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Explicit Expressions and Statistical Inference of Generalized Rayleigh Distribution Based on Lower Record Values

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Abstract—This article addresses the problem of frequentist and Bayesian estimation of the parameters of the generalized Rayleigh distribution using lower record values. The explicit expressions for single and product moments of lower record values from this distribution are given. The maximum likelihood and Bayes estimates based on lower records are derived for the parameters of the distribution. We consider the Bayes estimators of the parameters under the assumption of Gamma priors with respect to the shape and scale parameters. The Bayes estimators are inaccessible in explicit form. We analyze them with reference to both symmetric and asymmetric loss functions. We also derive the Bayes interval of this distribution. We carry out Monte Carlo simulations to compare the performance of the proposed methods.

Keywords: lower record values, single and product moments, characterization, generalized Rayleigh distribution, Bayes estimator, general entropy loss function, maximum likelihood estimator.

2000 Mathematics Subject Classification: primary 62M, 60F17.

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1. INTRODUCTION

Surles and Padgett (2001) introduced the two-parameter Burr Type X distribution and correctly named it as the generalized Rayleigh distribution. Note that the two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution.

Let $X_{L(1)}, X_{L(2)}, \dots, X_{L(n)}$ be the first n lower record values from generalized Rayleigh distribution (GR) with *pdf*

$$f(x;\alpha,\beta) = 2\alpha\beta^2 x\, e^{-(\beta x)^2}\big(1-e^{-(\beta x)^2}\big)^{\alpha-1}, \qquad x>0, \quad \alpha,\beta>0, \tag{1.1}$$

and the corresponding *cdf* is

$$F(x;\alpha,\beta) = \big(1-e^{-(\beta x)^2}\big)^{\alpha}, \qquad x>0, \quad \alpha,\beta>0. \tag{1.2}$$

The survival function is

$$S(x;\alpha,\beta) = 1-\big(1-e^{-(\beta x)^2}\big)^{\alpha}, \qquad x>0, \quad \alpha,\beta>0, \tag{1.3}$$

and the hazard function

$$h(x;\alpha,\beta) = \frac{2\alpha\beta^2 x\, e^{-(\beta x)^2}\big(1-e^{-(\beta x)^2}\big)^{\alpha-1}}{1-\big(1-e^{-(\beta x)^2}\big)^{\alpha}}. \tag{1.4}$$

Here α and β are the shape and scale parameters respectively. The GR distribution with shape parameter α and scale parameter β will be denoted by $GR(\alpha,\beta)$. An extensive study on the properties of the GR distribution was carried out by Surles and Padgett (2004). Due to its practicality, the two-parameter GR distribution can be used quite effectively in modeling strength data and general lifetime data. The various

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methods of estimation of this distribution have been studied by Kundu and Raqab (2005). The GR distribution has a decreasing survival function and right skewed unimodal density function for $\alpha \leq 1/2$ and $\alpha > 1/2$ respectively. The hazard function of the GR distribution can never be constant. It is either bathtub type or an increasing function, depending on the shape parameter α. For $\alpha \leq 1/2$, the hazard function of $GR(\alpha, \beta)$ is bathtub type and for $\alpha > 1/2$, it has an increasing hazard function. Plotted below are the probability density function (Fig. 1) and Cumulative Distribution Function (Fig. 2), for $\alpha = 1, 2, 3$ and $\beta = 1, 2, 3$, and the hazard functions (Fig. 3) and Survival Function (Fig. 4) for GR distribution when $\alpha = 0.2, 0.25, 0.3$ and $\beta = 0.2, 0.25, 0.3$.

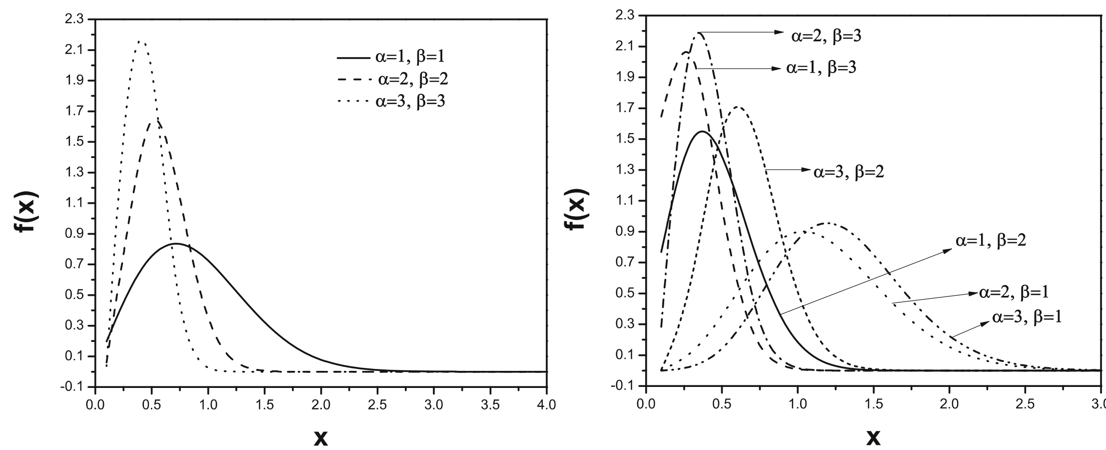

Fig. 1. GRD Density Function

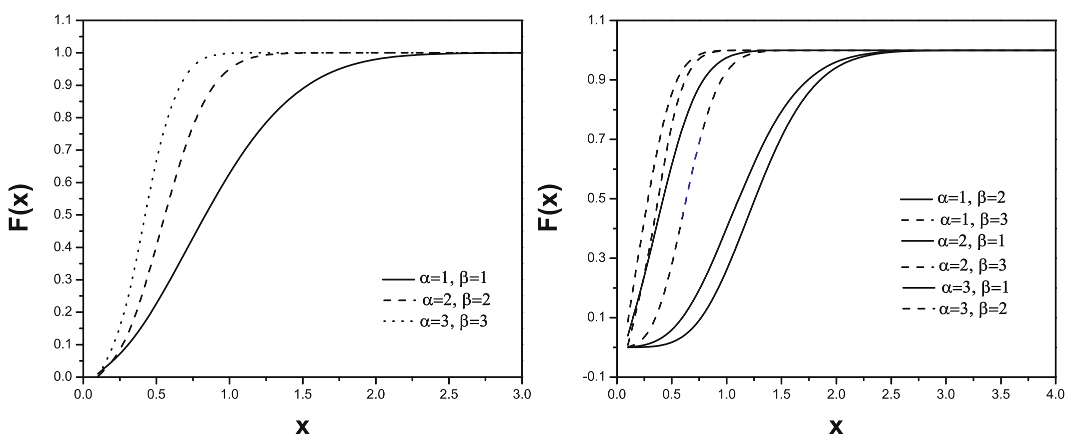

Fig. 2. GRD Cumulative Distribution Function

For $\alpha \geq 1$, the kth moments of the GR distribution are given by

$$\mu^k = \frac{\alpha}{\beta^k} \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma(p)\Gamma[(k/2)+1]}{p!\Gamma(\alpha - p)(p+1)^{(r/2)+1}}.$$

In the context of order statistics model and reliability theory, the life length of the r—out-of—n system is the $(n-r+1)$th order statistic in a sample of size n. Another related model is the model of record statistics defined by Chandler (1952) as a model for successive extremes in a sequence of independent and identically distributed (iid) random variables. This model takes a certain dependence structure into consideration. That is, the life-length distribution of the components in the system may change after each failure of the components. For this type of model, we consider the lower record statistics. If various voltages of equipment are considered, only the voltages less than the previous one can be recorded. These

