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Adaptive Minimax Test of Independence

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Abstract—The paper is concerned with the adaptive minimax problem of testing the independence of the components of a d-dimensional random vector. The functions under alternatives consist of smooth densities supported on $[0, 1]^d$ and separated away from the product of their marginals in L_2-norm. We are interested in finding the adaptive minimax rate of testing and a test that attains this rate. We focus mainly on the tests for which the error of the first kind α_n can decrease to zero as the number of observations increases. We show also how this property of the test affects its error of the second kind.

Keywords: independence hypothesis testing, minimax hypothesis testing, asymptotics of errors probabilities, density estimation.

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1. INTRODUCTION

Consider the statistical experiment generated by the observation $X^n = (X_1, \ldots, X_n)$, where $X_i = \big(X_i^{(1)}, \ldots, X_i^{(d)}\big)$, $i = 1, \ldots, n$, are independent identically distributed ($i.i.d.$) d-dimensional random vectors with density function f. Let Φ be the set of all probability densities on $\mathbb{R}^d$. Here and later we denote by f_j, $j = 1, \ldots, d$, the marginal densities of f. To the independence hypothesis there corresponds the set $\Phi_0 \subset \Phi$ of the densities of the form $f(x_1, \ldots, x_d) = f_1(x_1) \ldots f_d(x_d)$.

Given the observation X^n, we consider the nonparametric minimax problem of testing the null hypothesis

$$H_0 \colon f \in \Phi_0$$

against the alternative set

$$H_n \colon f \in \Phi_n(\psi_n) = \big\{f \in \Phi \colon \delta(f, \Phi_0) \geq \psi_n\big\},$$

where δ is a distance measure and ψ_n is a positive sequence tending to zero as $n \to \infty$. In other words, the set of alternatives $\Phi_n(\psi_n)$ is a subset of Φ of densities separated from the set Φ_0 by the distance at least ψ_n. Moreover, we will assume some smoothness properties of the density f. This problem named independence hypothesis testing is one of the classical problems in mathematical statistics. There have been papers, initiated by Ingster [8, 9] concerning independence hypothesis testing via an asymptotic minimax approach: Ermakov [3], Yodé [13]. Then, the goal of these papers is to determine the minimal (optimal) distance between the null hypothesis and the set of alternatives for which testing with prescribed error probabilities is still possible. However, the test procedures depend heavily on the smoothness assumption, which is typically unknown as well as the density function f: this seems unnatural and unattractive from a practical point of view. It is our first motivation to extend the non-adaptive case of our paper [13] to the adaptive case, i.e., the case, where the smoothness parameter is also supposed unknown.

As in Yodé [13], in this paper we deal with the problem of independence testing when the error of the first kind is bounded by a positive sequence α_n, which can decrease to zero as the number of

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observations increases, contrary to the usual context, where this bound is an absolute fixed positive constant $\alpha \in (0,1)$. We show also how this property affects the error of the second kind. Such kind of testing problems appear in the concept of random normalizing factor initiated by Lepski [11] and extended by Hoffmann and Lepski [10], Yodé [14], and Chiabrando [2].

We need to make the following assumptions.

Assumption 1. *Assume that density f belongs to the functions class $\Sigma(\beta, L, Q)$, $\beta > \frac{d}{4}$, $L > 0$, $Q > 0$ defined as follows:*

$$\Sigma(\beta, L, Q) \overset{\triangle}{=} \Sigma(\kappa) = \{f \in \Lambda(\beta, L)\colon \|f\|_\infty \le Q\},$$

where $\|\cdot\|_\infty$ is the supremum norm and $\Lambda(\beta, L)$ is the isotropic Hölder class, i.e., the class of all functions $f\colon \mathbb{R}^d \to \mathbb{R}$ having on $\mathbb{R}^d$ all partial derivatives of order $m = \lfloor \beta \rfloor$ and such that

$$\left| f(x) - \sum_{0 \le i_1 + \dots + i_d \le m} \frac{\partial^{i_1 + \dots + i_d} f(y)}{\partial y_1^{i_1} \dots \partial y_d^{i_d}} \prod_{j=1}^{d} \frac{(x_j - y_j)^{i_j}}{i_j!} \right| \le L \|x - y\|^\beta, \tag{1}$$

for any $x, y \in \mathbb{R}^d$, where x_j and y_j are the jth components of x and y and $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^d$. Moreover, we assume that f is compactly supported in $[0,1]^d$. The parameter $\kappa = (\beta, L, Q)$ is called the nuisance parameter.

To define the test statistic, we use a classical Parzen–Rosenblatt estimator based on the observation X^n and defined by

$$\hat{f}_n(x) = \frac{1}{n h_n^d} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right), \qquad x \in [0,1]^d, \tag{2}$$

where h_n is a positive sequence such that $h_n \to 0$, $nh_n^d \to \infty$, when $n \to \infty$ and $K\colon \mathbb{R}^d \to \mathbb{R}$ is a kernel function, i.e., a function such that $\int_{\mathbb{R}^d} K(x)\,dx = 1$. Under the hypothesis H_0, each univariate density f_k, $k = 1, \dots, d$, can be estimated separately using only the corresponding observations $\left(X_1^{(k)}, \dots, X_n^{(k)}\right)$. Let

$$\bar{f}_{kn}(x_k) = \frac{1}{n b_n} \sum_{k=1}^{n} K_0\left(\frac{x_k - X_i^{(k)}}{b_n}\right), \qquad x_k \in [0,1], \tag{3}$$

be the Parzen–Rosenblatt estimator attaining the univariate minimax rate of convergence $n^{-\frac{\beta}{2\beta+1}}$. The positive sequence $b_n \to 0$ is such that $nb_n \to +\infty$, $n \to +\infty$, and K_0 is a univariate kernel function. Thus, $\bar{f}_{kn}$ are *i.i.d* random variables, and therefore the estimator

$$\bar{f}_n^{(0)}(x) = \prod_{k=1}^{d} \bar{f}_{kn}(x_k), \qquad x = (x_1, \dots, x_d) \in [0,1]^d, \tag{4}$$

attains the univariate minimax rate of convergence $n^{-\frac{\beta}{2\beta+1}}$ (see Lepski [11]).

Assumption 2. *K and K_0 are Lipschitz-functions with compact support on $\mathbb{R}^d$ and $\mathbb{R}$ and Lipschitz-constants Q_1 and Q_2, respectively.*

Assumption 3. *For any $f \in \Sigma(\kappa)$ and univariate density g, we have*

$$\sup_x \left| \frac{1}{h} \int_{\mathbb{R}^d} K\left(\frac{x - t}{h}\right) f(t)\,dt - f(x) \right| \le L_0 h^\beta,$$

$$\sup_x \left| \frac{1}{b} \int_{\mathbb{R}} K_0\left(\frac{x - t}{b}\right) g(t) dt - g(x) \right| \le L_0 b^\beta.$$

where $L_0 > 0$ and h and b are positive sequences.

Remark 1. For example, if

$$\int_{\mathbb{R}^d} u_1^{i_1} \dots u_d^{i_d} K(u)\,du = 0 \quad \text{for any} \quad 1 \le i_1 + \dots + i_d \le m \quad \text{and} \quad \int_{\mathbb{R}^d} \|u\|^\beta |K(u)|\,du < +\infty,$$

we have

$$\left|\frac{1}{h}\int_{\mathbb{R}^d} K\Big(\frac{x-t}{h}\Big) f(t)\,dt - f(x)\right| = \left|\int_{\mathbb{R}^d} K(u)\big(f(x-hu) - f(x)\big)\,du\right|$$

$$\le \int_{\mathbb{R}^d} |K(u)| \left| f(x-hu) - \sum_{0 \le i_1+\dots+i_d \le m} \frac{\partial^{i_1+\dots+i_d} f(x)}{\partial x_1^{i_1} \dots \partial x_d^{i_d}} \prod_{j=1}^d \frac{(-hu_j)^{i_j}}{i_j!}\right| du$$

$$+ \left| \sum_{1 \le i_1+\dots+i_d \le m} \frac{\partial^{i_1+\dots+i_d} f(x)}{\partial x_1^{i_1} \dots \partial x_d^{i_d}} \frac{(-h)^{i_1+\dots+i_d}}{i_1! \dots i_d!} \int_{\mathbb{R}^d} u_1^{i_1} \dots u_d^{i_d} K(u)\,du \right| \le L_0 h^\beta$$

with

$$L_0 = L \int_{\mathbb{R}^d} \|u\|^\beta |K(u)|\,du.$$

For $m \ge 2$, K necessarily takes negative values. Despite their negative values, kernels of higher order are viable (see Gajek [5]).

2. MINIMAX AND ADAPTIVE MINIMAX FRAMEWORK

2.1. Minimax Testing Approach

Given the observation X^n, if we suppose κ known, one could consider the following hypotheses testing problem

$$H_{0,\kappa}\colon\ f \in \Sigma_0(\kappa) \stackrel{\triangle}{=} \Sigma(\kappa) \cap \Phi_0$$

against the alternative set

$$H_{n,\kappa}\colon\ f \in \Phi_n(C\varphi_\kappa(n)) = \big\{f \in \Sigma(\kappa)\colon \|f - f_0\|_2 \ge C\varphi_\kappa(n)\big\},$$

where $f_0(x_1, \dots, x_d) = f_1(x_1) \dots f_d(x_d)$ is the product of marginals of f, $\varphi_\kappa(n) \to 0$ when $n \to +\infty$, $C > 0$ and $\|\cdot\|_2$ denotes the usual L_2-norm on $[0,1]^d$.

A test $\Delta_n = \Delta_n(X^n)$ is a measurable function depending on observation X^n with values in the two-point set $\{0,1\}$. The value $\Delta_n = 0$ means that H_0 is accepted and $\Delta_n = 1$ means that H_0 is rejected. Introduce the first-kind error

$$\alpha_n(\Delta_n) \stackrel{\triangle}{=} \sup_{f \in \Sigma_0(\kappa)} \mathbb{P}_f^n\{\Delta_n = 1\},$$

and the second-kind error

$$\gamma\big(\Delta_n, C\varphi_\kappa(n)\big) \stackrel{\triangle}{=} \sup_{f \in \Phi_n(C\varphi_\kappa(n))} \mathbb{P}_f^n\{\Delta_n = 0\},$$

where $\mathbb{P}_f^n$ is the probability measure of the sample X^n, $\mathbb{P}_f$ being the probability measure on $(\mathbb{R}^d, \mathcal{B}_d)$ with density f. The properties of the test are characterized by both types of errors:

Definition 1. Let $\alpha_n \in (0,1)$. We call Δ_n an asymptotically α_n-level test if

$$\limsup_{n \to +\infty} \alpha_n^{-1} \sup_{f \in \Sigma_0(\kappa)} \mathbb{P}_f^n\{\Delta_n = 1\} \le 1.$$

Let $\mathcal{T}(\alpha_n)$ be the set of asymptotically α_n-level tests.

Definition 2. Let α_n and γ_n be two positive sequences in $(0,1)$. The positive sequence $\varphi_\kappa(n)$ is called minimax rate of testing if

— there exists $C_* > 0$ such that for any $C < C_*$, we have

$$\liminf_{n\to+\infty} \gamma_n^{-1} \inf_{\Delta_n \in \mathcal{T}(\alpha_n)} \gamma(\Delta_n, C\varphi_\kappa(n)) \geq 1; \quad (5)$$

— there exists $C^* > 0$ and $\Delta_n^* \in \mathcal{T}(\alpha_n)$ such that for any $C > C^*$

$$\limsup_{n\to+\infty} \gamma_n^{-1} \gamma(\Delta_n^*, C\varphi_\kappa(n)) \leq 1; \quad (6)$$

— $\Delta_{n,\kappa}^*$ is called asymptotically optimal test.

