{(1 - e^{-\lambda_i t})^2}}{\sum_{i=1}^n \dfrac{\lambda_i e^{-\lambda_i t}}{1 - e^{-\lambda_i t}}},$$

and similarly for $\varphi'_t(\boldsymbol{\gamma}; t)/\varphi(\boldsymbol{\gamma}; t)$.

Consider the function

$$\Phi(x_1, \ldots, x_n) = \frac{\sum_{i=1}^n \dfrac{x_i^2 e^{-x_i}}{(1 - e^{-x_i})^2}}{\sum_{i=1}^n \dfrac{x_i e^{-x_i}}{1 - e^{-x_i}}}, \qquad 0 < x_1 \leq x_2 \leq \cdots \leq x_n.$$

Denote the numerator part of $\Phi(x_1, \ldots, x_n)$ by N, and the denominator part by D. Let $b(x) = x/(1 - e^{-x})$, $c(x) = xe^{-x}/(1 - e^{-x})$, and $d(x) = b(x)c(x)$. In terms of these functions,

$$N = \sum_{i=1}^n d(x_i), \qquad D = \sum_{i=1}^n c(x_i), \qquad \frac{\partial \Phi}{\partial x_i} \overset{\text{sgn}}{=} \frac{\partial N}{\partial x_i} D - N \frac{\partial D}{\partial x_i} = d'(x_i)D - c'(x_i)N.$$

By Lemma 2.1, for $x > 0$, $c(x)$ is decreasing, $c'(x)$ is increasing, $b(x)$ is increasing, and the function $b'(x)c(x)$ is decreasing. Hence,

$$\begin{aligned}
\triangledown_{\boldsymbol{\epsilon}_k} \Phi = \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Phi}{\partial x_k} &= \big[d'(x_1) - d'(x_k)\big] \sum_{j=1}^n c(x_j) - \big[c'(x_1) - c'(x_k)\big] \sum_{j=1}^n b(x_j)c(x_j) \\
&\geq \Big\{\big[d'(x_1) - d'(x_k)\big] - \big[c'(x_1) - c'(x_k)\big]b(x_1)\Big\} \sum_{j=1}^n c(x_j) \\
&\overset{\text{sgn}}{=} \big[d'(x_1) - d'(x_k)\big] - \big[c'(x_1) - c'(x_k)\big]b(x_1) \\
&= b'(x_1)c(x_1) + b(x_1)c'(x_1) - b'(x_k)c(x_k) - b(x_k)c'(x_k) - c'(x_1)b(x_1) + c'(x_k)b(x_1)
\end{aligned}$$

$$= b'(x_1)c(x_1) - b'(x_k)c(x_k) - b(x_k)c'(x_k) + c'(x_k)b(x_1)$$
$$= b'(x_1)c(x_1) - b'(x_k)c(x_k) - c'(x_k)[b(x_k) - b(x_1)]$$
$$\geq b'(x_1)c(x_1) - b'(x_k)c(x_k) \geq 0.$$

For $\bigtriangledown_{\boldsymbol{\delta}_k}\Phi$, we have,

$$\begin{aligned}
\bigtriangledown_{\boldsymbol{\delta}_k}\Phi &= \frac{\partial\Phi}{\partial x_1} + \frac{1}{2}\frac{\partial\Phi}{\partial x_k} \\
&= \Big[d'(x_1) + \frac{1}{2}d'(x_k)\Big]\sum_{j=1}^{n} c(x_j) - \Big[c'(x_1) + \frac{1}{2}c'(x_k)\Big]\sum_{j=1}^{n} b(x_j)c(x_j) \\
&\geq \Big\{\Big[d'(x_1) + \frac{1}{2}d'(x_k)\Big] - \Big[c'(x_1) + \frac{1}{2}c'(x_k)\Big]b(x_1)\Big\}\sum_{j=1}^{n} c(x_j) \\
&\overset{\text{sgn}}{=} \Big[d'(x_1) + \frac{1}{2}d'(x_k)\Big] - \Big[c'(x_1) + \frac{1}{2}c'(x_k)\Big]b(x_1) \\
&= b'(x_1)c(x_1) + b(x_1)c'(x_1) + \frac{1}{2}b'(x_k)c(x_k) + \frac{1}{2}b(x_k)c'(x_k) - c'(x_1)b(x_1) - \frac{1}{2}c'(x_k)b(x_1) \\
&= b'(x_1)c(x_1) + \frac{1}{2}b'(x_k)c(x_k) + \frac{1}{2}c'(x_k)\big[b(x_k) - b(x_1)\big] \\
&\geq b'(x_1)c(x_1) + \frac{1}{2}c'(x_k)\big[b(x_k) - b(x_1)\big] \\
&\overset{\text{sgn}}{=} 2b'(x_1)c(x_1) + c'(x_k)\big[b(x_k) - b(x_1)\big] \geq 0,
\end{aligned}$$

where the last inequality comes from Lemma 2.1.