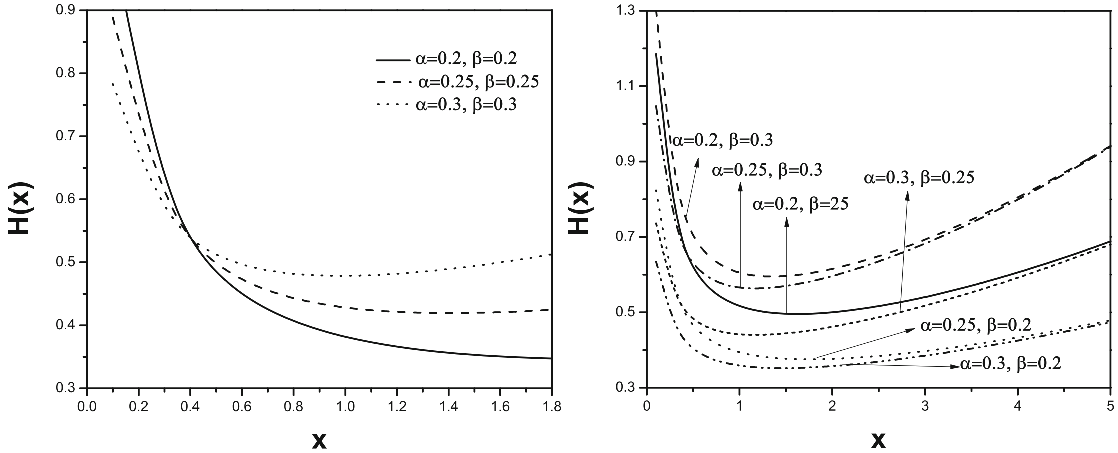

Fig. 3. GRD Hazard Function

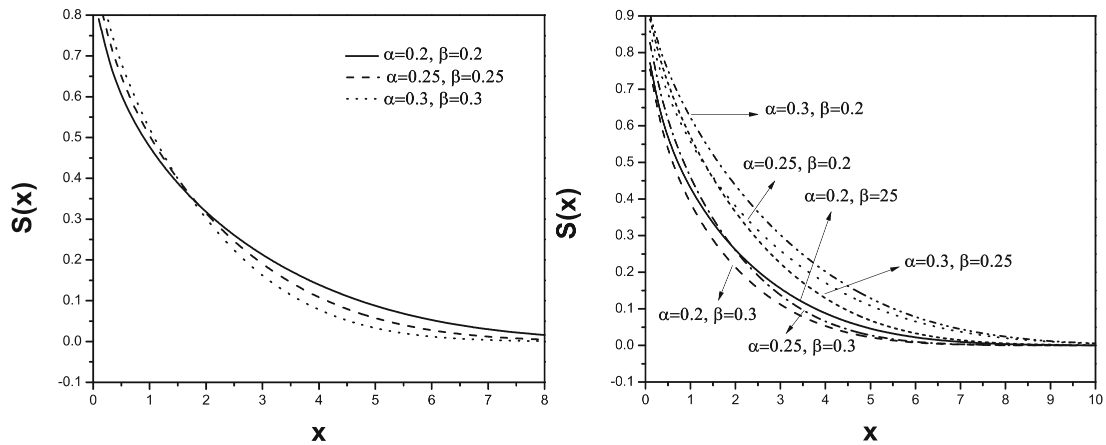

Fig. 4. GRD Survival Function

recorded voltages are the lower record value sequence. Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them, for example, Olympic records or world records in sport. Record values are also used in reliability theory. Moreover, these statistics are closely connected with the occurrence times of some corresponding nonhomogeneous Poisson process used in shock models.

The study of record values and associated statistics is of great significance in many real life situations such as meteorology, seismology, athletic events, economics, and life testing. Theory of record values and its distributional properties have been extensively studied in the literature, Ahsanullah (1995), Balakrishnan and Ahsanullah (1994, 1995), Grudzien and Szynal (1997) and Arnold et al. (1992, 1998). Kumar and Kulshrestha (2013) have established recurrence relations for moments of record values from generalized Pareto distribution.

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (*iid*) random variables with cumulative distribution function (*cdf*) $F(x)$ and probability density function (*pdf*) $f(x)$. The jth order statistic of the sample $(X_1, X_2, \ldots, X_n)$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$, we define the sequence $\{L^{(k)}(n), n \geq 1\}$ of kth lower record times of $X_1, X_2, \ldots$ as follows:

$$L^{(k)}(1) = 1,$$
$$L^{(k)}(n+1) = \min\{j > L^{(k)}(n) \colon X_{k:L^{(k)}(n)+k-1} > X_{k:j+k-1}\}.$$

The sequence $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L^{(k)}(n)+k-1}$, $n = 1, 2, \ldots$, is called the sequence of kth lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Z_0^{(k)} = 0$. Note that for $k = 1$ we have $Y_n^{(1)} = X_{L(n)}$, $n \geq 1$, i.e., record values of $\{X_n, n \geq 1\}$.

The joint *pdf* of kth lower record values $Z_1^{(k)}, \ldots, Z_n^{(k)}$ can be obtained as the joint *pdf* of kth upper record values of $\{-X_n, n \geq 1\}$ and is given by (Pawlas and Szynal (1998))

$$f_{z_1^{(k)},\ldots,z_n^{(k)}}(z_1, \ldots, z_n) = k^n \left(\prod_{i=1}^{n-1} \frac{f(z_i)}{F(z_i)}\right) [F(z_n)]^{k-1} f(z_n), \quad z_1 > z_2 > \ldots > z_n.$$

In view of the above equation, the marginal *pdf* of $X_{L(n)}^{(k)}$, $n \geq 1$, is given by

$$f_{X_{L(n)}^{(k)}}(x) = \frac{k^n}{\Gamma(n)}[-\log(F(x))]^{n-1}[F(x)]^{k-1}f(x), \quad n \geq 1, \tag{1.5}$$

and the joint *pdf* of $X_{L(m)}^{(k)}$ and $X_{L(n)}^{(k)}$, $1 \leq m < n$, $n > 2$, is given by

$$f_{X_{L(m)}^{(k)},X_{L(n)}^{(k)}}(x, y) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}[-\log(F(x))]^{m-1}[-\log(F(y)) + \log(F(x))]^{n-m-1} \times [F(y)]^{k-1}\frac{f(x)}{F(x)}f(y), \qquad x > y, \quad 1 \leq m < n, \quad n \geq 2, \tag{1.6}$$

where $\Gamma(x)$ is the gamma function. When x is a positive integer, $\Gamma(x) = (x-1)!$.

Let $X_1, X_2, \ldots, X_n$ be a random sample of the GR distribution with *pdf* and *cdf* as in (1.1) and (1.2) respectively, and let $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ be the first n lower record values obtained from this sample. Let us denote the single moments $E\big((X_{L(n)}^{(k)})^r\big)$ by $\mu_{L(n):k}^{(r)}$, $r, n = 1, 2, \ldots$, and the product moments $E\big((X_{L(m)}^{(k)})^r, (X_{L(n)}^{(k)})^s\big)$ by $\mu_{L(m,n):k}^{(r,s)}$ for $1 \leq m \leq n-1$. For convenience, let us also use $\mu_{L(n)}$ for $\mu_{L(n)}^{(1)}$.