Definition 2 implies that $\varphi_\kappa(n)$ is a critical rate for testing. If the alternative is too close to the null hypothesis set of functions then (5) ensures that no asymptotically α_n-level test procedure can asymptotically achieve a second kind error lower than γ_n. Though, (6) states that it is possible to construct a test that detects H_0 against a local alternative separated away from the null hypothesis by a distance asymptotically equal to $\varphi_\kappa(n)$.

2.2. Adaptive Minimax Approach

The test constructed above is based on the prior knowledge of the nuisance parameter κ.

Assumption 4. *Thereafter, we assume that $f \in \Sigma(\kappa)$, where κ is unknown and*

$$\kappa = (\beta, L, Q) \in \Psi = [\beta_*, \beta^*] \times (0, L^*] \times (0, Q^*],$$

where $0 < \frac{d}{4} < \beta_* < \beta^*$, $L^* > 0$ and $Q^* > 0$.

For each $\kappa \in \Psi$, the optimal rate of testing is $\varphi_\kappa(n)$ and the asymptotically optimal test is $\Delta_{n,\kappa}$. Now, for the problem of adaptive testing, we expect to construct a universal test function Δ_n (free of κ) that achieves the optimal rate of testing for all $\kappa \in \Psi$. The existence of such test may depend on the model. In case no test can fulfill this condition, we may introduce the following rule to compare the testing rate families (Spokoiny [12]). First, we need to introduce

$$\mathcal{I}(\Psi, \alpha_n) = \Big\{\Delta_n \colon \limsup_{n\to\infty} \alpha_n^{-1} \sup_{\kappa\in\Psi} \sup_{f\in\Sigma_0(\kappa)} P_f^n\{\Delta_n = 1\} \leq 1\Big\}.$$

This set is to be understood as the family of tests for which the maximal first kind error over Ψ is controlled by a prescribed scale α_n.

Definition 3. Let positive sequences α_n, $\gamma_n \in (0,1)$. The factor t_n is said to be adaptive (α_n, γ_n)-optimal w.r.t. the family $\big\{\Sigma_0(\kappa), \Sigma(\kappa), \kappa \in \Psi\big\}$ if

(1) there exists $C_* = C_*(\Psi) > 0$ such that for all $C < C_*$,

$$\liminf_{n\to+\infty} \gamma_n^{-1} \inf_{\Delta_n\in\mathcal{I}(\Psi,\alpha_n)} \sup_{\kappa\in\Psi} \sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))} \mathbb{P}_f^n\{\Delta_n = 0\} \geq 1;$$

(2) there exists a constant $C^* = C^*(\Psi) > 0$ and a test $\Delta_{n,\Psi}^* \in \mathcal{I}(\Psi, \alpha_n)$ such that for any $C > C^*$

$$\limsup_{n\to+\infty} \gamma_n^{-1} \sup_{\kappa\in\Psi} \sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))} \mathbb{P}_f^n\{\Delta_{n,\Psi}^* = 0\} \leq 1.$$

In this case, $\{\varphi_\kappa(t_n^{-1}n)\}_{\kappa\in\Psi}$ is called the adaptive minimax rate of testing and $\Delta_{n,\Psi}^* \in \mathcal{I}(\Psi, \alpha_n)$ is called an adaptive optimal test for the family $\big\{\Sigma_0(\kappa), \Sigma(\kappa), \kappa \in \Psi\big\}$.

Following Definition 3, t_n must be interpreted as the smallest penalization for which one can construct a test that detects, simultaneously on Ψ, the null hypothesis $\Sigma_0(\kappa)$ at a distance of order $\varphi_\kappa(nt_n^{-1})$. In this approach, t_n is a uniform penalization in the sense that t_n does not depend on $\kappa \in \Psi$.

3. MAIN RESULTS

3.1. Minimax procedure

In this section, we recall the results obtained in [13]. We set

$$\varphi_\kappa(n) = \left(n^{-1}\sqrt{\log\left(\frac{\alpha_0}{\alpha_n}\right)}\right)^{\frac{2\beta}{4\beta+d}}, \qquad \Delta_{n,\kappa} = \mathbf{1}_{\{T_n > (\lambda\varphi_\kappa(n))^2\}}, \tag{7}$$

where

$$\alpha_0 \geq 2e^4, \qquad \lambda \geq \sqrt{2Q\sqrt{eK_1\Gamma} + L_0^2},$$

Γ and K_1 are positive constants. We define

$$T_n = \left\| \hat{f}_n - \prod_{k=1}^{d} \bar{f}_{kn} \right\|_2^2 - \frac{1}{n^2 h_n^{2d}(\kappa)} \sum_{i=1}^{n} \int\limits_{[0,1]^d} K^2\left(\frac{x - X_i}{h_n(\kappa)}\right) dx$$

with

$$h_n(\kappa) = h_n(\beta) \triangleq \left(n^{-1}\sqrt{\log\left(\frac{\alpha_0}{\alpha_n}\right)}\right)^{\frac{2}{4\beta+d}}. \tag{8}$$

We fix A_* and B_* and τ such that

$$A_* > \frac{8Q\|K_*\|^2(\beta+2)}{2\beta+1}, \qquad B_* > \frac{4Q\|K\|_2^2(4\beta+3d+4)}{(4\beta+d)},$$

$$\tau < \min\left\{1, -\frac{4\beta+3d+4}{2(4\beta+d)} + \frac{B_*}{8Q\|K\|_2^2}, -\frac{\beta+2}{2\beta+1} + \frac{A_*}{Q\|K_*\|_2^2}\right\}.$$

Let us put

$$\gamma_n(\alpha_n) \triangleq 2e^4\left(\frac{\alpha_n}{\alpha_0}\right)^{\frac{(2C^2 - 2\sqrt{2}\lambda C - \lambda)^2}{\lambda^4 eK_1Q\Gamma}} + \frac{8dQ_2}{A_*^{1/2}} n^{\frac{\beta+2}{2\beta+1} - \frac{A_*}{8Q\|K_*\|_2^2}} + \frac{8Q_1}{B_*^{1/2}} n^{\frac{4\beta+3d+4}{2(4\beta+d)} - \frac{B_*}{8Q\|K\|_2^2}} + \frac{16}{n}.$$

We have the following results.

Theorem 1 (Yodé [13]). *Assume that Assumptions* 1, 2, *and* 3 *are satisfied. If* $d > 2$ *and* $\alpha_n^{-1} = O_n(n^\tau)$*, then*

— $\displaystyle\sup_{f\in\Sigma_0(\kappa)} \mathbb{P}_f^n\{\Delta_{n,\kappa} = 1\} \leq \alpha_n(1 + o_n(1))$;

— [*Upper bound*] *there exists* $C^* = (1 + \frac{\sqrt{2}}{2})\lambda$ *such that for any* $C > C^*$

$$\sup_{f\in\Phi_n(C\varphi_\kappa(n))} \mathbb{P}_f^n\{\Delta_{n,\kappa} = 0\} \leq \gamma_n(\alpha_n)(1 + o_n(1));$$

— [*Lower bound*] *there exists* $C_*(\kappa) > 0$ *such that for any* $C < C_*(\kappa)$

$$\liminf_{n\to\infty} \inf_{\Delta_n \in \mathcal{T}(\alpha_n)} \sup_{f\in\Phi_n(C\varphi_\kappa(n))} \mathbb{P}_f^n\{\Delta_{n,\kappa} = 0\} = 1.$$

Remark 2. According to Theorem 1, $\varphi_\kappa(n)$ is the minimax rate of testing and $\Delta_{n,\kappa}$ is an asymptotically optimal test.

This test function depends obviously on $\kappa = (\beta, L, Q)$. The purpose of this paper is to solve the above testing problem in an adaptive framework, i.e., assuming that κ is unknown.

3.2. Adaptive Minimax Procedure

We use a naive adaptive method consisting of two steps: we first construct a grid on the set of nuisance parameters; then, we accept the independence hypothesis if only if each of the tests attached to the nodes of the grid accepts and reject as soon as at least one of them rejects. The problem here is that each test has a finite type 1 error probability but its type 2 error is too large. To cope with this, we take the threshold value for each test with an extra growth factor. We refer to Abramovich *et al.* [1], Chiabrando [2], Fromont and Laurent [4], Gayraud and Pouet [6], and Spokoiny [12] for similar approaches.

We put

$$t_n \triangleq \sqrt{1 + \frac{\log(\log(n))}{\log(\frac{\alpha_0}{\alpha_n})}},$$

$$h_{*n} = h_{*n}(\beta) \triangleq \left(n^{-1} t_n \sqrt{\log\left(\frac{\alpha_0}{\alpha_n}\right)}\right)^{\frac{2}{4\beta+d}} \quad \text{with } \alpha_0 \geq 2,$$

and we introduce the statistic

$$T_n^*(\kappa) = \left\| f_{n,\kappa}^* - \prod_{k=1}^{d} \bar{f}_{kn,\kappa} \right\|_2^2 - \frac{1}{n^2 h_{*n}^{2d}} \sum_{i=1}^{n} \int_{[0,1]^d} K^2\left(\frac{x - X_i}{h_{*n}}\right) dx,$$

where

$$f_{n,\kappa}^*(x) = \frac{1}{n h_{*n}} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_{*n}}\right).$$

We fix $r > 1$ and define $J_n(\beta)$ such that $r^{-J_n(\beta)} = h_n(\beta)$, that is

$$J_n(\beta) = \frac{2}{4\beta + d} \frac{\log\left(n \sqrt{\log(\frac{\alpha_0}{\alpha_n})}^{-1}\right)}{\log(r)}.$$

We assume without loss of generality that $J_n(\beta)$ is an integer. Otherwise, one can take its integer part. For each parameter $\kappa = (\beta, L, Q) \in \Psi$, we may determine the level $h_n(\beta)$ and the corresponding test procedure $\Delta_{n,\kappa}$ defined as (8) and (7) respectively. Therefore, the range of adaptation Ψ can be translated into a grid of $[\beta_*, \beta^*]$. To do that, we need to introduce the set

$$\mathcal{J}(\Psi) = \{J_n(\beta), \beta \in [\beta_*, \beta^*]\}.$$

Therefore, we arbitrarily construct an injective mapping, denoted by $\mathcal{V}$, from $\mathcal{J}(\Psi)$ to $[\beta_*, \beta^*]$ such that for any $j \in \mathcal{J}(\Psi)$, we have $J_n(\mathcal{V}(j)) = j$. Hence we define the grid

$$\mathcal{B}_n \triangleq \left\{\kappa_j = (\mathcal{V}(j) \triangleq \beta_j, L^*, Q^*) \colon j \in \mathcal{J}(\Psi)\right\} \subset \Psi.$$

Since $\beta_* \leq \beta^*$, we have

$$\frac{2}{4\beta^* + d} \frac{\log\left(n \sqrt{\log(\frac{\alpha_0}{\alpha_n})}^{-1}\right)}{\log(r)} \leq J_n(\beta) \leq \frac{2}{4\beta_* + d} \frac{\log\left(n \sqrt{\log(\frac{\alpha_0}{\alpha_n})}^{-1}\right)}{\log(r)}.$$