By Lemma 2.1, $\boldsymbol{\lambda} \overset{d}{\succeq} \boldsymbol{\gamma}$ implies $\boldsymbol{v} = \boldsymbol{\gamma} - \boldsymbol{\lambda} = \sum a_i\boldsymbol{\epsilon}_i + \sum b_j\boldsymbol{\delta}_j$, with $a_i, b_j \geq 0$. Thus, $\bigtriangledown_{\boldsymbol{v}}\Phi = \sum a_i \bigtriangledown_{\boldsymbol{\epsilon}_i}\Phi + \sum b_j \bigtriangledown_{\boldsymbol{\delta}_j}\Phi \geq 0$. Hence, in the direction $\boldsymbol{v} = \boldsymbol{\gamma} - \boldsymbol{\lambda}$, the function $\Phi(x_1, \ldots, x_n)$ is increasing. Therefore,

$$\begin{aligned}
\psi'(t) \overset{\text{sgn}}{=} \frac{\varphi_t'(\boldsymbol{\lambda}; t)}{\varphi(\boldsymbol{\lambda}; t)} - \frac{\varphi_t'(\boldsymbol{\gamma}; t)}{\varphi(\boldsymbol{\gamma}; t)} &= -\frac{\sum_{i=1}^{n} \frac{\lambda_i^2 t^2 e^{-\lambda_i t}}{(1 - e^{-\lambda_i t})^2}}{\sum_{i=1}^{n} \frac{\lambda_i t e^{-\lambda_i t}}{1 - e^{-\lambda_i t}}} + \frac{\sum_{i=1}^{n} \frac{\gamma_i^2 t^2 e^{-\gamma_i t}}{(1 - e^{-\gamma_i t})^2}}{\sum_{i=1}^{n} \frac{\gamma_i t e^{-\gamma_i t}}{1 - e^{-\gamma_i t}}} \\
&= \Phi(\gamma_1 t, \ldots, \gamma_n t) - \Phi(\lambda_1 t, \ldots, \lambda_n t) \geq 0.
\end{aligned}$$

This shows that $\psi'(t) \geq 0$, and thus $T(\lambda_1, \ldots, \lambda_n) \geq_{lr} T(\gamma_1, \ldots, \gamma_n)$. □

Remark 3.2. For $n = 2$, $(\lambda_1, \lambda_2) \overset{m}{\succ} (\gamma_1, \gamma_2)$ is equivalent to $(\lambda_1, \lambda_2) = (\gamma_1, \gamma_2)$, or, $\gamma_2 - \lambda_2 = -(\gamma_1 - \lambda_1) < 0$. Excluding the trivial condition $(\lambda_1, \lambda_2) = (\gamma_1, \gamma_2)$, the majorization order $(\lambda_1, \lambda_2) \overset{m}{\succ} (\gamma_1, \gamma_2)$ is equivalent to $(\gamma_1, \gamma_2) - (\lambda_1, \lambda_2) = (1, -1)$. Hence, Theorem 3.1 generalizes and extends the classical DKR theorem to the case of $n \geq 2$.

Remark 3.3. By Theorem 1.C.8. in Shaked and Shanthikumar (2007), the likelihood ratio order is closed under increasing transformations. It is self-evident that an increasing transformation keeps the maximum order. Hence, Theorem 3.1 can be readily extended to the case of the proportional hazard rate (PHR) models and the case of Weibull distributed variables.

4. DISCUSSION

For a parallel system with two exponential components, the DKR theorem shows that the reliability of the system (in terms of likelihood ratio order) is decreasing in the direction $(1, -1)$. Wang and Laniado (2015) proved that the reliability of the system is decreasing in the directions $(1, -1)$, $(1, \frac{1}{2})$, and their combinations. As pointed out by Boland *et al.* (1994), the DKR theorem cannot be extended directly to the cases of $n \geq 3$. In this paper, we find a way to extend the result of Wang and Laniado (2015) to general n, and incidentally, extend the DKR theorem to general n.

Based on the result of Wang and Laniado (2015), the reliability of the system is decreasing in the direction $(1, a)$, $-1 \leq a \leq 1/2$. As we have mentioned, the reliability of the system is not decreasing in the direction $(1, \infty)$. It is an interesting question, what is the maximum value of a such that the reliability of the system is decreasing in the direction $(1, a)$. Further investigation is needed for this question.