The presentation of the content of this work is as follows. In Section 2, we obtain explicit expressions for single and product moments of lower record values from GR distribution. In Section 3, we characterize this distribution by conditional expectation of record values. We use maximum likelihood estimators ($MLEs$) as a part of frequentist methodology for parameter estimation in Section 4. The asymptotic confidence intervals based on the observed Fisher's information matrix are also discussed here. Next, we consider Bayesian estimation of the unknown parameters in Section 5. The Bayesian inference mainly depends on two features: choice of prior distribution of the parameters and the loss function to be used for Bayesian computations. In this article, we use Gamma priors for both scale and shape parameters and they are assumed to be independent of each other. For Bayesian inference, we use a general entropy loss function. The main idea behind using this loss function is that with particular choices of the parameter involved in the form of loss function this method produces estimates under several well-known loss functions, which are both symmetric and asymmetric in nature. A brief discussion of this loss function is presented later in this article in Section 6. The joint posterior distribution is complicated and thus the posterior sampling is not straightforward to implement. Here we propose a Markov Chain Monte Carlo technique which involves Metropolis-Hasting algorithm for posterior sampling. Besides Bayes estimates, we also obtain a two-sided Bayes probability intervals as a Bayesian counterpart of the asymptotic confidence intervals in Section 6. Bayes estimation heavily depends on the choice of hyperparameters involved in the prior distributions, which is quite sensitive for the Bayesian inference. An alternative way to avoid this issue is to use an empirical Bayes estimation procedure for parameter estimation; Section 7 introduces this procedure for the aforementioned distribution.

Another important problem in life-testing experiments involving record value is the prediction of unknown observation based on currently available samples, sometimes referred to as informative samples. The prediction of future record value based on given records is a useful research component involved in many applications. Section 8 introduces both frequentist and Bayesian prediction procedures for future record value given the informative records. Besides estimating the future record, we also obtain predictive interval, which is quite effective in statistical applications. Section 8 contains a brief conclusion.

2. RELATIONS FOR MOMENTS

In this section we will derive the explicit expressions for single and product moments of the kth lower record values from the GR distribution.

For the GR distribution as given in (1.2), the rth moments $E(X_{L(n)}^{(k)})^r$ are given as

$$\mu_{L(n):k}^{(r)} = \frac{k^n}{\Gamma(n)} \int_0^\infty x^r [F(x)]^{k-1} [-\log(F(x))]^{n-1} f(x)\, dx. \tag{2.1}$$

By setting $z = [F(x)]^{1/\alpha}$ in (2.1), and the fact that $-\log(1-z) = \sum_{p=1}^\infty \frac{z^p}{p}$, we get

$$\mu_{L(n):k}^{(r)} = \frac{(\alpha k)^n}{\beta^r} \sum_{p=0}^\infty \frac{\phi_p(r/2)}{[\alpha k + p + (r/2)]^n}, \tag{2.2}$$

and hence for lower records

$$\mu_{L(n)}^{(r)} = \frac{\alpha^n}{\beta^r} \sum_{p=0}^\infty \frac{\phi_p(r/2)}{[\alpha + p + (r/2)]^n}. \tag{2.3}$$

As a special case of (2.3), the first single moment (mean) and the second single moment are, respectively

$$\mu_{L(n)} = \frac{\alpha^n}{\beta} \sum_{p=1}^\infty \frac{1}{p(\alpha+p)^n} \tag{2.4}$$

and

$$\mu_{L(n)}^{(2)} = \frac{\alpha^n}{\beta^2} \sum_{p=1}^\infty \sum_{q=1}^\infty \frac{1}{pq(\alpha+p+q)^n}. \tag{2.5}$$

Next, the rth and sth product moments are

$$\mu_{L(m,n):k}^{(r,s)} = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty x^r [-\log(F(x))]^{m-1} \frac{f(x)}{F(x)} G(x)\, dx, \tag{2.6}$$

where

$$G(x) = \int_0^x y^s [-\log(F(y)) + \log(F(x))]^{n-m-1} [F(y)]^{k-1} f(y)\, dy, \tag{2.7}$$

which upon making the transformation $z = [F(y)]^{1/\alpha}$ in (2.7) and using (2.6) yields

$$\mu_{L(m,n):k}^{(r,s)} = \frac{(\alpha k)^n}{\beta^{r+s}} \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{\phi_p(s/2)\phi_q(r/2)}{[\alpha k + p + (s/2)]^{m-n} [\alpha k + p + q + (r+s)/2]^m} \tag{2.8}$$

and hence for lower records

$$\mu_{L(m,n)}^{(r,s)} = \frac{\alpha^n}{\beta^{r+s}} \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{\phi_p(s/2)\phi_q(r/2)}{[\alpha + p + (s/2)]^{m-n} [\alpha + p + q + (r+s)/2]^m}. \tag{2.9}$$

For $r = s = 1$ then

$$\mu_{L(m,n)} = \frac{\alpha^n}{\beta^2} \sum_{p=1}^\infty \sum_{q=1}^\infty \frac{1}{pq(\alpha+p)^{m-n}(\alpha+p+q)^m}. \tag{2.10}$$

Making use of (2.4), (2.5) and (2.10) we can evaluate the means $\mu_{L(n)}$, variances $\sigma_{L(n)}^2 = \mu_{L(n)}^{(2)} - \mu_{L(n)}^2$, $1 \le m < n-1$, and covariances $\sigma_{L(m,n)} = \mu_{L(m,n)} - \mu_{L(m)}\mu_{L(n)}$ of record statistics, respectively.

The explicit expressions for the first single moments of record statistics given in (2.4) allow us to evaluate the means of all record statistics. Table 1 presents the means of $X_{L(n)}$, $n = 1(1)5$, for $\beta = 1(1)4$

Table 1. Means of record Statistics

n	$\beta = 1$			$\beta = 2$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	0.9941520	1.4883338	1.8158910	0.4970760	0.7441690	0.9079455
2	0.3550489	0.7100641	0.9817132	0.1775244	0.3550321	0.4908566
3	0.1530090	0.4019037	0.6216839	0.0765045	0.2009519	0.3108420
4	0.0706858	0.2433184	0.4197917	0.0353429	0.1216592	0.2098958
5	0.0337580	0.1524743	0.2932269	0.0168790	0.0762372	0.1466134
n	$\beta = 3$			$\beta = 4$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	0.3313840	0.4961127	0.6052970	0.2485380	0.3720845	0.4539728
2	0.1183496	0.2366880	0.3272377	0.0887622	0.1775160	0.2454283
3	0.0510030	0.1339679	0.2072280	0.0382522	0.1004759	0.1554210
4	0.0235619	0.0811061	0.1399306	0.0176714	0.0608296	0.1049479
5	0.0112527	0.0508248	0.0977423	0.0084395	0.0381186	0.0733067

and $\alpha = 1(1)3$ to seven decimal places. For the computation of variances and covariances, the product moments $\mu_{L(m,n)}$, $1 \le m \le n$, were computed first. The diagonal elements $\sigma_{L(m,n)} = \sigma_n^2$ are obtained from the explicit expressions given in (2.4) and (2.5). Next, the explicit expression (2.10) was used for the computation of product moments of any two record statistics. For $m > n$, the values of $\sigma_{L(m,n)}$ were filled in by using the symmetry of the variance-covariance matrix $((\sigma_{L(m,n)}))$. Tables 2 and 3 provide the variance and product moments of record statistics to seven decimal places for $\beta = 1(1)4$ and $\alpha = 1(1)3$. The values of means, variances and product moments for $n \ge 6$ are evaluated but not presented here.