Obviously, the cardinality of $\mathcal{B}_n$, denoted by π_n, is controlled as follows:

$$\pi_n \leq C_1(\Psi) \log(n),$$

where $C_1(\Psi) > 0$ is a constant that only depends on the length $\beta^* - \beta_*$. Now, we introduce the following test:

$$\Delta_{n,\Psi}^* = \mathbf{1}_{\left\{\max\limits_{\kappa_j \in \mathcal{B}_n} \left\{\frac{T_n^*(\kappa_j)}{\varphi_{\kappa_j}^2(n t_n^{-1})}\right\} > \lambda\right\}}.$$

where $\lambda \geq \sqrt{eK_1\Gamma_1}Q^* + L_0^2$ with

$$\Gamma_1 = \int\limits_{\mathbb{R}^{3d}} |K(\omega_1)||K(\omega_2)||K(\omega_1+\omega_3)||K(\omega_2+\omega_3)|\,d\omega_1 d\omega_2 d\omega_3.$$

Fix positive constants A_*, B_*, and τ such that

$$A_* > \frac{8Q^*\|K_*\|^2(\beta_*+2)}{2\beta_*+1}, \qquad B_* > \frac{4Q^*\|K\|_2^2(4\beta_*+3d+4)}{(4\beta_*+d)}, \tag{9}$$

$$\tau < \min\left\{1, -\frac{4\beta_*+3d+4}{2(4\beta_*+d)} + \frac{B_*}{8Q^*\|K\|_2^2}, -\frac{\beta_*+2}{2\beta_*+1} + \frac{A_*}{Q^*\|K_*\|_2^2}\right\}. \tag{10}$$

Theorem 2. *Assume that Assumptions* 1, 2, 3, *and* 4 *are satisfied. If* $d > 2$ *and* $\alpha_n^{-1} = O_n\big(\frac{n^\tau}{\log(n)}\big)$, *then*

$$\sup_{\kappa\in\Psi}\sup_{f\in\Sigma_0(\kappa)} \mathbb{P}_f^n\{\Delta_{n,\Psi}^* = 1\} \leq \alpha_n(1+o_n(1)).$$

Let us put

$$\gamma_n(\alpha_n) \triangleq 2\left(\frac{\alpha_n}{\alpha_0\log(n)}\right)^{\frac{(\lambda-(L_0-C)^2)^2}{eK_1Q^{*2}\Lambda}} + C\left(n^{\frac{\beta_*+2}{2\beta_*+1} - \frac{A_*}{8Q^*\|K_*\|_2^2}} + n^{\frac{4\beta_*+3d+4}{2(4\beta_*+d)} - \frac{B_*}{8Q^*\|K\|_2^2}}\right) + \frac{16}{n},$$

$$C^*(\Psi) = L_0 + \sqrt{\lambda},$$

$$C_*(\Psi) = 2^{\frac{\beta^*}{4\beta_*+d}}.$$

Theorem 3 (Upper bound)**.** *Assume that Assumptions* 1, 2, 3, and 4 *are satisfied. If* $d > 2$ *and* $\alpha_n^{-1} = O_n\big(\frac{n^\tau}{\log(n)}\big)$, *then for any* $C > C^*(\Psi)$

$$\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(t_n^{-1}n))} \mathbb{P}_f^n\{\Delta_{n,\Psi}^* = 0\} \leq \gamma(\alpha_n)(1+o_n(1)).$$

Theorem 4 (Lower bound)**.** *Assume that Assumptions* 1, 2, *and* 3 *are satisfied. If* $d > 2$, *then for any* $C < C_*(\Psi)$

$$\liminf_{n\to\infty}\inf_{\Delta_n\in\mathcal{I}(\Psi,\alpha_n)}\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(t_n^{-1}n))} \mathbb{P}_f^n\{\Delta_{n,\Psi}^* = 0\} = 1.$$

Remark 3. According to Theorems 2, 3 and 4, the family of rates of testing $\{\varphi_\kappa(nt_n^{-1}), \kappa\in\Psi\}$ is the adaptive minimax rate of testing and $\Delta_{n,\Psi}^*$ is adaptive optimal for the family $\{\Sigma_0(\kappa), \Sigma(\kappa), \kappa\in\Psi\}$.

4. PROOFS OF THEOREMS

Throughout this section, if u_n and v_n are two real sequences such that $v_n \neq 0$, we put

$$u_n = o_n(v_n) \iff \lim_{n\to+\infty}\frac{u_n}{v_n} = 0,$$

$$u_n = O_n(v_n) \iff \limsup_{n\to+\infty}\left|\frac{u_n}{v_n}\right| < \infty.$$

All positive constants appearing in the proofs are called C, although they may vary from one occurrence to another.

Put

$$\eta(x) = K\left(\frac{x-X}{h_{*n}}\right) - \mathbb{E}_f^n K\left(\frac{x-X}{h_{*n}}\right),$$

$$\eta_i(x) = K\left(\frac{x-X_i}{h_{*n}}\right) - \mathbb{E}_f^n K\left(\frac{x-X_i}{h_{*n}}\right), \quad i = 1,\dots,n,$$

where $X_1,\dots,X_n$ and X are i.i.d. random vectors.

4.1. Asymptotics of the Test Statistic under the Null Hypothesis

In the sequel, we use the following decomposition:

$$T_n^*(\kappa) = \Big\| f_{n,\kappa}^* - \prod_{k=1}^d \bar f_{kn,\kappa} \Big\|_2^2 - \frac{1}{n^2 h_{*n}^{2d}} \sum_{i=1}^n \int_{[0,1]^d} K^2\Big(\frac{x - X_i}{h_{*n}}\Big) dx = \sum_{l=1}^6 S_{l,n,\kappa},$$

where

$$S_{1,n,\kappa} = \int_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big)^2 dx - \frac{1}{n^2 h_{*n}^{2d}} \sum_{i=1}^n \int_{[0,1]^d} K^2\Big(\frac{x - X_i}{h_{*n}}\Big) dx,$$

$$S_{2,n,\kappa} = \int_{[0,1]^d} \Big(\mathbb{E}_f^n(f_{n,\kappa}^*(x)) - \prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}(x_k))\Big)^2 dx,$$

$$S_{3,n,\kappa} = \int_{[0,1]^d} \Big(\prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}(x_k)) - \prod_{k=1}^d \bar f_{kn,\kappa}(x_k)\Big)^2 dx,$$

$$S_{4,n,\kappa} = 2\int_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big)\Big(\mathbb{E}_f^n(f_{n,\kappa}^*(x)) - \prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}(x_k))\Big) dx,$$

$$S_{5,n,\kappa} = 2\int_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big)\Big(\prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}(x_k)) - \prod_{k=1}^d \bar f_{kn,\kappa}(x_k)\Big) dx,$$

$$S_{6,n,\kappa} = 2\int_{[0,1]^d} \Big(\mathbb{E}_f^n(f_{n,\kappa}^*(x)) - \prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}(x_k))\Big)\Big(\prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}(x_k)) - \prod_{k=1}^d \bar f_{kn,\kappa}(x_k)\Big) dx.$$

4.1.1. Study of $\mathbf{S_{1,n,\kappa}(f)}$. We have

$$S_{1,n,\kappa}(f) = \chi_{n,\kappa} + U_{n,\kappa} + \zeta_{n,\kappa,f} + \mathbb{E}_f^n(\hat\theta_{n,\kappa}) - \hat\theta_{n,\kappa},$$

where

$$\chi_{n,\kappa} = \frac{1}{n^2 h_{*n}^{2d}} \sum_{i=1}^n \int_{[0,1]^d} \Big[\eta_i^2(x) - \mathbb{E}_f^n \eta_i^2(x)\Big] dx,$$

$$\hat\theta_{n,\kappa} = \frac{1}{n^2 h_{*n}^{2d}} \sum_{i=1}^n \int_{[0,1]^d} K^2\Big(\frac{x - X_i}{h_{*n}}\Big) dx,$$

$$\zeta_{n,\kappa,f} = -\frac{1}{n h_{*n}^{2d}} \int_{[0,1]^d} \Big(\mathbb{E}_f^n K\Big(\frac{x - X_1}{h_{*n}}\Big)\Big)^2 dx,$$

$$U_{n,\kappa} = \frac{1}{n^2} \sum_{1 \le i \ne j \le n} H_{n,\kappa}(X_i, X_j), \tag{11}$$

where

$$H_{n,\kappa}(X_i, X_j) = \frac{1}{h_{*n}^{2d}} \int_{[0,1]^d} \eta_i(x)\eta_j(x) dx. \tag{12}$$

We have the following results:

Lemma 1 (Yodé [13]). *There exists $C > 0$ such that for n large enough, we have*

$$\sup_{f\in\Sigma(\kappa)} \mathbb{P}\left(|\chi_{n,\kappa}| \geq C\sqrt{\frac{\log(n)}{n^3 h_{*n}^{2d}}}\right) \leq \frac{2}{n},$$

$$\sup_{f\in\Sigma(\kappa)} \mathbb{P}\left(|\hat{\theta}_{n,\kappa} - \mathbb{E}_f^n(\hat{\theta}_{n,\kappa})| \geq C\sqrt{\frac{\log(n)}{n^5 h_{*n}^{2d}}}\right) \leq \frac{2}{n},$$

$$\sup_{f\in\Sigma(\kappa)} |\zeta_{n,\kappa,f}| \leq \frac{Q^{*2}}{n}.$$

4.1.2. Study of $\mathbf{S_{2,n,\kappa}(f)}$, $\mathbf{S_{3,n,\kappa}(f)}$, and $\mathbf{S_{6,n,\kappa}(f)}$. The estimations are based on the following Lemmas:

Lemma 2 (Yodé [13]). *For any $A_* > 0$, for n large enough*

$$\sup_{f\in\Sigma(\kappa)} \mathbb{P}_f^n\left\{\sup_{k=1,\dots,d} \|\bar{f}_{kn,\kappa} - \mathbb{E}_f^n(\bar{f}_{kn,\kappa})\|_\infty \geq \sqrt{\frac{A_*\log(n)}{nb_n}}\right\} \leq Cn^{\frac{\beta_*+2}{2\beta_*+1} - \frac{A_*}{8Q^*\|K_*\|_2^2}}(1 + o_n(1)).$$

Here and in the sequel, if A_* satisfies (9), we put

$$A_{1,n,\kappa} = \left\{\sup_{k=1,\dots,d} \|\bar{f}_{kn,\kappa} - \mathbb{E}_f^n(\bar{f}_{kn,\kappa})\|_\infty \leq \sqrt{\frac{A_*\log(n)}{nb_n}}\right\}.$$

Lemma 3 (Yodé [13]). *We have*

$$\sup_{f\in\Sigma(\kappa)} \left|\left\|\mathbb{E}_f^n(f_{n,\kappa}^*) - \prod_{k=1}^d \mathbb{E}_f^n(\bar{f}_{kn,\kappa})\right\|_2 - \left\|f - \prod_{k=1}^d f_k\right\|_2\right|^2 \leq L_0^2 h_{*n}^{2\beta}(1 + o_n(1)). \tag{13}$$

For any $x \in [0,1]^d$, we have

$$\prod_{k=1}^d \mathbb{E}_f^n(\bar{f}_{kn,\kappa}(x_k)) - \prod_{k=1}^d \bar{f}_{kn,\kappa}(x_k) = -\prod_{k=1}^d \left(\bar{f}_{kn,\kappa}(x_k) - \mathbb{E}_f^n(\bar{f}_{kn,\kappa}(x_k))\right) - \mathcal{H}_{n,f,\kappa}(x),$$

where

$$\mathcal{H}_{n,f,\kappa}(x) = \sum_{l=1}^{d-1} \sum_{k_1\neq\ldots\neq k_l\neq\ldots\neq k_d} \prod_{s=1}^l \left(\bar{f}_{k_s n,\kappa}(x_{k_s}) - \mathbb{E}_f^n(\bar{f}_{k_s n,\kappa}(x_{k_1}))\right) \prod_{s=l+1}^d \mathbb{E}_f^n(\bar{f}_{k_s n,\kappa}(x_{k_s})).$$