APPENDIX: PROOFS OF THE LEMMAS

Proof of Lemma 2.1. Denote $\boldsymbol{v} = \boldsymbol{\gamma} - \boldsymbol{\lambda} = (v_1, v_2, \ldots, v_n)$. Clearly, $\boldsymbol{\lambda} \overset{w}{\succ} \boldsymbol{\gamma}$ if and only if $\sum_{i=1}^{k} v_i \geq 0$ for any $k = 1, \ldots, n$. When $\boldsymbol{\lambda} \overset{d}{\succ} \boldsymbol{\gamma}$, it is a combination of those $\boldsymbol{\epsilon}_k$ and $\boldsymbol{\delta}_j$. The lemma is self-evident now. $\square$

Proof of Lemma 2.2. We just establish that the function $b'(x)c(x)$ is decreasing in $x > 0$. Denote $p(x) = 1 - e^{-x} - xe^{-x}$. Then, $b'(x)c(x) = [p(x)xe^{-x}]/(1 - e^{-x})^3$. We have,

$$\begin{aligned}
[b'(x)c(x)]' &= [p(x)xe^{-x}]'(1 - e^{-x})^{-3} - 3(1 - e^{-x})^{-4}[p(x)xe^{-x}] \\
&\overset{\text{sgn}}{=} [p(x)xe^{-x}]'(1 - e^{-x}) - 3e^{-x}[p(x)xe^{-x}] \\
&\overset{\text{sgn}}{=} [x^2e^{-x} + p(x)(1 - x)](1 - e^{-x}) - 3xp(x)e^{-x} \\
&= 1 - x - 2e^{-x} - 2xe^{-x} + 2x^2e^{-x} + e^{-2x} + 3xe^{-2x} + x^2e^{-2x} \\
&\overset{\text{def}}{=} I(x).
\end{aligned}$$

Easily,

$$\begin{aligned}
I'(x) &= -1 + 6xe^{-x} - 2x^2e^{-x} + e^{-2x} - 4xe^{-2x} - 2x^2e^{-2x} \\
&\overset{\text{sgn}}{=} -e^{2x} + 6xe^{x} - 2x^2e^{x} + 1 - 4x - 2x^2 \\
&\overset{\text{def}}{=} -J(x).
\end{aligned}$$

We have,

$$\begin{aligned}
J'(x) &\overset{\text{sgn}}{=} e^{2x} - 3e^{x} - xe^{x} + x^2e^{x} + 2 + 2x, \\
J''(x) &= 2e^{2x} - 4e^{x} + xe^{x} + x^2e^{x} + 2 \\
&\geq 2e^{2x} - 4e^{x} + 2 \geq 0.
\end{aligned}$$

So, $J'(x) \geq J'(0) = 0$ and thus, $J(x) \geq J(x) = 0$. Hence, $I'(x) \leq 0$, which indicates that the function $b'(x)c(x)$ is decreasing in $x > 0$. $\square$

Proof of Lemma 2.3. It is clear that the conclusion is valid when $x_1 > x_2 > 0$, since the function $b(x)$ is increasing. Consider the function $f(x) = 2b'(x_1)c(x_1) + b(x_1)c(x) - b(x)c(x)$, $x > x_1 > 0$. Clearly, $f(x_1) = f(\infty) > 0$, so this function has an extreme value at a point $x_0 \in (x_1, \infty)$. We have,

$$\begin{aligned}
f'(x) &= b(x_1)c'(x) - [b(x)c(x)]' \\
&= b(x_1)c'(x) - b'(x)c(x) - b(x)c'(x) \\
&= b(x_1)[b'(x)e^{-x} - c(x)] - b'(x)c(x) - b(x)[b'(x)e^{-x} - c(x)] \\
&= b(x_1)b'(x)e^{-x} - b(x_1)c(x) - 2b'(x)c(x) + b(x)c(x).
\end{aligned}$$

So, x_0 satisfies

$$b(x_1)b'(x_0)e^{-x_0} - b(x_1)c(x_0) - 2b'(x_0)c(x_0) + b(x_0)c(x_0) = 0.$$

Thus,

$$f_{\min} = f(x_0) = 2b'(x_1)c(x_1) + b(x_1)c(x_0) - b(x_0)c(x_0)$$

$$= 2b'(x_1)c(x_1) + b(x_1)c(x_0) + b(x_1)b'(x_0)e^{-x_0} - b(x_1)c(x_0) - 2b'(x_0)c(x_0)$$
$$= 2b'(x_1)c(x_1) + b(x_1)b'(x_0)e^{-x_0} - 2b'(x_0)c(x_0)$$
$$> 2b'(x_1)c(x_1) - 2b'(x_0)c(x_0) > 0.$$

This completes the proof of the lemma. □

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