Table 2. Variances of record Statistics

n	$\beta = 1$			$\beta = 2$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	0.9411758	1.1442041	1.2143144	0.2352908	0.2860513	0.3035658
2	0.1797892	0.3173054	0.3947753	0.0449470	0.0793240	0.0986864
3	0.0474583	0.1219664	0.1759941	0.0118684	0.0304890	0.0439980
4	0.0141434	0.0537651	0.0906063	0.0035410	0.0134391	0.0226524
5	0.0044504	0.0252732	0.0500560	0.0011153	0.0063174	0.0125163
n	$\beta = 3$			$\beta = 4$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	0.1045770	0.1271353	0.1349220	0.0588181	0.071516	0.0759012
2	0.0199732	0.0352581	0.0438643	0.0112424	0.0198270	0.0246743
3	0.0052690	0.0135523	0.0195562	0.0029671	0.0076242	0.0110050
4	0.0015751	0.0059710	0.0100704	0.0008884	0.0033603	0.0056651
5	0.0004934	0.0028073	0.0055673	0.0002790	0.0015772	0.0031264

Table 3. Product moments of record Statistics

m	n	$\beta = 1$			$\beta = 2$		
		$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	1	1.0394452	0.9883383	0.9313388	0.2598613	0.2470846	0.2328347
1	2	0.3081528	0.4254467	0.4748024	0.0770382	0.1063617	0.1187006
1	3	0.1223841	0.2363410	0.3026547	0.0305960	0.0590853	0.0756637
1	4	0.0543617	0.1432744	0.2071323	0.0135904	0.0358186	0.0517831
1	5	0.0254768	0.0902913	0.1463156	0.0063692	0.0225728	0.0365789
2	2	0.0994734	0.0788283	0.0663106	0.0248684	0.0197071	0.0165777
2	3	0.0323559	0.0416457	0.0417339	0.0080890	0.0104114	0.0104335
2	4	0.0135013	0.0251327	0.0288325	0.0033753	0.0062832	0.0072081
2	5	0.0061740	0.0159081	0.0205771	0.0015435	0.0039770	0.0051443
3	3	0.0106201	0.0087501	0.0067826	0.0026550	0.0021875	0.0016956
3	4	0.0040330	0.0052122	0.0046992	0.0010082	0.0013030	0.0011748
3	5	0.0017852	0.0033016	0.0033772	0.0004463	0.0008254	0.0008443
4	4	0.0013045	0.0011615	0.0008210	0.0003261	0.0002904	0.0002053
4	5	0.0005548	0.0007342	0.0005927	0.0001387	0.0001836	0.0001482
5	5	0.0001792	0.0001695	0.0001080	0.0007167	0.0006782	0.0004319
m	n	$\beta = 3$			$\beta = 4$		
		$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	1	0.1154939	0.1098154	0.1034821	0.0649653	0.0617711	0.0582087
1	2	0.0342392	0.0472719	0.0527558	0.0192596	0.0265904	0.0296752
1	3	0.0135982	0.0262601	0.0336283	0.0076490	0.0147713	0.0189159
1	4	0.0060402	0.0159194	0.0230147	0.0033976	0.0089547	0.0129458
1	5	0.0028308	0.0100324	0.0162573	0.0015923	0.0056432	0.0091447
2	2	0.0110526	0.0087587	0.0073678	0.0062171	0.0049268	0.0041444
2	3	0.0035951	0.0046273	0.0046371	0.0020222	0.0026029	0.0026084
2	4	0.0015001	0.0027925	0.0032036	0.0008438	0.0015708	0.0018020
2	5	0.0006860	0.0017676	0.0022863	0.0003859	0.0009943	0.0012861
3	3	0.0011800	0.0009722	0.0007536	0.0006638	0.0005469	0.0004239
3	4	0.0004481	0.0005791	0.0005221	0.0002521	0.0003258	0.0002937
3	5	0.0001984	0.0003668	0.0003752	0.0001116	0.0002063	0.0002111
4	4	0.0001449	0.0001291	0.0000912	0.0001235	0.0001100	0.0000777
4	5	0.0001096	0.0001450	0.0001171	0.0001600	0.0000180	0.0001712
5	5	0.0000301	0.0000400	0.0000600	0.0000101	0.0000002	0.0000001

3. CHARACTERIZATION

This section contains characterizations of GR distribution by conditional expectation of lower record values.

Let $\{X_n, n \geq 1\}$ be a sequence of $i.i.d$ continuous random variables with *df* $F(x)$ and *pdf* $f(x)$. Let $X_{L(n)}$ be the nth lower record values, then the conditional *pdf* of $X_{L(n)}$ given $X_{L(m)} = x$, $1 \leq m < n$ in view of (1.1) and (1.2) for $k = 1$, is

$$f(X_{L(n)} \,|\, X_{L(m)} = x) = \frac{1}{(n-m-1)!}[-\log(F(y)) + \log(F(x))]^{n-m-1}\frac{f(y)}{F(x)}, \quad x > y. \tag{3.1}$$

Theorem 3.1. *Let X be an absolutely continuous random variable with df $F(x)$ and pdf $f(x)$ on the support $(0,\infty)$, then for $m < n$,*

$$E[X_{L(n)} \,|\, X_{L(m)} = x] = \frac{(2\alpha)^{n-m}}{\beta}\sum_{p=0}^{\infty}\phi_p(1/2)\frac{(1-e^{(\beta x)^2})^{p+(1/2)}}{[2(\alpha+p)+1]^{n-m}} \tag{3.2}$$

if and only if

$$F(x;\alpha,\beta) = \left(1 - e^{-(\beta x)^2}\right)^{\alpha}, \qquad x > 0, \quad \alpha,\beta > 0.$$

Proof. From (3.1), we have

$$E[X_{L(n)} \,|\, X_{L(m)} = x] = \frac{1}{(n-m-1)!}\int_0^x y\Big[\log\Big(\frac{F(x)}{F(y)}\Big)\Big]^{n-m-1}\frac{f(y)}{F(x)}\,dy \tag{3.3}$$

By setting $t = \log\left(\frac{F(x)}{F(y)}\right)$ from (1.4) in (3.3), we obtain

$$E[X_{U(n)} \,|\, X_{U(m)} = x] = \frac{1}{\alpha(n-m-1)!}\int_0^{\infty}\big[-\log\big\{1 - \big(1 - e^{-(\beta x)^2}\big)e^{-t/\alpha}\big\}\big]^{1/2} t^{n-m-1}e^{-t}\,dt.$$

Simplifying the above expression, we derive the relation given in (3.2).

To prove the sufficiency part, we have from (3.1) and (3.2)

$$\frac{1}{(n-m-1)!}\int_x^{\infty} y[-\log(F(y)) + \log(F(x))]^{n-m-1}f(y)\,dy = F(x)H_r(x), \tag{3.4}$$

where

$$H_r(x) = \frac{(2\alpha)^{n-m}}{\beta}\sum_{p=0}^{\infty}\phi_p(1/2)\frac{(1-e^{(\beta x)^2})^{p+(1/2)}}{[2(\alpha+p)+1]^{n-m}}.$$

Differentiating both sides of (3.4) with respect to x, we get

$$\frac{1}{(n-m-2)!}\int_0^x y[-\log(F(y)) + \log(F(x))]^{n-m-2}\frac{f(x)}{F(x)}f(y)\,dy = f(x)H_r(x) + F(x)H_r'(x),$$

$$\frac{f(x)}{F(x)} = \frac{H_r'(x)}{[H_{r+1}(x) - H_r(x)]} = \frac{2\alpha\beta^2 x}{e^{(\beta x)^2} - 1},$$

which proves that

$$F(x;\alpha,\beta) = \left(1 - e^{-(\beta x)^2}\right)^{\alpha}, \qquad x > 0, \quad \alpha,\beta > 0.$$

□

4. NON-BAYESIAN ESTIMATION

In this section we discuss the process of obtaining the maximum likelihood estimators of the parameters α and β based on lower record values. Let $X_1, X_2, \ldots$ be a sequence of *iid* random variables with *cdf* $F(x)$ and *pdf* $f(x)$ on positive support. Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. The observation $X_j,\ j \geq 1$, is a lower record value of this sequence, if it is greater than all preceding observations, that is $Y_j < Y_{j-1}$ for $j > 1$.

Suppose we observe n lower record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ from a sequence of *iid* random variables following a $GR(\alpha, \beta)$ with *pdf* (1.1). Arnold et al. (1998) gives the likelihood function based on the random sample of size n, which is obtained from

$$L(\alpha, \beta \,|\, x) = f(X_{L(n)}; \alpha, \beta) \prod_{i=1}^{n-1} \frac{f(X_{L(i)}; \alpha, \beta)}{F(X_{L(i)}; \alpha, \beta)}. \tag{4.1}$$

By using (1.1) and (1.2), (4.1) can be rewritten as

$$L(\alpha, \beta \,|\, x) = 2^n \alpha^n \beta^{2n} \exp\left\{\alpha \log\left(1 - e^{-(\beta x_{L(n)})^2}\right)\right\} \prod_{i=1}^{n} \frac{x_{L(i)} e^{-(\beta x_{L(i)})^2}}{\left(1 - e^{-(\beta x_{L(i)})^2}\right)}. \tag{4.2}$$

The maximum likelihood estimates are the values of α and β that maximize this likelihood function.