Therefore, we can state that

$$\sup_{f\in\Sigma(\kappa)} \left[\mathbf{1}_{A_{1,n,\kappa}}\left\|\prod_{k=1}^d \mathbb{E}_f^n(\bar{f}_{kn,\kappa}) - \prod_{k=1}^d \bar{f}_{kn,\kappa}\right\|_\infty\right] \leq C\sqrt{\frac{\log(n)}{nb_n}}$$

for n large enough. Hence we get

$$\sup_{f\in\Sigma(\kappa)} [\mathbf{1}_{A_{1,n,\kappa}} S_{3,n,\kappa}] \leq C\mu_n h_{*n}^{2\beta}$$

for n large enough and

$$\mu_n = \frac{\log(n)}{nb_n h_{*n}^{2\beta}} \longrightarrow 0, \qquad n \to +\infty, \tag{14}$$

because of the choice of α_n and $d > 2$. Finally, we deduce

$$\sup_{f\in\Sigma(\kappa)} [\mathbf{1}_{A_{1,n,\kappa}} S_{3,n,\kappa}] = o_n(h_{*n}^{2\beta}). \tag{15}$$

Under the null hypothesis, we deduce from Lemma 3 that

$$\sup_{f\in\Sigma_0(\kappa)} \left\|\mathbb{E}_f^n(f_{n,\kappa}^*) - \prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa})\right\|_2^2 \le L_0^2 h_{*n}^{2\beta}\left(1+o_n(1)\right).$$

Therefore, we obtain

$$\sup_{f\in\Sigma_0(\kappa)} S_{2,n,\kappa} \le L_0^2 h_{*n}^{2\beta}(1+o_n(1)). \tag{16}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\sup_{f\in\Sigma_0(\kappa)} |\mathbf{1}_{A_{1,n,\kappa}} S_{6,n,\kappa}| &\le \sup_{f\in\Sigma_0(\kappa)} \left[2\mathbf{1}_{A_{1,n,\kappa}} \left\|\mathbb{E}_f^n(f_n^*) - \prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn})\right\|_2 \left\|\prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn,\kappa}) - \prod_{k=1}^d \bar f_{kn,\kappa}\right\|_\infty\right]\\ &\le C\sqrt{\mu_n} h_{*n}^{2\beta},\end{aligned}$$

for n large enough, μ_n is defined by (14). Hence we deduce that

$$\sup_{f\in\Sigma_0(\kappa)} |\mathbf{1}_{A_{1,n,\kappa}} S_{6,n,\kappa}| = o_n(h_{*n}^{2\beta}). \tag{17}$$

4.1.3. Study of $\mathbf{S_{4,n,\kappa}(f)}$. We need the following lemma.

Lemma 4 (Yodé [13]). *Let the sequence $R_{n,f}\colon \Sigma(\kappa)\times[0,1]^d \to \mathbb{R}$ be such that*

$$R_n^* = \sup_{f\in\Sigma(\kappa)} \|R_{n,f}\|_\infty = \sup_{f\in\Sigma(\kappa)} \sup_{x\in[0,1]^d} |R_{n,f}(x)| < \infty.$$

For any positive sequence

$$z_n = o_n(R_n^*),$$

one has

$$\sup_{f\in\Sigma(\kappa)} \mathbb{P}_f^n\left\{\left|\int\limits_{[0,1]^d} \left(f_n^*(x) - \mathbb{E}_f^n(f_n^*(x)\right) R_{n,f}(x)\,dx\right| \ge z_n\right\} \le 2\exp\left\{\frac{-Cnz_n^2}{R_n^{*2}}\right\},$$

for n large enough.

Under the null hypothesis, we have

$$\sup_{f\in\Sigma_0(\kappa)} \left\|\mathbb{E}_f^n(f_{*n}) - \prod_{k=1}^d \mathbb{E}_f^n(\bar f_{kn})\right\|_\infty \le L_0 h_{*n}^\beta$$

for n large enough. Thus, we obtain

$$\sup_{f\in\Sigma_0(\kappa)} \mathbb{P}_f^n\left\{|S_{4,n,\kappa}| \ge Ch_{*n}^\beta \sqrt{\frac{\log(n)}{n}}\right\} \le \frac{2}{n}, \tag{18}$$

for n large enough.

4.1.4. Study of $\mathbf{S_{5,n,\kappa}(f)}$. We need the following lemma.

Lemma 5 (Yodé [13]). *For any $B_*>0$,*

$$\sup_{f\in\Sigma(\kappa)} \mathbb{P}_f^n\left\{\|f_{n,\kappa}^* - \mathbb{E}_f^n(f_{n,\kappa}^*)\|_\infty \ge \sqrt{\frac{B_*\log(n)}{nh_{*n}^d}}\right\} \le Cn^{\frac{4\beta_*+3d+4}{2(4\beta_*+d)} - \frac{B_*}{8Q_*\|K\|_2^2}}(1+o_n(1)).$$

Here and in the sequel, if B_* satisfies (9), we consider

$$A_{2,n,\kappa} = \left\{ \|f^*_{n,\kappa} - \mathbb{E}^n_f(f^*_{n,\kappa})\|_\infty \le \sqrt{\frac{B_* \log(n)}{n h^d_{*n}}} \right\}.$$

For any $f \in \Sigma(\kappa)$, the following decomposition holds

$$S_{5,n,\kappa} = S^{(1)}_{5,n,\kappa} + S^{(2)}_{5,n,\kappa} + S^{(3)}_{5,n,\kappa},$$

where

$$S^{(1)}_{5,n,\kappa} = -2 \int\limits_{[0,1]^d} \left(f^*_{n,\kappa}(x) - \mathbb{E}^n_f(f^*_{n,\kappa}(x))\right) \prod_{k=1}^{d} \left(\bar{f}_{kn,\kappa}(x_k) - \mathbb{E}^n_f \bar{f}_{kn,\kappa}(x_k)\right) dx,$$

$$S^{(2)}_{5,n,\kappa} = -2 \sum_{l=2}^{d-1} \sum_{k_1 \ne \cdots \ne k_d} \int\limits_{[0,1]^d} \left(f^*_{n,\kappa}(x) - \mathbb{E}^n_f(f^*_{n,\kappa}(x))\right) \prod_{s=1}^{l} \left(\bar{f}_{k_s n,\kappa}(x_{k_s}) - \mathbb{E}^n_f \bar{f}_{k_s n,\kappa}(x_{k_s})\right) \times \prod_{s=l+1}^{d} \mathbb{E}^n_f \bar{f}_{k_s n,\kappa}(x_{k_s})\, dx,$$

$$S^{(3)}_{5,n,\kappa} = -2 \sum_{k_1 \ne \cdots \ne k_d} \int\limits_{[0,1]^d} \left(f^*_{n,\kappa}(x) - \mathbb{E}^n_f(f^*_{n,\kappa}(x))\right) \left(\bar{f}_{k_1 n,\kappa}(x_{k_1}) - \mathbb{E}^n_f \bar{f}_{k_1 n}(x_{k_1})\right) \times \prod_{s=2}^{d} \mathbb{E}^n_f \bar{f}_{k_s n,\kappa}(x_{k_s})\, dx.$$

Thus, we get

$$\begin{aligned}
&\sup_{f \in \Sigma(\kappa)} |\mathbf{1}_{A_{1,n,\kappa}} \mathbf{1}_{A_{2,n,\kappa}} S^{(1)}_{5,n,\kappa}| \\
&\quad \le \sup_{f \in \Sigma(\kappa)} \left[\mathbf{1}_{A_{2,n,\kappa}} \|f^*_{n,\kappa} - \mathbb{E}^n_f(f^*_{n,\kappa})\|_\infty \Big(\mathbf{1}_{A_{1,n,\kappa}} \sup_{k=1,\ldots,d} \|\bar{f}_{kn,\kappa} - \mathbb{E}^n_f \bar{f}_{kn,\kappa}\|_\infty \Big)^d \right] \\
&\quad \le C \eta_n h^{2\beta}_{*n},
\end{aligned}$$

where

$$\eta_n = \frac{(\log(n))^{\frac{d+1}{2}}}{n^{\frac{d+1}{2}} h_{*n}^{\frac{d}{2}+2\beta} b_n^{\frac{d}{2}}} \overset{n\to\infty}{\longrightarrow} 0.$$

Finally, we deduce that

$$\sup_{f \in \Sigma(\kappa)} \left| \mathbf{1}_{A_{1,n,\kappa}} \mathbf{1}_{A_{2,n,\kappa}} S^{(1)}_{5,n,\kappa} \right| = o_n(h^{2\beta}_{*n}). \tag{19}$$

Likewise, we state

$$\sup_{f \in \Sigma(\kappa)} \left| \mathbf{1}_{A_{1,n,\kappa}} \mathbf{1}_{A_{2,n,\kappa}} S^{(2)}_{5,n,\kappa} \right| = o_n(h^{2\beta}_{*n}). \tag{20}$$

To estimate $S^{(3)}_{5,n,\kappa}$, we use the following decomposition

$$S^{(3)}_{5,n,\kappa} = \sum_{k_1 \ne \ldots \ne k_d} (S^{(3,k,1)}_{5,n,\kappa} + S^{(3,k,2)}_{5,n,\kappa}),$$

where $k = (k_1, \dots, k_d)$ and

$$S^{(3,k,1)}_{5,n,\kappa} = \frac{2}{n^2}\sum_{i=1}^{n}\psi_n(X_i, X_i), \qquad S^{(3,k,2)}_{5,n,\kappa} = \frac{2}{n^2}\sum_{i\neq j}\psi_n(X_i, X_j) \tag{21}$$

with

$$\psi_n(X_i, X_j) \approx \frac{1}{nh^d_{*n}b_n}\int\limits_{[0,1]^d}\eta_i(x)\biggl(K_*\biggl(\frac{x_{k_1} - X^{(k_1)}_j}{b_n}\biggr) - \mathbb{E}^n_f K_*\biggl(\frac{x_{k_1} - X^{(k_1)}_j}{b_n}\biggr)\biggr)\,dx. \tag{22}$$

The statistic $S^{(3,k,2)}_{5,n,\kappa}$ is a degenerate U-statistic of order 2. Then, using Proposition 1, we state the following result:

$$\sup_{f\in\Sigma(\kappa)}\mathbb{P}^n_f\biggl\{|S^{(3,k,2)}_{5,n,\kappa}| \geq C\sqrt{\frac{\log(n)}{n^2h_{*n}}}\biggr\} \leq \frac{2}{n} \tag{23}$$

for n large enough. For the statistic $S^{(3,k,1)}_{5,n,\kappa}$, the following decomposition holds

$$S^{(3,k,1)}_{5,n,\kappa} = \frac{1}{n^2}\sum_{i=1}^{n}\bigl(\psi_n(X_i, X_i) - \mathbb{E}^n_f\psi_n(X_i, X_i)\bigr) + \frac{1}{n}\mathbb{E}^n_f\psi_n(X_1, X_1).$$