The log likelihood function $l(\alpha, \beta \,|\, x) = \log L(\alpha, \beta \,|\, x)$, dropping terms that do not involve α and β, is

$$l(\alpha, \beta \,|\, x) = n(\log \alpha + 2 \log \beta) + \alpha \log\left(1 - e^{-(\beta x_{L(n)})^2}\right) - \beta^2 \sum_{i=1}^{n} x_{L(i)}^2 - \sum_{i=1}^{n} \log\left(1 - e^{-(\beta x_{L(i)})^2}\right). \tag{4.3}$$

We assume that the parameters α and β are unknown. To obtain the normal equations for the unknown parameters, we differentiate (4.3) partially with respect to α and β and equate to zero. The resulting equations are

$$0 = \frac{\partial l(\alpha, \beta \,|\, x)}{\partial \alpha} = \frac{n}{\alpha} + \log\left(1 - e^{-(\beta x_{L(n)})^2}\right), \tag{4.4}$$

and

$$0 = \frac{\partial l(\alpha, \beta \,|\, x)}{\partial \beta} = \frac{2n}{\beta} + \frac{2\alpha\beta x_{L(n)}^2 e^{-(\beta x_{L(n)})^2}}{\left(1 - e^{-(\beta x_{L(n)})^2}\right)} - 2\beta \sum_{i=1}^{n} x_{L(i)}^2 - \sum_{i=1}^{n} \frac{2\beta x_{L(i)}^2 e^{-(\beta x_{L(i)})^2}}{\left(1 - e^{-(\beta x_{L(i)})^2}\right)}. \tag{4.5}$$

The solutions of the above equations are the maximum likelihood estimators of the GR distribution parameters α and β, denoted $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$, respectively. As the equations expressed in (4.4) and (4.5) cannot be solved analytically, one must use a numerical procedure to solve them.

Since the $MLEs$ of the unknown parameters α and β cannot be derived in closed form, it is not easy to derive the exact distributions of the $MLEs$. Hence, we cannot obtain exact confidence intervals for the parameters. We must use the large sample approximation. It is known that the asymptotic distribution of the $MLEs$ is $[\sqrt{n}(\hat{\alpha}_{MLE} - \alpha), \sqrt{n}(\hat{\beta}_{MLE} - \beta)] \to N_2(0, I^{-1}(\alpha, \beta))$, we can refer Lawless (1982), where $I^{-1}(\alpha, \beta)$, the inverse of observed information matrix of the unknown parameters $\Theta = (\alpha, \beta)$, is

$$I^{-1}(\Theta) = \begin{pmatrix} -\frac{\partial^2 l(\alpha,\beta)}{\partial^2 \alpha} & -\frac{\partial^2 l(\alpha,\beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 l(\alpha,\beta)}{\partial \alpha \partial \beta} & -\frac{\partial^2 l(\alpha,\beta)}{\partial^2 \beta} \end{pmatrix}^{-1}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} = \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{Var}(\hat{\alpha}) \end{pmatrix}.$$

The derivatives in $I(\Theta)$ are given by

$$\frac{\partial^2 l(\alpha, \beta \,|\, x)}{\partial \alpha^2} = -\frac{n}{\alpha^2}, \tag{4.6}$$

$$\frac{\partial^2 l(\alpha,\beta\,|\,x)}{\partial\alpha\,\partial\beta} = \frac{2\beta x_{L(n)}^2 e^{-(\beta x_{L(n)})^2}}{\left(1-e^{-(\beta x_{L(n)})^2}\right)} = \frac{\partial^2 l(\alpha,\beta\,|\,x)}{\partial\beta\,\partial\alpha}, \tag{4.7}$$

$$\frac{\partial^2 l(\alpha,\beta\,|\,x)}{\partial\beta^2} = -\frac{2n}{\beta^2} - 2\sum_{i=1}^{n} x_{L(i)}^2 - \frac{4\alpha\beta^2\, x_{L(n)}^4\, e^{-2(\beta x_{L(n)})^2}}{\left(1-e^{-(\beta x_{L(n)})^2}\right)^2} + \frac{2\alpha\, x_{L(n)}^2\, e^{-(\beta x_{L(n)})^2}(1-2\beta^2 x_{L(n)}^2)}{\left(1-e^{-(\beta x_{L(n)})^2}\right)} + 4\beta^2\sum_{i=1}^{n}\frac{x_{L(i)}^4\, e^{-2(\beta x_{L(i)})^2}}{\left(1-e^{-(\beta x_{L(i)})^2}\right)^2} - \alpha\sum_{i=1}^{n}\frac{x_{L(i)}^2\, e^{-(\beta x_{L(i)})^2}(1-2\beta^2 x_{L(i)}^2)}{\left(1-e^{-(\beta x_{L(i)})^2}\right)^2}. \tag{4.8}$$

The above approach is used to derive approximate $100(1-\tau)\%$ confidence intervals of the parameters α and β of the forms

$$\hat{\alpha} \pm z_{\tau/2}\sqrt{\mathrm{Var}(\hat{\alpha})}$$

and

$$\hat{\beta} \pm z_{\tau/2}\sqrt{\mathrm{Var}(\hat{\beta})},$$

where $z_{\tau/2}$ is the upper $(\tau/2)$th percentile of the standard normal distribution.

5. BAYESIAN ESTIMATION

In this section we consider Bayesian inference of the unknown parameters of the GR distribution. It is assumed that α and β has the independent Gamma prior distributions with probability density functions

$$h(\alpha) \propto \alpha^{a-1}e^{-b\alpha}, \quad \alpha > 0 \tag{5.1}$$

and

$$h(\beta) \propto \beta^{c-1}e^{-d\beta}, \quad \beta > 0. \tag{5.2}$$

The hyper-parameters a, b, c, and d are known and nonnegative. If both the parameters α and β are unknown, joint conjugate priors do not exist. It is not unreasonable to assume independent Gamma priors on the shape and scale parameters for a two-parameter GR distribution, because Gamma distributions are very flexible, and the Jeffreys (non-informative) prior, introduced by Jeffreys (1946) is a special case of this. The joint prior distribution in this case is

$$h(\alpha,\beta) \propto \alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta}, \quad \alpha,\beta > 0. \tag{5.3}$$

Combining (5.3) with (4.2) and using the Bayes theorem, the joint posterior distribution is derived as

$$\pi(\alpha,\beta\,|\,x) = 2^n\alpha^{n+a-1}\beta^{2n+c-1}e^{-b\alpha-d\beta}\left(1-e^{-(\beta x_{L(n)})^2}\right)^{\alpha}\frac{1}{I_0}\prod_{i=1}^{n}\frac{x_{L(i)}e^{-(\beta x_{L(i)})^2}}{\left(1-e^{-(\beta x_{L(i)})^2}\right)}, \tag{5.4}$$

where

$$I_0 = \int_0^{\infty}\int_0^{\infty} 2^n\alpha^{n+a-1}\beta^{2n+c-1}e^{-b\alpha-d\beta}\left(1-e^{-(\beta x_{L(n)})^2}\right)^{\alpha}\prod_{i=1}^{n}\frac{x_{L(i)}e^{-(\beta x_{L(i)})^2}}{\left(1-e^{-(\beta x_{L(i)})^2}\right)}. \tag{5.5}$$

The marginal posterior distribution of a parameter is obtained by integrating the joint posterior distribution with respect to the other parameter. Hence, the marginal posterior probability density functions of α and β are given, respectively, by

$$\pi_1(\alpha\,|\,x) = \frac{2^n\alpha^{n+a-1}e^{-b\alpha}}{I_0}\int_0^{\infty}\beta^{2n+c-1}e^{-d\beta}\left(1-e^{-(\beta x_{L(n)})^2}\right)^{\alpha}\prod_{i=1}^{n}\frac{x_{L(i)}e^{-(\beta x_{L(i)})^2}}{\left(1-e^{-(\beta x_{L(i)})^2}\right)}\,d\beta \tag{5.6}$$

and

$$\pi_2(\beta \,|\, x) = \frac{2^n \beta^{2n+c-1} e^{-d\beta}}{I_0} \prod_{i=1}^{n} \frac{x_{L(i)} e^{-(\beta x_{L(i)})^2}}{\left(1 - e^{-(\beta x_{L(i)})^2}\right)} \int_0^\infty \alpha^{n+a-1} e^{-b\alpha} \left(1 - e^{-(\beta x_{L(n)})^2}\right)^\alpha d\alpha. \tag{5.7}$$

Next, we must consider the question of what loss function will be used to derive the estimators from the marginal posterior distributions.

5.1. Bayes Estimators under the General Entropy Loss Function

Calabria and Pulcini (1996) have derived the point estimation under asymmetric loss function from left-truncated exponential samples. According to this theory, the Bayes estimators for the parameters α and β for the probability density function (1.1) under the general entropy loss function may be defined as

$$\hat{\alpha}_{BGE} = [E(\alpha)^{-q}]^{-1/q} \tag{5.8}$$

and

$$\hat{\beta}_{BGE} = [E(\beta)^{-q}]^{-1/q} \tag{5.9}$$

respectively, provided that $E(\alpha)^{-q}$ and $E(\beta)^{-q}$ exist and are finite. These estimators can be expressed as

$$\hat{\alpha}_{BGE} = \left[\frac{I_\alpha}{I_0}\right]^{-1/q} \qquad \text{and} \qquad \hat{\beta}_{BGE} = \left[\frac{I_\beta}{I_0}\right]^{-1/q},$$

where

$$I_\alpha = \int_0^\infty \int_0^\infty 2^n \alpha^{n+a-p-1} \beta^{2n+c-1} e^{-d\beta} e^{-\alpha[b - \log(1 - e^{-(\beta x_{L(i)})^2})]} \prod_{i=1}^{n} \frac{x_{L(i)} e^{-(\beta x_{L(i)})^2}}{\left(1 - e^{-(\beta x_{L(i)})^2}\right)} \, d\alpha \, d\beta$$

and

$$I_\beta = \int_0^\infty \int_0^\infty 2^n \alpha^{n+a-1} \beta^{2n+c-p-1} e^{-d\beta} e^{-\alpha[b - \log(1 - e^{-(\beta x_{L(i)})^2})]} \prod_{i=1}^{n} \frac{x_{L(i)} e^{-(\beta x_{L(i)})^2}}{\left(1 - e^{-(\beta x_{L(i)})^2}\right)} \, d\alpha \, d\beta.$$

All the double integrals above have no closed form. Therefore, we will implement the Metropolis–Hastings (M-H) algorithm to compute the estimators. The M-H algorithm is a powerful Markov Chain Monte Carlo algorithm. The M-H algorithm was introduced by Metropolis et al. (1953). For a discussion of the algorithm, the reader is referred to any Bayesian statistics textbook. In this paper, we consider three special cases of the general entropy loss function, corresponding to $q = -1$, $q = 1$ and $q = -2$. It should be mentioned that for $q = -1$ the general entropy loss function simplifies to the squared-error loss function. The weighted squared-error loss function results from $q = 1$. For $q = -2$, the general entropy loss function is referred to as the precautionary loss function which is an asymmetric loss function.

5.2. Two-Sided Bayes Probability Intervals

The Bayesian method to interval estimation is much more direct than the frequentist method based on confidence intervals. Once the marginal posterior distribution of α has been obtained, a symmetric $100(1-\tau)\%$ two-sided Bayes probability interval estimate of α, denoted by $[\alpha_L, \alpha_U]$, can be obtained by solving the two equations [see Martz and Waller (1982), pp. 208–209]

$$\int_0^{\alpha_L} \pi_1(w \,|\, x)\, dw = \frac{\tau}{2} \tag{5.10}$$

and

$$\int_{\alpha_U}^{\infty} \pi_1(w \,|\, x)\, dw = \frac{\tau}{2} \tag{5.11}$$

for the limits α_L and α_u. Similarly, a symmetric $100(1-\tau)\%$ two-sided Bayes probability interval estimate of β, denoted by $[\beta_L, \beta_U]$, can be obtained by solving

$$\int_0^{\beta_L} \pi_2(w \mid x)\, dw = \frac{\tau}{2} \tag{5.12}$$

and

$$\int_{\beta_U}^{\infty} \pi_2(w \mid x)\, dw = \frac{\tau}{2} \tag{5.13}$$

for the limits β_L and β_U. Again, these equations cannot be solved in a closed form.

6. EMPIRICAL BAYES ESTIMATION

In the preceding sections, we assumed that the hyper-parameters a, b, c, and d are known. Empirical Bayes estimation addresses the question of estimating the hyper-parameters from existing data. When the current sample is observed, assume that p past samples $X_{jL(1)}, X_{jL(2)}, \ldots, X_{jL(n)}$, for $j = 1, 2, \ldots, p$, are available. Each sample is assumed to be a sample of size n from a GR distribution. The likelihood function for each sample j is given by

$$L(\alpha, \beta \mid x) = 2^n \alpha^n \beta^{2n} \exp\left\{\alpha \log(1 - e^{-(\beta x_{jL(n)})^2})\right\} \prod_{i=1}^{n} \frac{x_{jL(i)} e^{-(\beta x_{jL(i)})^2}}{\left(1 - e^{-(\beta x_{jL(i)})^2}\right)}. \tag{6.1}$$

For each sample j, let $\hat{\alpha}_j$ and $\hat{\beta}_j$ be the maximum likelihood estimates for α and β, respectively, based on sample j, which are obtained from (6.1). We then calculate the mean and variance of the maximum likelihood estimators for each of the j samples, equate these to the mean and variance of the Gamma prior distribution, and solve for the hyper-parameters. We can find $\hat{a}$ and $\hat{b}$, estimators for a and b, by solving

$$\frac{1}{p} \sum_{i=1}^{p} \hat{\alpha}_j = \frac{b}{a}$$

and

$$\frac{1}{(p-1)} \sum_{i=1}^{p} \left(\hat{\alpha}_j - \frac{1}{p} \sum_{i=1}^{p} \hat{\alpha}_j\right)^2 = \frac{b}{a^2}.$$

We can find $\hat{c}$ and $\hat{d}$, estimators for c and d, by solving

$$\frac{1}{p} \sum_{i=1}^{p} \hat{\beta}_j = \frac{d}{c}$$

and

$$\frac{1}{(p-1)} \sum_{i=1}^{p} \left(\hat{\beta}_j - \frac{1}{p} \sum_{i=1}^{p} \hat{\beta}_j\right)^2 = \frac{d}{c^2}.$$

Solving the above equations yields the estimators for the hyper-parameters

$$\hat{a} = \frac{\left(\frac{1}{p} \sum_{i=1}^{p} \hat{\alpha}_j\right)}{\left(\frac{1}{p-1} \sum_{i=1}^{p} \left(\hat{\alpha}_j - \frac{1}{p} \sum_{i=1}^{p} \hat{\alpha}_j\right)^2\right)}$$

and

$$\hat{b} = \frac{\left(\frac{1}{p} \sum_{i=1}^{p} \hat{\alpha}_j\right)^2}{\left(\frac{1}{p-1} \sum_{i=1}^{p} \left(\hat{\alpha}_j - \frac{1}{p} \sum_{i=1}^{p} \hat{\alpha}_j\right)^2\right)}$$

for the prior distribution for α. Similarly, estimators for the hyper-parameters for the prior distribution for β can be found as

$$\hat{c} = \frac{\left(\frac{1}{p}\sum_{i=1}^{p}\hat{\beta}_j\right)}{\left(\frac{1}{p-1}\sum_{i=1}^{p}\left(\hat{\beta}_j - \frac{1}{p}\sum_{i=1}^{p}\hat{\beta}_j\right)^2\right)}$$

and

$$\hat{d} = \frac{\left(\frac{1}{p}\sum_{i=1}^{p}\hat{\beta}_j\right)^2}{\left(\frac{1}{p-1}\sum_{i=1}^{p}\left(\hat{\beta}_j - \frac{1}{p}\sum_{i=1}^{p}\hat{\beta}_j\right)^2\right)}.$$

The empirical Bayes estimators of α and β are found by substituting $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ into (7.6) and (7.7) and proceeding as before.