Therefore, we have the following results

$$\sup_{f\in\Sigma(\kappa)}\mathbb{P}^n_f\biggl\{|\psi_n(X_i, X_i) - \mathbb{E}^n_f\psi_n(X_i, X_i)| \geq C\sqrt{\frac{\log(n)}{n^3h^2_{*n}}}\biggr\} \leq \frac{2}{n} \tag{24}$$

for n large enough and

$$\sup_{f\in\Sigma(\kappa)}\left[\frac{1}{n}\mathbb{E}^n_f\psi_n(X_1, X_1)\right] = o_n\left(h^{2\beta}_{*n}\right). \tag{25}$$

4.1.5. Conclusion. Using the estimations above, we deduce the behavior of test statistic $T^*_n(\kappa)$ under the null hypothesis:

$$T^*_n(\kappa) \leq U_{n,\kappa} + L^2_0h^{2\beta}_{*n}(1 + o_n(1)).$$

4.2. Exponential Inequality for Completely Degenerate U-Statistic of Order 2

Let

$$U_n = \frac{1}{n(n-1)}\sum_{1\leq i\neq j\leq n}H_n(X_i, X_j)$$

be a completely degenerated U-statistic of order 2, that is

$$\mathbb{E}(H_n(X_i, X_j) \mid X_i) = \mathbb{E}(H_n(X_i, X_j) \mid X_j) = 0,$$

where the kernel H_n is a symmetric function. We introduce the following functions:

$$\underline{H}(x, y) = \mathbb{E}\bigl(H(X_1, X)H(X_2, X) \mid X_1 = x, X_2 = y\bigr),$$
$$\underline{H}_{(0)}(x) = \underline{H}(x, x) - \mathbb{E}(\underline{H}(X, X)).$$

Proposition 1 (Chiabrando [2]). *Assume that there exist $u_{s,n}$, $k_{s,n}$ and $c_s > 0$, $s = 1, 2, 3$, such that for any $p \in 4\mathbb{N}$*

(i) $\mathbb{E}\left(H_n(X_1, X_2)\right) \leq k_{1,n}p^{c_1p}u^p_{1,n}$,

(ii) $\max\left\{\mathbb{E}(\max_{i\neq j}|\underline{H}_n(X_i,X_j)|),\mathbb{E}\underline{H}^p_{(0)}\right\}\le k_{2,n}p^{c_2p}u^p_{2,n}$,

(iii) $\mathbb{E}\Big(\max_i\{(\mathbb{E}(\underline{H}^2_n(X_i,X)\mid X_i))^p\}\Big)\le k_{3,n}p^{c_3p}u^p_{3,n}$.

Then, for any $x>0$,

$$\mathbb{P}\left(\frac{1}{n(n-1)}\left|\sum_{1\le i\neq j\le n}H_n(X_i,X_j)\right|\ge x\right)\le c_n(x)e^{4-m_n(x)}.$$

The mixture term is defined by

$$m_n(x)=\min\{m_{i,n}(x),1\le i\le 7\},$$

where $K_1,\dots,K_5$ are universal constants and

$$m_{1,n}(x)=\left(\frac{nx}{eK_1\sigma_n}\right)^2,\qquad m_{2,n}(x)=\frac{nx}{eK_2\sqrt{\underline{\sigma}_n}},\qquad m_{3,n}(x)=\left(\frac{n\sqrt{n}x}{eK_3\sigma_n}\right)^{\frac{2}{3}},$$

$$m_{4,n}(x)=\left(\frac{n^{\frac{5}{4}}x}{eK_4\sqrt{\underline{\sigma}_{(0)}}}\right)^{\frac{4}{3}},\qquad m_{5,n}(x)=\left(\frac{n^2x}{eK_2u_{n,1}}\right)^{\delta(c_1)},$$

$$m_{6,n}(x)=\left(\frac{n\sqrt{n}x}{e(K_2\vee K_5)\sqrt{u_{n,2}}}\right)^{\delta(c_2)},\qquad m_{7,n}(x)=\left(\frac{n^{\frac{4}{5}}x}{eK_2u^{\frac{1}{4}}_{3,n}}\right)^{\delta(c_3)},$$

where

$$\sigma^2_n=\mathbb{E}(H^2_n(X_1,X_2)),\qquad \underline{\sigma}^2_n=\mathbb{E}(\underline{H}^2_n(X_1,X_2)),\qquad \underline{\sigma}^2_{(0)}=\mathbb{E}(\underline{H}^2_{(0)}(X,X)),$$

$$\delta(c_1)=\frac{1}{2+c_1},\qquad \delta(c_2)=\frac{2}{3+c_2},\qquad \delta(c_3)=\frac{4}{5+c_3}.$$

Moreover,

$$c_n(x)\overset{\triangle}{=}\max\{c_{j,n}\colon j\mid m_n(x)=m_{j,n}(x)\},$$

where

$$c_{1,n}=c_{2,n}=c_{4,n}=1,\quad c_{3,n}=n,\quad c_{5,n}=2nk_{1,n},\quad c_{6,n}=n^2k_{2,n},\quad c_{7,n}=k_{3,n}.$$

The proof of this result is a slight modification of that of the exponential inequality in Giné *et al.* [7]. We consider the U-statistic $U_{n,\kappa}$ defined by (11) and (12). For n large enough, we have

$$\sigma^2_n=\mathbb{E}(H^2_{n,\kappa}(X_1,X_2))\le Q^{*2}\Gamma_1h^{-d},$$
$$\mathbb{E}(H^p_{n,\gamma}(X_1,X_2))\le Q^{*2}\Gamma_2h^dp^{\nu p}h^{-pd}$$

for all $p\in\mathbb{N}^*$, where $\nu>0$ and

$$\Gamma_1=\int\limits_{\mathbb{R}^{3d}}|K(\omega_1)K(\omega_2)K(\omega_1+\omega_3)K(\omega_2+\omega_3)|\,d\omega_1d\omega_2d\omega_3,$$

$$\Gamma_2=\int\limits_{\mathbb{R}^d}\left(\prod_{s=1}^{d}\int\limits_{\mathbb{R}^d}|K(\omega_s)K(\omega_s+\omega)|\,d\omega_s\right)d\omega.$$

We have

$$\underline{H}_{n,\kappa}(X_1, X_2) = \frac{1}{h^{4d}} \int_{\mathbb{R}^{2d}} \eta_1(x)\eta_2(y)\mathbb{E}(\eta(x)\eta(y))\, dxdy.$$

Therefore, for n large enough, for all $p \in \mathbb{N}^*$, we have

$$\mathbb{E}(\underline{H}^2_{n,\kappa}(X_i, X) \mid X_i) \leq Q^{*3}\, \|K\|^8_\infty\, h^{-d},$$
$$\mathbb{E}\big(\max_i\{(\mathbb{E}(\underline{H}^2_{n,\kappa}(X_i, X) \mid X_i))^p\}\big) \leq Q^{*3p}\|K\|^{8p}_\infty p^{\nu p} h^{-pd},$$
$$\mathbb{E}\big(\max_{i\neq j} |\underline{H}^p_{n,\kappa}(X_i, X_j)|\big) \leq Q^{*p}\|K\|^{4p}_\infty p^{\nu p} h^{-pd},$$
$$\mathbb{E}(\underline{H}^p_{(0),\kappa}(X)) \leq 2^{p+1}\|K\|^{4p}_\infty Q^{*2p} p^{\nu p} h^{-pd},$$

where $\nu > 0$. We obtain

$$\mathbb{P}\left(\frac{1}{n^2}\Big|\sum_{1\leq i\neq j\leq n} H_{n,\kappa}(X_i, X_j)\Big| \geq x\right) \leq c_n(x)e^{4-m_n(x)},$$

where

$$m_n(x) = \min\{m_{i,n}(x), 1 \leq i \leq 7\}$$

with

$$m_{1,n}(x) = \frac{n^2h^dx^2}{eK_1Q^{*2}\Gamma_1}, \qquad m_{2,n}(x) = \frac{nh^{\frac{d}{4}}x}{eK_2Q_*\|K\|^2_\infty}, \qquad m_{3,n}(x) = \left(\frac{n\sqrt{n}h^{\frac{d}{2}}x}{eK_3Q^*\Gamma_1^{\frac{1}{2}}}\right)^{\frac{2}{3}},$$

$$m_{4,n}(x) = \left(\frac{n^{\frac{5}{4}}h^{\frac{d}{2}}x}{eK_4\Gamma_2^{\frac{1}{4}}}\right)^{\frac{4}{3}}, \qquad m_{5,n}(x) = \left(\frac{n^2h^dx}{eK_2Q^*\Gamma_2}\right)^{\delta(c_1)},$$

$$m_{6,n}(x) = \left(\frac{n\sqrt{n}h^{\frac{d}{2}}x}{e(K_2 \vee K_5)\Delta}\right)^{\delta(c_2)}, \qquad m_{7,n}(x) = \left(\frac{n^{\frac{4}{5}}h^{\frac{d}{5}}x}{eK_2Q^{*\frac{3p}{4}}}\right)^{\delta(c_3)}.$$

4.3. Proof of Theorem 2

Let us introduce the random events for any $f \in \Sigma_0(\kappa)$,

$$A_{1,n,\kappa} = \left\{\sup_{k=1,\ldots,d} \|\bar{f}_{kn,\kappa} - \mathbb{E}^n_f(\bar{f}_{kn,\kappa})\|_\infty \leq \sqrt{\frac{A_*\log(n)}{nb_n}}\right\},$$

$$A_{2,n,\kappa} = \left\{\|f^*_{n,\kappa} - \mathbb{E}^n_f(f^*_{n,\kappa})\|_\infty \leq \sqrt{\frac{B_*\log(n)}{nh^d_{*n}}}\right\},$$

$$A_{3,n,\kappa} = \left\{|S^{(3,k,2)}_{5,n,\kappa}| \leq C\sqrt{\frac{\log(n)}{n^2h_{*n}}}\right\},$$

$$A_{4,n,\kappa} = \left\{|\psi_n(X_i, X_i) - \mathbb{E}^n_f(\psi_n(X_i, X_i))| \leq C\sqrt{\frac{\log(n)}{n^3h^2_{*n}}}\right\},$$

$$A_{5,n,\kappa} = \left\{|S_{4,n,\kappa}| \leq Ch^\beta_{*n}\sqrt{\frac{\log(n)}{n}}\right\},$$

$$A_{6,n,\kappa} = \left\{|\chi_{n,\kappa}| \leq C\sqrt{\frac{\log(n)}{n^3h^{2d}_{*n}}}\right\},$$

$$A_{7,n,\kappa} = \left\{ |\hat{\theta}_{n,\kappa} - \mathbb{E}_f^n(\hat{\theta}_{n,\kappa})| \le C\sqrt{\frac{\log(n)}{n^3 h_{*n}^{2d}}} \right\},$$

where $S_{5,n,\kappa}^{(3,k,2)}$ and ψ are defined by (21) and (22) respectively.