7. PREDICTION OF FUTURE RECORD VALUES

Next, we consider the problem of predicting future record values given a sample of observed record values.

7.1. Non-Bayesian Prediction

Suppose that we observe the first n lower record values from a population with *pdf* $f(x;\Theta)$. Our aim is to predict $z = X_{L(m)}$, $m > n$, having observed records $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$. The joint predictive likelihood function of $z = X_{L(m)}$, and the possibly vector-valued parameter Θ can be written, see Basak and Balakrishnan (2003), as

$$L(z,\Theta,X) = \prod_{i=1}^{n} h(X_{L(i)};\Theta)\frac{[H(z,\Theta) - H(X_{L(n)};\Theta)]^{m-n-1}}{\Gamma(m-n)} f(z;\Theta),$$

where

$$H(z,\Theta) = -\log[F(z,\Theta)]$$

and

$$h(X_{L(i)};\Theta) = \frac{f(X_{L(i)};\alpha,\beta)}{S(X_{L(i)};\alpha,\beta)} = \frac{2\alpha\beta^2\, z_{L(i)}e^{-(\beta z_{L(i)})^2}\left(1-e^{-(\beta z_{L(i)})^2}\right)^{\alpha-1}}{1-\left(1-e^{-(\beta z_{L(i)})^2}\right)^{\alpha}}.$$

The predictive likelihood function for the GR distribution is

$$L(y;\alpha,\beta) = 2^{n+1}\alpha^m\beta^{2(n+1)}\prod_{i=1}^{n}\frac{y_{L(i)}e^{-(\beta y_{L(i)})^2}\left(1-e^{-(\beta y_{L(i)})^2}\right)^{\alpha-1}}{1-\left(1-e^{-(\beta y_{L(i)})^2}\right)^{\alpha}}$$
$$\times\frac{\left[\log\left(1-e^{-(\beta y_{L(n)})^2}\right) - \log\left(1-e^{-(\beta y)^2}\right)\right]^{m-n-1}}{\Gamma(m-n)} y\left(1-e^{-(\beta y)^2}\right)^{\alpha-1}e^{-(\beta y)^2}.$$

The log predictive likelihood is given by

$$\begin{aligned}\log L = {} & (n+1)\log 2 + m\log\alpha + 2(n+1)\log\beta \\ & + \sum_{i=1}^{n}\log\ y_{L(i)} + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-(\beta y_{L(i)})^2}\right) \\ & + \sum_{i=1}^{n}\log\left(1-e^{-(\beta y_{L(i)})^2}\right) - \sum_{i=1}^{n}\log\left[1-\left(1-e^{-(\beta y_{L(i)})^2}\right)^{\alpha}\right] \\ & + (m-n-1)\log\left[\log\left(1-e^{-(\beta y_{L(n)})^2}\right) - \log\left(1-e^{-(\beta y)^2}\right)\right] \\ & - \log\Gamma(m-n-1) + \log y - (\beta y)^2 + (\alpha-1)\log\left(1-e^{-(\beta y)^2}\right).\end{aligned} \tag{7.1}$$

To obtain the normal equations for the unknown parameters, we differentiate (7.1) partially with respect to α and β and equate to zero. The resulting equations are

$$0 = \frac{\partial \log L}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{n} \log\left(1 - e^{-(\beta y_{L(i)})^2}\right) + \sum_{i=1}^{n} \frac{\left(1 - e^{-(\beta y_{L(i)})^2}\right)^{\alpha} \log\left(1 - e^{-(\beta y_{L(i)})^2}\right)}{1 - \left(1 - e^{-(\beta y_{L(i)})^2}\right)^{\alpha}},$$

$$0 = \frac{\partial \log L}{\partial \beta} = \frac{2(n+1)}{\beta} - (\alpha - 1)\sum_{i=1}^{n} \frac{2\beta y_{L(i)}^2 \, e^{-(\beta y_{L(i)})^2}}{\left(1 - e^{-(\beta y_{L(i)})^2}\right)} - \sum_{i=1}^{n} \frac{2\beta y_{L(i)}^2 \, e^{-(\beta y_{L(i)})^2}}{\left(1 - e^{-(\beta y_{L(i)})^2}\right)}$$

$$+ \sum_{i=1}^{n} \frac{2\alpha\beta y_{L(i)}^2 \, e^{-(\beta y_{L(i)})^2} \left(1 - e^{-(\beta y_{L(i)})^2}\right)^{\alpha-1}}{1 - \left(1 - e^{-(\beta y_{L(i)})^2}\right)^{\alpha}} - \frac{(\alpha-1) 2\beta y^2 e^{-(\beta y)^2}}{\left(1 - e^{-(\beta y_{L(i)})^2}\right)} - 2\beta y^2$$

$$+ (m - n - 1) \frac{\left(\frac{2\beta y_{L(n)}^2 e^{-(\beta y_{L(n)})^2}}{1 - e^{-(\beta y_{L(n)})^2}}\right) - \left(\frac{2\beta y^2 e^{-(\beta y)^2}}{1 - e^{-(\beta y)^2}}\right)}{\log\left(1 - e^{-(\beta y_{L(n)})^2}\right) - \log\left(1 - e^{-(\beta y)^2}\right)}$$

and

$$0 = \frac{\partial \log L}{\partial y} = -(m - n - 1) \frac{\left(\frac{2\beta^2 y e^{-(\beta y)^2}}{1 - e^{-(\beta y)^2}}\right)}{\log\left(1 - e^{-(\beta y_{L(n)})^2}\right) - \log\left(1 - e^{-(\beta y)^2}\right)}$$

$$+ \frac{1}{y} + (\alpha - 1) \frac{2\beta^2 y e^{-(\beta y)^2}}{\left(1 - e^{-(\beta y)^2}\right)} - 2\beta^2 y.$$

The above equations cannot be solved in a closed form. Thus, we must use a numerical procedure to find the maximum likelihood predictor.

7.2. Conditional Median Prediction

We now consider the conditional median prediction of future record values. Suppose that we have n lower records $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ from a $GR(\alpha, \beta)$ and we are interested in predicting $z = X_{L(m)}$, the mth lower record, for some $m > n$. It is well known that the distribution of $z = X_{L(m)}$ depends only on the current lower record, $y = X_{L(n)}$. The conditional median predictor is found as the median of the conditional distribution of z given y.