Using a Bonferroni argument, we have

$$\mathbb{P}_f^n(\Delta_{n,\Psi}^* = 1) = \mathbb{P}_f^n\big(\max_{\kappa_j \in \mathcal{B}_n} T_n^*(\kappa_j) > \lambda\varphi_{\kappa_j}^2(nt_n^{-1})\big) \le \sum_{\kappa_j \in \mathcal{B}_n} \mathbb{P}_f^n\big(T_n^*(\kappa_j) > \lambda\varphi_{\kappa_j}^2(nt_n^{-1})\big).$$

For any $f \in \Sigma_0(\kappa)$, using Lemmas 1, 2, 5, (15)–(20), and (23)–(25), we get

$$\begin{aligned}
\mathbb{P}_f^n\big(T_n^*(\kappa_j) > \lambda\varphi_{\kappa_j}^2(nt_n^{-1})\big) &\le \mathbb{P}_f^n\big(T_n^*(\kappa_j) > \lambda\varphi_{\kappa_j}^2(nt_n^{-1})\big) + \sum_{j=1}^{7} \mathbb{P}_f^n(A_{l,n,\kappa_j}^c) \\
&\le \mathbb{P}_f^n\big(U_{n,\kappa_j} \ge (\lambda - L_0^2)\varphi_{\kappa_j}^2(nt_n^{-1})\big) + \sum_{l=1}^{7} \mathbb{P}_f^n(A_{l,n,\kappa_j}^c) \\
&\le 2\exp\left(- \frac{(\lambda - L_0^2)^2 \varphi_{\kappa_j}^4(nt_n^{-1}) n^2 h_n^d(\kappa_j)}{eK_1 Q_*^2 \Gamma} \right) + \frac{16}{n} \\
&\quad + C\left(n^{\frac{4\beta_j+3d+4}{2(4\beta_j+d)} - \frac{B_*}{8Q\|K\|_2^2}} + n^{\frac{\beta_j+2}{2\beta_j+1} - \frac{A_*}{8Q\|K_*\|_2^2}} \right)(1 + o_n(1)).
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
\mathbb{P}_f^n\big(T_n^*(\kappa_j) > \lambda\varphi_{\kappa_j}^2(nt_n^{-1})\big) &\le 2e^4 \left(\frac{\alpha_n}{\alpha_0 \log(n)} \right)^{\frac{(\lambda - L_0^2)^2}{eK_1 Q_*^2 \Gamma_1}} + \frac{16}{n} \\
&\quad + C\left(n^{\frac{4\beta_*+3d+4}{2(4\beta_*+d)} - \frac{B_*}{8Q^*\|K\|_2^2}} + n^{\frac{\beta^*+2}{2\beta^*} - \frac{A_*}{8Q^*\|K_*\|_2^2}} \right)(1 + o_n(1)).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{P}_f^n(\Delta_{n,\Psi}^* = 1) &\le 2\pi_n e^4 \left(\frac{\alpha_n}{\alpha_0 \log(n)} \right)^{\frac{(\lambda - L_0^2)^2}{eK_1 Q_*^2 \Gamma_1}} + \frac{16\pi_n}{n} \\
&\quad + C\pi_n\left(n^{\frac{4\beta_*+3d+4}{2(4\beta_*+d)} - \frac{B_*}{8Q^*\|K\|_2^2}} + n^{\frac{\beta^*+2}{2\beta^*} - \frac{A_*}{8Q^*\|K_*\|_2^2}} \right)(1 + o_n(1)) \\
&\le 2e^4 C_1(\Psi) \log(n) \left(\frac{\alpha_n}{\alpha_0 \log(n)} \right)^{\frac{(\lambda - L_0^2)^2}{eK_1 Q_*^2 \Gamma}} + \frac{16C_1(\Psi)\log(n)}{n} \\
&\quad + CC_1(\Psi)\log(n)(\Psi)\left(n^{\frac{4\beta_*+3d+4}{2(4\beta_*+d)} - \frac{B_*}{8Q^*\|K\|_2^2}} + n^{\frac{\beta^*+2}{2\beta^*+1} - \frac{A_*}{8Q^*\|K_*\|_2^2}} \right)(1 + o_n(1)).
\end{aligned}$$

It suffices to choose constants $\alpha_0 \ge 2e^4 C_1(\Psi)$, $\lambda \ge \sqrt{eK_1\Gamma_1} Q_* + L_0^2$, and $\alpha_n^{-1} = O_n\big(\frac{n^\tau}{\log(n)}\big)$ to conclude that

$$\sup_{\kappa \in \Psi} \sup_{f \in \Sigma_0(\kappa)} \mathbb{P}_f^n\big\{\Delta_{n,\Psi}^* = 1\big\} \le \alpha_n(1 + o_n(1))$$

using (10). □

4.4. Asymptotics of the Test Statistic under Alternative

Now, we aim to study the behavior of the test $\Delta_{\Psi,n}^*$ under the alternative $f \in \Phi_n(C\varphi_\kappa(nt_n^{-1}))$ for a given $\kappa \in \Psi$ and $C > 0$.

4.4.1. Study of $S_{2,n,\kappa}$. Here and in the sequel, for any $f \in \Sigma(\kappa)$, $x \in [0,1]^d$, we put

$$R_f(x) = f(x) - \prod_{k=1}^{d} f_k(x_k).$$

Using Lemma 3, for any $f \in \Sigma$, for any $\beta \in [\beta_*, \beta^*]$

$$\|R_f\|_2 - L_0 h_{*n}^{\beta} \leq \Big\| \mathbb{E}_f^n(f_{n,\kappa}^*) - \prod_{k=1}^{d} \mathbb{E}_f^n(\bar{f}_{kn}) \Big\|_2.$$

Thus, we get

$$S_{2,n,\kappa} \geq \big(\|R_f\|_2 - L_0 h_{*n}^{\beta}\big)^2. \tag{26}$$

4.4.2. Study of $S_{1,n,\kappa}$ and $S_{5,n,\kappa}$. To estimate $S_{1,n,\kappa}$ and $S_{5,n,\kappa}$, we use the same results as in subsections 4.1.1 and 4.1.4.

4.4.3. Study of $S_{4,n,\kappa}$. We use Lemma 4 and the following result.

Lemma 6 (Yodé [13]). *For any $f \in \Sigma$, there exists a positive sequence*

$$z_n = o_n\big(h_n^{\frac{3d}{2}} \|R_f\|_2\big)$$

such that

$$\mathbb{P}_f^n\bigg\{\bigg| \int\limits_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big) R_f(x)\, dx \bigg| \geq z_n \bigg\} \leq 2 \exp\bigg\{ - \frac{C n z_n^2}{\|R_f\|_2} \bigg\}$$

for n large enough.

We use the following decomposition

$$S_{4,n,\kappa} = J_{1,n,\kappa} + J_{2,n,\kappa} + J_{3,n,\kappa} + J_{4,n,\kappa},$$

where

$$J_{1,n,\kappa} = 2 \int\limits_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big) R_{1,n,f}(x)\, dx,$$

$$J_{2,n,\kappa} = 2 \int\limits_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big) R_{2,n,f}(x)\, dx,$$

$$J_{3,n,\kappa} = 2 \int\limits_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big) R_f(x)\, dx,$$

$$J_{4,n,\kappa} = 2 \int\limits_{[0,1]^d} \big(f_{n,\kappa}^*(x) - \mathbb{E}_f^n(f_{n,\kappa}^*(x))\big) B_{n,f}(x)\, dx$$

with

$$B_{n,f}(x) = \sum_{l=1}^{d-1} \sum_{k_1 \neq \cdots \neq k_d} \prod_{s=1}^{l} \big(\mathbb{E}_f^n \bar{f}_{k_s n}(x_{k_s}) - f_{k_s}(x_{k_s})\big) \prod_{s=l+1}^{d} f_{k_s}(x_{k_s}),$$

$$\mathbb{R}_{1,n,f}(x) = \mathbb{E}_f^n(f_{n,\kappa}^*) - f(x),$$

$$\mathbb{R}_{2,n,f}(x) = \prod_{k=1}^{d} \big(\mathbb{E}_f^n \bar{f}_{kn}(x_k) - f_k(x_k)\big).$$

It is easy to prove that

$$R^*_{1,n} = \sup_{f\in\Sigma(\kappa)} \|R_{1,n,f}\|_\infty \le h^\beta_{*n},$$
$$\mathbb{R}^*_{2,n} = \sup_{f\in\Sigma(\kappa)} \|R_{2,n,f}\|_\infty \le Cb_n^{\beta d},$$
$$B^*_n = \sup_{f\in\Sigma(\kappa)} \|B_{n,f}\|_\infty \le Cb_n^\beta$$

for n large enough. Thus, we get the following results uniformly on $\Phi_n(C\varphi_\kappa(nt_n^{-1}))$:

$$\mathbb{P}^n_f\big\{|J_{1,n,\kappa}| \ge z_n\big\} \le 2\exp\bigg\{-\frac{Cnz_n^2}{h_{*n}^{2\beta}}\bigg\} \quad \text{for any} \quad z_n = o_n(h^\beta_{*n}),$$
$$\mathbb{P}^n_f\big\{|J_{2,n,\kappa}| \ge z_n\big\} \le 2\exp\bigg\{-\frac{Cnz_n^2}{b_n^{2\beta d}}\bigg\} \quad \text{for any} \quad z_n = o_n(b_n^{\beta d}),$$
$$\mathbb{P}^n_f\big\{|J_{4,n,\kappa}| \ge z_n\big\} \le 2\exp\bigg\{-\frac{Cnz_n^2}{b_n^{2\beta}}\bigg\} \quad \text{for any} \quad z_n = o_n(b_n^{\beta}).$$

For any $\Phi_n(C\varphi_\kappa(nt_n^{-1}))$ there exists a positive sequence $z_{n,f} = o_n\big(h_{*n}^{\frac{3d}{2}}\|R_f\|_2\big)$ such that

$$\mathbb{P}^n_f\left\{|J_{3,n,\kappa}| \ge z_{n,f}\right\} \le \exp\bigg\{-\frac{Cnz^2_{n,f}}{\|R_f\|_2}\bigg\}.$$

In particular, putting $z_{n,f} = C\|R_f\|_2\sqrt{\frac{\log n}{n}}$, we have

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))} \mathbb{P}^n_f\big\{|J_{3,n,\kappa}| \ge z_{n,f}\big\} \le \frac{2}{n} \tag{27}$$

for n large enough.

4.4.4. Study of $S_{6,n,\kappa}$. We have the following decomposition

$$S_{6,n,\kappa_{j_0}}(f) = K_{1,n,\kappa} + K_{2,n,\kappa} + K_{3,n,\kappa} + K_{4,n,\kappa},$$

where

$$K_{1,n,\kappa} = 2\int\limits_{[0,1]^d}\bigg(\prod_{k=1}^d \mathbb{E}(\bar f_{kn}(x_k)) - \prod_{k=1}^d \bar f_{kn}(x_k)\bigg)R_{1,n,f}(x)\,dx,$$
$$K_{2,n,\kappa} = -2\int\limits_{[0,1]^d}\bigg(\prod_{k=1}^d \mathbb{E}(\bar f_{kn}(x_k)) - \prod_{k=1}^d \bar f_{kn}(x_k)\bigg)R_{2,n,f}(x)\,dx,$$
$$K_{3,n,\kappa} = 2\int\limits_{[0,1]^d}\bigg(\prod_{k=1}^d \mathbb{E}(\bar f_{kn}(x_k)) - \prod_{k=1}^d \bar f_{kn}(x_k)\bigg)R_{f}(x)\,dx,$$
$$K_{4,n,\kappa} = 2\int\limits_{[0,1]^d}\bigg(\prod_{k=1}^d \mathbb{E}(\bar f_{kn}(x_k)) - \prod_{k=1}^d \bar f_{kn}(x_k)\bigg)B_{n,f}(x)\,dx.$$

Since

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\bigg\|\mathbf{1}_{A_{1,n,\kappa}}\bigg(\prod_{k=1}^d \mathbb{E}(\bar f_{kn}) - \prod_{k=1}^d \bar f_{kn}\bigg)\bigg\|_\infty \le C\sqrt{\frac{\log n}{nb_n}},$$

we obtain

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\left[\mathbf{1}_{A_{1,n,\kappa}}|K_{1,n,\kappa_{j_0}}(f)|\right]\leq Ch_{*n}^{\beta}\sqrt{\frac{\log n}{nb_n}},$$

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\left[\mathbf{1}_{A_{1,n,\kappa}}|K_{2,n,\kappa_{j_0}}(f)|\right]\leq Cb_n^{d\beta}\sqrt{\frac{\log n}{nb_n}},$$

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\left[\mathbf{1}_{A_{1,n,\kappa}}|K_{4,n,\kappa_{j_0}}|\right]\leq Cb_n^{\beta}\sqrt{\frac{\log n}{nb_n}}$$

for n large enough. Therefore, we deduce that

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\left[\mathbf{1}_{A_{1,n,\kappa}}|K_{1,n,\kappa}|\right]=o_n(h_{*n}^{2\beta}),$$

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\left[\mathbf{1}_{A_{1,n,\kappa}}|K_{2,n,\kappa}|\right]=o_n(h_{*n}^{2\beta}),$$

$$\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\left[\mathbf{1}_{A_{1,n,\kappa}}|K_{4,n,\kappa}|\right]=o_n(h_{*n}^{2\beta}).$$

Using the Cauchy–Schwarz inequality, we have

$$|\mathbf{1}_{A_{1,n,\kappa}}K_{3,n,\kappa}|\leq\mathbf{1}_{A_{1,n,\kappa}}\Big\|\prod_{k=1}^{d}\bar{f}_{kn}-\prod_{k=1}^{d}\mathbb{E}_f^n\bar{f}_{kn}\Big\|_\infty\Big\|f-\prod_{k=1}^{d}f_k\Big\|_2\leq C\|R_f\|_2\sqrt{\frac{\log n}{nb_n}}$$

for n large enough, for any $f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))$.

4.5. Proof of Theorem 3

Now, we consider the test $\Delta^*_{\Psi,n}$ under the alternative $f\in\Phi(C\varphi_\kappa(nt_n^{-1}))$ for a given $\kappa\in\Psi$ and $C>0$. We denote by $j_0\in\mathcal{B}_n$ the unique index such that $J_n(\mathcal{V}(j_0))=J_n(\beta)$. Recall that $J_n(\mathcal{V}(j_0))=J_n(\beta)$ is defined as in subsection 3.2. Moreover, we notice that

$$\varphi_{\kappa_{j_0}}(nt_n^{-1})=\varphi_\kappa(nt_n^{-1})(1+o_n(1)). \tag{28}$$

Thus, we get

$$\mathbb{P}_f^n(\Delta^*_{\Psi,n}=0)=\mathbb{P}_f^n\left(\max_{\kappa_j\in\mathcal{B}_n}\left\{\frac{T_n^*(\kappa_j)}{\varphi^2_{\kappa_j}(nt_n^{-1})}\right\}\leq\lambda\right)\leq\mathbb{P}_f^n(T_n^*(\kappa_{j_0})\leq\lambda\varphi^2_{\kappa_{j_0}}(nt_n^{-1})).$$

Now, we put, for any $f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))$,

$$\mathbb{P}_f^n(T_n^*(\kappa_{j_0})\leq\lambda\varphi^2_{\kappa_{j_0}}(nt_n^{-1}))=\mathbb{P}_f^n\left\{\sum_{l=1}^{6}S_{l,n,\kappa_{j_0}}(f)\leq\lambda\varphi^2_{\kappa_{j_0}}(nt_n^{-1})\right\}$$
$$\leq\mathbb{P}_f^n\left\{\sum_{l\notin\{2,3\}}S_{l,n,\kappa_{j_0}}(f)+\left(\|R_f\|_2-L_0^*h_n^{\beta_{j_0}}\right)^2\leq\lambda\varphi^2_{\kappa_{j_0}}(nt_n^{-1})\right\}.$$

Let us consider the random events for any $f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))$,

$$A_{8,n,\kappa_{j_0}}=\left\{|J_{1,n,\kappa_{j_0}}(f)|\leq Ch_n^{\beta_{j_0}}\sqrt{\frac{\log n}{n}}\right\},$$

$$A_{9,n,\kappa_{j_0}}=\left\{|J_{2,n,\kappa_{j_0}}(f)|\leq Cb_n^{d\beta_{j_0}}\sqrt{\frac{\log n}{n}}\right\},$$

$$A_{10,n,\kappa_{j_0}}=\left\{|J_{3,n,\kappa_{j_0}}(f)|\leq C\|R_f\|_2\sqrt{\frac{\log n}{n}}\right\},$$

$$A_{11,n,\kappa_{j_0}} = \left\{ |J_{4,n,\kappa_{j_0}}(f)| \le C b_n^{\beta_{j_0}} \sqrt{\frac{\log n}{n}} \right\}.$$

We have the following equalities:

$$a_n \overset{\triangle}{=} (\|R_f\|_2 - L_0^* h_n^{\beta_{j_0}})^2 - C_1 \|R_f\|_2 \sqrt{\frac{\log n}{n}} - C_2 \|R_f\|_2 \sqrt{\frac{\log n}{n b_n}}$$
$$= \|R_f\|_2^2 \left(\left(1 - \frac{L_0 h_n^{\beta}}{\|R_f\|_2}\right)^2 - \frac{C_1 \sqrt{\frac{\log n}{n}}}{\|R_f\|_2} - \frac{C_2 \sqrt{\frac{\log n}{n b_n}}}{\|R_f\|_2} \right).$$

For n large enough, since $f \in \Phi(C\varphi_\kappa(nt_n^{-1}))$, using (28), we get

$$a_n \ge (C\varphi_\kappa(nt_n^{-1}))^2 \left(\left(1 - \frac{L_0}{C}\right)^2 + o_n(1) \right) = (C\varphi_{\kappa_{j_0}}(nt_n^{-1}))^2 \left(1 - \frac{L_0}{C}\right)^2 (1 + o_n(1)).$$

Hence, for any $f \in \Phi_n(C\varphi_\kappa(nt_n^{-1}))$, for n large enough, we have

$$\begin{aligned}
\mathbb{P}_f^n\big(T_n^*(\kappa_{j_0}) \le \lambda \varphi^2_{\kappa_{j_0}}(nt_n^{-1})\big) &\le \mathbb{P}_f^n\big(U_n + a_n \le \lambda \varphi^2_{\kappa_{j_0}}(nt_n^{-1})\big) \\
&\quad + \sum_{l=6}^{11} \mathbb{P}_f^n\big(A^c_{l,n,\kappa_{j_0}}\big) + \mathbb{P}_f^n\big(A^c_{1,n,\kappa_{j_0}}\big) + \mathbb{P}_f^n\big(A^c_{2,n,\kappa_{j_0}}\big) \\
&\le \mathbb{P}_f^n\big(U_n + (\varphi_{\kappa_{j_0}}(nt_n^{-1}))^2 (C - L_0)^2 \le \lambda \varphi^2_{\kappa_{j_0}}(nt_n^{-1})\big) \\
&\quad + \sum_{l=6}^{11} \mathbb{P}_f^n\big(A^c_{l,n,\kappa_{j_0}}\big) + \mathbb{P}_f^n\big(A^c_{1,n,\kappa_{j_0}}\big) + \mathbb{P}_f^n\big(A^c_{2,n,\kappa_{j_0}}\big) \\
&\le \mathbb{P}_f^n\big(U_n \le (\lambda - (L_0 - C)^2) \varphi^2_{\kappa_{j_0}}(nt_n^{-1})\big) \\
&\quad + \sum_{l=6}^{11} \mathbb{P}_f^n\big(A^c_{l,n,\kappa_{j_0}}\big) + \mathbb{P}_f^n\big(A^c_{1,n,\kappa_{j_0}}\big) + \mathbb{P}_f^n\big(A^c_{2,n,\kappa_{j_0}}\big).
\end{aligned}$$

Putting

$$\gamma_n(\alpha_n) = 2e^4 \left(\frac{\alpha_n}{\alpha_0 \log(n)} \right)^{\frac{(\lambda - (C - L_0)^2)^2}{e K_1 Q^{*2} \Gamma_1}} + \frac{16}{n} + C \left(n^{\frac{4\beta_* + 3d + 4}{2(4\beta_* + d)} - \frac{B_*}{8 Q_* \|K\|_2^2}} + n^{\frac{\beta_* + 2}{2\beta_* + 1} - \frac{A_*}{8 Q_* \|K_*\|_2^2}} \right)$$

if $C > L_0 + \sqrt{\lambda}$, using Proposition 1 and results of subsection 4.4, we get

$$\mathbb{P}_f^n\big(T_n^*(\kappa_{j_0}) \le \lambda \varphi^2_{\kappa_{j_0}}(nt_n^{-1})\big) \le \gamma_n(\alpha_n)(1 + o_n(1)).$$

Therefore, we obtain

$$\sup_{\kappa \in \Psi} \sup_{f \in \Phi_n(C\varphi_\kappa(nt_n^{-1}))} \mathbb{P}_f^n(\Delta^*_{\Psi,n} = 0) \le \gamma_n(\alpha_n)(1 + o_n(1)). \quad \square$$

4.6. Proof of Theorem 4

4.6.1. Construction of discrete family of functions $\mathcal{F}_{n,k_j}$. Fix $\sigma > 0$ and put $\delta_{n,j} = \sigma h_{*n}(\beta_j)$, $M_{n,j} = \delta_{n,j}^{-1}$. The value of σ can be determined later. Suppose that $M_{n,j}$ is an integer. Otherwise, one can take its integer part. Let $\{u_1, \ldots, u_{M_{n,j}}\}$ be a regular subdivision of $[0,1]$ and put $A_{n,j,l} = [u_l, u_{l+1}[$, $l = 1, \ldots, M_{n,j} - 1$, $A_{n,j,M_{n,j}} = [u_{M_{n,j}}, 1]$, where

$$u_l = \frac{l - 1}{M_{n,j}}.$$

For a multi-index $s = (s_1, \ldots, s_d) \in \mathcal{M}_{n,j} \triangleq \{1, \ldots, M_{n,j}\}^d$, define $A_{n,j,s} = A_{n,j,s_1} \times \ldots \times A_{n,j,s_d}$. Thus, the family $\{A_{n,j,s}, s \in \mathcal{M}_{n,j}\}$ is a partition of $[0,1]^d$.

Let ψ be an infinitely differentiable function with support $[0,1]$ such that

$$\int_{[0,1]} \psi(x)\,dx = 0, \qquad \int_{[0,1]} \psi^2(x)\,dx = 1. \tag{29}$$

For any $s = (s_1, \ldots, s_d) \in \mathcal{M}_{n,j}$, consider the function

$$\psi_{n,j,s}(x_1, \ldots, x_d) = \frac{1}{\delta_{n,j}^{d/2}} \prod_{r=1}^{d} \psi\left(\frac{x_r - u_{s_r}}{\delta_n}\right)$$

such that

$$\delta_{n,j}^{\beta_j + d/2} \sum_{s \in \mathcal{M}_{n,j}} \left| \psi_{n,j,s}(x) - \sum_{0 \le |i| \le m} \frac{1}{i!}(x - v)^i D^i \psi_{n,j,s}(v) \right| \le \frac{L}{2} \|x - v\|^{\beta_j}. \tag{30}$$

The function $\psi_{n,j,s}$ is compactly supported in $A_{n,j,s}$ and using (29) we have

$$\int_{A_{nj,s}} \psi_{n,j,s}(x)\,dx = 0, \qquad \int_{A_{nj,s}} \psi_{n,j,s}^2(x)\,dx = 1. \tag{31}$$

Moreover, the functions $\psi_{n,j,s}$ have nonintersecting supports.