First, we consider the case where α and β are unknown. The conditional distribution of z given y is found in (7.4). To find the median, let the cumulative distribution function of z given y be

$$F(z \,|\, y) = \int_0^z f^*(\tau \,|\, y)\, d\tau.$$

Let $F^{-1}(u)$ be the inverse distribution function. Equating $F^{-1}(u) = 1/2$ and solving for u yields $\hat{z}_{CMP}$, the conditional median predictor of z when α and β are unknown. The solution does not exist in a closed form. In the case where α and β are known, the conditional median predictor can be found as the median of (7.3). Let the cumulative distribution function of z given y be

$$F(z \,|\, y; \alpha, \beta) = \int_0^z f^*(\tau \,|\, y; \alpha\beta)\, d\tau.$$

Again, let $F^{-1}(u)$ be the inverse distribution function. Equating $F^{-1}(u) = 1/2$ and solving for u yields $\hat{z}_{CMP}$, the conditional median predictor of z when α and β are known. Again, the solution does not exist in a closed form.

7.3. Bayes Prediction

In this section, we consider the prediction of future records based on a Bayesian approach. Prediction of future records has been studied by a number of statisticians, among them Ahmadi and Doostparast (2006), Ahsanullah (1998), Arnold et al (1992), Berred (1998) and Dunsmore (1983). Suppose that we have n lower records $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ from a GR distribution. Based on such a record sample, we are interested in obtaining a Bayesian prediction interval for the future lower record $X_{L(m)}$, for some $m > n$, with a certain confidence. The conditional *pdf* of $z = X_{L(m)}$ for a given $y = X_{L(n)}$ is given, Ahsanullah (1998), by

$$f(z\,|\,y;\alpha,\beta) = \frac{[\log S(y;\alpha,\beta) - \log S(z;\alpha,\beta)]^{m-n-1}}{\Gamma(m-n)} \frac{f(z;\alpha,\beta)}{S(y;\alpha,\beta)}, \quad z > y. \tag{7.2}$$

For the GR distribution, with *pdf* given in (1.1), the function $f(z\,|\,y;\alpha,\beta)$ can be shown to be

$$f(z\,|\,y;\alpha,\beta) = \frac{\alpha^{m-n}\beta e^{-(\beta z)^2}(1-e^{-(\beta z)^2})^{\alpha-1}}{(1-e^{-(\beta y)^2})^{\alpha}} \left[\log\left(\frac{1-e^{-(\beta y)^2}}{1-e^{-(\beta z)^2}}\right)\right]^{m-n-1}. \tag{7.3}$$

As we know, future record values satisfy the Markovian (memoryless) property, the future lower record $z = X_{L(m)}$ given the set of the first n lower records $X = \{X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}\}$ depends only on the current lower record $y = X_{L(n)}$. Therefore, the conditional distribution of z given x is the same as the conditional distribution of z given y. The predictive *pdf* of z given x is

$$f^*(z\,|\,x) = \int_0^\infty \int_0^\infty f(z\,|\,y;\alpha,\beta)\pi(\alpha,\beta\,|\,x)\,d\alpha\,d\beta, \tag{7.4}$$

where $f(z\,|\,y;\alpha,\beta)$ and $\pi(\alpha,\beta\,|\,x)$ are given respectively by (7.3) and (5.4).

The predictive limits of the $100(1-\tau)\%$ two-sided interval of the future lower record z can be obtained by solving

$$\int_y^{z_L} f^*(z\,|\,x)\,dz = \frac{\tau}{2} \tag{7.5}$$

and

$$\int_{z_U}^\infty f^*(z\,|\,x)\,dz = \frac{\tau}{2} \tag{7.6}$$

with respect to the lower and upper limits z_L and z_U.

The Bayesian prediction bounds for $Y = X_{U(m)}$ are obtained by evaluating $P(Y \geq \eta\,|\,x)$, for some given positive value of η. From (7.4), we have

$$P(Y \geq \eta\,|\,x) = \int_\eta^\infty f^*(z\,|\,x)\,dz.$$

The $100(1-\tau)\%$ predictive interval for $Y = X_{U(m)}$ is obtained by evaluating both the lower $L(x)$ and the upper $U(x)$ limits which satisfy $P(Y \geq L(x)\,|\,x) = 1 - \tau/2$ and $P(Y \geq U(x)\,|\,x) = \tau/2$.

8. SIMULATION STUDY

In this section, we examine and compare the performance of maximum likelihood and Bayes estimators for the two parameters GR distribution based on record values by conducting various simulations for different sizes $n = 5, 10, 20$. We simulate 1000 samples for the true parameters values $\alpha = 1.2$ and $\beta = 2.2$. First we compute the maximum likelihood estimators using the methods described in Section 4.1. We report the average bias and mean squared error (MSE) over 1000 replications. The Bayes estimators cannot be found in a closed form. Therefore, we use the Metropolis–Hastings algorithm to compute Bayes estimates. We use informative priors for both α and β. The chosen hyper-parameters are $a = c = 4$ and $b = d = 1$. Bayes estimators are computed using the general entropy loss function with $q = -2, -1, 1$. This allows us to consider the Bayes estimators under both symmetric

Table 4. Average bias and MSE of estimators for α and β

n	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{WSEL}$	$\hat{\alpha}_{PL}$	$\hat{\beta}_{MLE}$	$\hat{\beta}_{SEL}$	$\hat{\beta}_{WSEL}$	$\hat{\beta}_{PL}$
5	0.8320	0.0334	0.0185	0.0443	0.4283	0.1972	0.3312	0.1467
	(2.902)	(0.0019)	(0.0010)	(0.028)	(1.7203)	(0.8104)	(0.6914)	(0.8101)
10	0.4914	0.0196	0.1508	0.0209	0.4094	0.1614	0.2667	0.1182
	(0.4501)	(0.0012)	(0.0071)	(0.0032)	(1.6304)	(0.0413)	(0.0831)	(0.0401)
20	0.2617	0.0093	0.0790	0.0107	0.2189	0.1473	0.3162	0.1206
	(0.2205)	(0.0001)	(0.0003)	0.0052	(1.1012)	(0.0394)	(0.0053)	(0.0153)

Table 5. Average bias and MSE of empirical Bayes estimators for α and β

n	$\hat{\alpha}_{SEL}$	$\hat{\alpha}_{WSEL}$	$\hat{\alpha}_{PL}$	$\hat{\beta}_{SEL}$	$\hat{\beta}_{WSEL}$	$\hat{\beta}_{PL}$
5	0.6165	0.4917	0.7130	1.3514	0.2243	2.1621
	(0.4689)	(0.2042)	(0.5961)	(2.9017)	(0.2602)	(7.2031)
10	0.4715	0.3721	0.5104	0.4620	0.1834	0.8257
	(0.2043)	(0.1644)	(0.2304)	(1.3296)	(0.0010)	(2.4604)
20	0.2107	0.1682	0.2506	0.1942	0.1145	0.3478
	(0.0140)	(0.01023)	(0.0231)	(0.0117)	(0.0031)	(3.2613)

and asymmetric loss functions. The proposal distribution used for the M-H algorithm is a chi-square distribution. We generate 6000 samples after 6000 burn-in samples.

Table 4 contains the bias and MSE for the maximum likelihood estimators and the Bayes estimators under three different loss functions for α and β. Average bias is listed first and the corresponding MSE is listed second in parentheses. Table 5 includes the average bias and MSE of the empirical Bayes estimators for α and β.

It is observed in Tables 4 and 5 that for each method the $MLEs$ decrease as the sample size increases. It indicates that both the methods deduce asymptotically unbiased and consistent estimators of the parameters. Table 4 shows that the performance of the Bayes estimates is better than that of the MLEs in terms of the bias and MSE. The Bayes estimates under precautionary loss function (which is an asymmetric loss function) exhibit better performance than when under symmetric loss functions.

9. CONCLUSION

The GR distribution provided excellent model for life time data for a variety of situations. Thus, it is important for the analyst to have reliable statistical methods to use for this distribution. We have provided in this study new explicit expressions for single and product moments of lower record values from the GR distribution. Further, a characterizing result of this distribution on using the conditional expectation of lower record values is discussed. We have provided both frequentist and Bayesian methods of estimating the parameters based on samples of upper record values and methods of predicting future record values. We have examined and compared the different methods, including Bayesian methods under different loss functions.

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