We fix the density $g(x_1, \ldots, x_d) = g_1(x_1) \ldots g_d(x_d) \in \Sigma(\beta_j, \frac{L}{2}, \frac{Q}{2}) \cap \Phi_0$ and we consider the collection of functions

$$\mathcal{F}_{n,\kappa_j} = \left\{ f_{j,k}, k = 1, \ldots, 2^{M_{n,j}^d} \right\},$$

where

$$f_{j,k}(x_1, \ldots, x_d) = g(x_1, \ldots, x_d) + \delta_n^{\beta_j + d/2} \sum_{s \in \mathcal{M}_{n,j}} a_{k,s} \psi_{n,j,s}(x_1, \ldots, x_d),$$

where $a_{k,s}$, $s \in \mathcal{M}_{n,j}$, are $i.i.d.$ taking values 1 and -1 with probability $1/2$. Moreover, we suppose that

$$\int_{[0,1]^d} \frac{\psi_{n,j,s}^2(x)}{g(x)}\,dx = 1. \tag{32}$$

For any $j = 1, \ldots, 2^{M_n^d}$, $f_{j,k} \in \Phi_n(C\varphi_{\kappa_j}(nt_n^{-1}))$ for any $\kappa_j \in \mathcal{B}_n$:

1. for n large enough $f_{j,k}$, $k = 1, \ldots, 2^{M_n^d}$, are nonnegative and are densities belonging to $\Sigma(\kappa_j)$ according to (30);

2. the marginal distributions of $f_{j,k}$ are the same as for the density g; then, using (31), we get

   $$\|f_{j,k} - g\|_2^2 = \sigma^{2\beta_j} (\varphi_{\kappa_j}(nt_n^{-1}))^2 \ge (C\varphi_{\kappa_j}(nt_n^{-1}))^2,$$

   choosing

   $$0 < C \le \sigma^{\beta_j}; \tag{33}$$

3. moreover, we have $\|f_{j,k}\|_\infty \le Q$ for n large enough.

4.6.2. The whole parametric family $\mathcal{F}$. We will consider the whole parametric family defined as

$$\mathcal{F} = \bigcup_{k_j \in \mathcal{B}_n} \mathcal{F}_{n,\kappa_j}.$$

We define a probability measure μ_n on $\mathcal{F}$ as

$$\mu_n = \frac{1}{\pi_n} \sum_{k_j \in \mathcal{B}_n} \mu_{\kappa_j},$$

where μ_{κ_j} is the uniform measure on $\mathcal{F}_{n,\kappa_j}$. Note that we have

$$Z_n = \frac{d\mathbb{P}_\mu^n}{d\mathbb{P}_g^n} = \frac{1}{\pi_n} \sum_{k_j \in \mathcal{B}_n} \frac{d\mathbb{P}_{\mu_{\kappa_j}}^n}{d\mathbb{P}_g^n} = \frac{1}{\pi_n} \sum_{k_j \in \mathcal{B}_n} Z_{n,\kappa_j},$$

where

$$Z_{n,\kappa_j} = \frac{d\mathbb{P}_{\mu_{\kappa_j}}^n}{d\mathbb{P}_g^n}(X^n) = \mathbb{E}_{\mu_{\kappa_j}}^n \left[\frac{d\mathbb{P}_{f_{j,k}}^n}{d\mathbb{P}_g^n}(X^n) \right] = \frac{1}{2^{M_{n,j}^d}} \sum_{k=1}^{2^{M_{n,j}^d}} \frac{d\mathbb{P}_{f_{j,k}}^n}{d\mathbb{P}_g^n}(X^n)$$

$$= \frac{1}{2^{M_n^d}} \sum_{k=1}^{2^{M_n^d}} \prod_{i=1}^{n} \left(1 + \delta_{n,j}^{\beta_j + d/2} \sum_{s \in \mathcal{M}_{n,j}} a_{k,s} \frac{\psi_{n,j,s}(X_i)}{g(X_i)} \right).$$

Since the random variables X_i are independent, elementary computation and use of (31) show that

$$\mathbb{E}_{f_0}^n(Z_{n,\kappa_j}^2) = \frac{1}{2^{2M_{n,j}^d}} \sum_{k=1}^{2^{M_{n,j}^d}} \sum_{l=1}^{2^{M_{n,j}^d}} \left(1 + \delta_{n,j}^{2\beta_j + d} \sum_{s \in \mathcal{M}_{n,j}} a_{k,s} a_{l,s} \right)^n.$$

Therefore, putting $a_{k,s}^{'} = a_{k,s} a_{l,s}$, we get

$$\mathbb{E}_{f_0}^n(Z_{n,\kappa_j}^2) = \frac{1}{2^{M_{n,j}^d}} \sum_{k=1}^{2^{M_{n,j}^d}} \left[\frac{1}{2^{M_{n,j}^d}} \sum_{l=1}^{2^{M_{n,j}^d}} \left(1 + \delta_{n,j}^{2\beta_j + d} \sum_{s \in \mathcal{M}_{n,j}} a_{k,s} a_{l,s} \right)^n \right]$$

$$= \frac{1}{2^{M_{n,j}^d}} \sum_{k=1}^{2^{M_{n,j}^d}} \left(1 + \delta_{n,j}^{2\beta_j + d} \sum_{s \in \mathcal{M}_{n,j}} a_{k,s}^{'} \right)^n$$

$$= \frac{1}{2^{M_{n,j}^d}} \sum_{i=0}^{M_{n,j}^d} \binom{M_{n,j}^d}{i} \left(1 + \delta_{n,j}^{2\beta_j + d} (M_{n,j}^d - 2i) \right)^n$$

$$\leq \frac{1}{2^{M_{n,j}^d}} \sum_{i=0}^{M_{n,j}^d} \binom{M_{n,j}^d}{i} \left(\exp(n\delta_n^{2\beta_j + d}) \right)^{M_n^d - i} \left(\exp(-n\delta_n^{2\beta_j + d}) \right)^i$$

$$= \left(\frac{\exp(n\delta_n^{2\beta_j + d}) + \exp(-n\delta_n^{2\beta_j + d})}{2} \right)^{M_{n,j}^d}$$

$$\leq \exp\left(n^2 M_{n,j}^d \delta_{n,j}^{4\beta_j + 2d} \right) \leq \exp\left(\sigma^{4\beta_j + d} t_n^2 \log\left(\frac{\alpha_0}{\alpha_n} \right) \right).$$

Therefore, we obtain

$$\mathbb{E}_{f_0}^n(Z_{n,\kappa_j}^2) \leq \left(\frac{\alpha_0}{\alpha_n} \log(n) \right)^{\sigma^{4\beta_j + d}} \leq \left(\frac{\alpha_0}{\alpha_n} \log(n) \right)^{\sigma^{C(\Psi)}}, \tag{34}$$

where $C(\Psi)$ is a constant depending only on Ψ.

Thus, we can write by using the Cauchy–Schwarz inequality

$$\mathbb{E}_f^n(Z_n^2) = \frac{1}{\pi_n^2}\mathbb{E}_f^n\Big(\sum_{\kappa_j\in\mathcal{B}_n} Z_{n,\kappa_j}\Big)^2 \le \frac{1}{\pi_n}\sum_{\kappa_j\in\mathcal{B}_n}\mathbb{E}_f^n(Z_{n,\kappa_j}^2)$$
$$\le \sup_{\kappa_j\in\mathcal{B}_n}\mathbb{E}_f^n(Z_{n,\kappa_j}^2) \le \left(\frac{\alpha_0}{\alpha_n}\log(n)\right)^{\sigma^{C(\Psi)}}. \tag{35}$$

4.6.3. Proof of the result. For any $\Delta_n \in \mathcal{I}(\Psi,\alpha_n)$, since $f_0 \in \Sigma_0(\kappa)$, we have for all $c > 0$

$$\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\mathbb{P}_f^n\{\Delta_n = 0\} \ge \mathbb{P}_{\mu_n}^n(\Delta_n = 0) + c\alpha_n^{-1}\mathbb{P}_{f_0}^n\{\Delta_n = 1\} - c$$
$$= \mathbb{E}_{f_0}^n\big(Z_n\mathbf{1}_{\{\Delta_n=0\}} + c\alpha_n^{-1}\mathbf{1}_{\{\Delta_n=1\}}\big) - c.$$

We have the following result

$$\inf_{\mathcal{F}_n}\big(Z_n\mathbf{1}_{\mathcal{F}_n} + c\alpha_n^{-1}\mathbf{1}_{\mathcal{F}_n^c}\big) = Z_n\mathbf{1}_{Z_n<c\alpha_n^{-1}} + c\alpha_n^{-1}\mathbf{1}_{Z_n\ge c\alpha_n^{-1}}.$$

Its follows that

$$\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\mathbb{P}_f^n\{\Delta_n = 0\} \ge \mathbb{E}_{f_0}^n(Z_n\mathbf{1}_{\{Z_n<c\alpha_n^{-1}\}}) - c$$
$$= \frac{1}{\pi_n}\sum_{\kappa_j\in\mathcal{B}_n}\mathbb{E}_{f_0}^n\left(\frac{1}{2^{M_{n,j}^d}}\sum_{k=1}^{2^{M_{n,j}^d}}\frac{d\mathbb{P}_{f_{j,k}}^n}{d\mathbb{P}_{f_0}^n}(X^n)\mathbf{1}_{\{Z_n<c\alpha_n^{-1}\}}\right) - c$$
$$= \frac{1}{\pi_n}\sum_{\kappa_j\in\mathcal{B}_n}\left(\frac{1}{2^{M_{n,j}^d}}\sum_{k=1}^{2^{M_{n,j}^d}}\mathbb{P}_{f_{j,k}}^n\{Z_n < c\alpha_n^{-1}\}\right) - c$$
$$= (1-c) - \frac{1}{\pi_n}\sum_{\kappa_j\in\mathcal{B}_n}\left(\frac{1}{2^{M_{n,j}^d}}\sum_{k=1}^{2^{M_{n,j}^d}}\mathbb{P}_{f_{j,k}}^n\{Z_n \ge c\alpha_n^{-1}\}\right).$$

Applying Markov's inequality yields

$$\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\mathbb{P}_f^n\{\Delta_n = 0\} \ge (1-c) - c^{-2}\alpha_n^2\mathbb{E}_{f_0}^n(Z_n^2).$$

According to (35), we write

$$\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\mathbb{P}_f^n\{\Delta_n = 0\} \ge (1-c) - c^{-2}\alpha_n^{2-\sigma^{C(\Psi)}}\alpha_0^{\sigma^{C(\Psi)}}(\log(n))^{\sigma^{C(\Psi)}}.$$

Choosing $\sigma^{C(\Psi)} < 2$, i.e., $\sigma < 2^{\frac{1}{C(\Psi)}} \le 2^{\frac{1}{4\beta_j+d}} \le 2^{\frac{1}{4\beta_*+d}}$, we deduce that

$$\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\mathbb{P}_f^n\{\Delta_n = 0\} \ge (1-c) + o_n(1)$$

for all $c > 0$, for n large enough. Therefore, according to (33), for all $C < 2^{\frac{\beta^*}{4\beta_*+d}}$

$$\liminf_{n\to\infty}\inf_{\Delta_n\in\mathcal{I}(\Psi,\alpha_n)}\sup_{\kappa\in\Psi}\sup_{f\in\Phi_n(C\varphi_\kappa(nt_n^{-1}))}\mathbb{P}_f^n(\Delta_n = 0) = 1.$$

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