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Curvilinear Three-Webs with Automorphisms

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Abstract—A general form of the equation of a curvilinear three-web admitting a one-parameter family of automorphisms (AW-webs) is found. It is proved that the trajectories of automorphisms of an AW-web are geodesics of its Chern connection. All AW-webs are found for which one of the covariant derivatives of curvature is zero.

Keywords: curvilinear three-web, regular three-web, automorphism of a three-web, infinitesimal automorphism, Chern connection of a three-web, geodesic line

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1. An arbitrary curvilinear three-web does not, generally speaking, admit automorphisms. A parallel three-web admits a three-parameter group of automorphisms. Indeed, a parallel three-web is given by the equation $z = x + y$. Its automorphisms have the form $x = a\tilde{x} + b_1$, $y = a\tilde{y} + b_2$, and the corresponding autotopies (A_1, A_2, A_3) of a three-web (that is, admissible transformations of the parameters of the families forming the web) are $x = a\tilde{x} + b_1$, $y = a\tilde{y} + b_2$, $z = a\tilde{z} + b_1 + b_2$. It follows that any regular web (that is, a web locally diffeomorphic to a parallel one) also admits a three-parameter group of automorphisms. Therefore, webs that admit smaller families of automorphisms are of interest.

Three-webs with automorphisms have been studied by many authors, starting with Cartan [1]. We do review this topic and just mention two of the latest works in this direction [2, 3].

The following statements were proven in [4].

Theorem 1. *If a curvilinear three-web on the real plane admits a one-parameter family of automorphisms* (AW*-webs*), *then in some local coordinates its equation can be reduced to the form*

$$z = x + y + \lambda(x - y). \tag{1}$$

Here, $\lambda(x - y)$ *is an arbitrary smooth function of the variable* $x - y$.

Theorem 2. *If a curvilinear three-web on the real plane admits a two-parameter family of automorphisms*, *then it is regular*.

The proofs in [4] were based on the simple fact that the absolute invariants of a web are constant along the trajectories of automorphisms. In the current paper, we prove Theorem 1 in a different way, directly integrating the corresponding system of differential equations, and also show that the trajectories of automorphisms of an AW-web are geodesics of the Chern connection of a three-web. In addition, in this paper we found all AW-webs for which one of the covariant derivatives (with respect to the canonical Chern connection of this web) is equal to zero.

2. Let W be an arbitrary curvilinear three-web formed in a certain region D of the plane by lines $x = \text{const}$, $y = \text{const}$, and $f(x, y) = \text{const}$, then the equation of this web has the form $z = f(x, y)$. We put

$$\omega_1 = f_x dx, \quad \omega_2 = f_y dy. \tag{2}$$

By performing exterior differentiation of the forms ω_1 and ω_2, we get

$$d\omega_1 = f_{xy} dy \wedge dx = \Gamma\omega_1 \wedge \omega_2, \quad d\omega_2 = f_{xy} dx \wedge dy = \Gamma\omega_2 \wedge \omega_1, \tag{3}$$

where

$$\Gamma = -\frac{f_{xy}}{f_x f_y}.$$

Putting $\omega = \Gamma(\omega_1 + \omega_2)$, we rewrite equalities (3) as

$$d\omega_1 = \omega_1 \wedge \omega, \quad d\omega_2 = \omega_2 \wedge \omega. \tag{4}$$

Exterior differentiation of the form ω leads to the equality

$$d\omega = b\omega_1 \wedge \omega_2, \tag{5}$$

where

$$b = \frac{\Gamma_x}{f_x} - \frac{\Gamma_y}{f_y}.$$

Equations (4) and (5) are called structure equations of a three-web W, and the function b is referred to as the curvature of this web. The condition $b = 0$ characterizes the class of regular webs.

On the other hand, Eqs. (4) and (5) are equations of some torsion-free affine connection, and the forms of connection are given by

$$(\omega_1, \omega_2), \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}.$$

This connection is called the *Chern connection*.

The geodesic lines of the Chern connection are given by the equations

$$d\omega_1 + \omega_1\omega = \Theta\omega_1, \quad d\omega_2 + \omega_2\omega = \Theta\omega_2, \tag{6}$$

where d is the ordinary differentiation and Θ is some 1-form depending on the choice of parameter on the geodesic. Excluding Θ from Eqs. (6), we arrive at the geodesic equation in another form:

$$\omega_2 d\omega_1 - \omega_1 d\omega_2 = 0.$$

From here we get $\omega_1 = C\omega_2$, $C = \text{const}$, or

$$f_x dx = C f_y dy. \tag{7}$$

At $C = 0, \infty, -1$ we get lines of the first, second, and third families of the web W. At $C = 1$ we obtain a family of geodesic lines, which, together with the lines of the third family, harmonically divide (at each point) a pair of lines of the first and second families. We can also say that the family $C = 1$ is associated with the third family of web lines relative to the pair of the first two. Therefore, we call this family of lines 3-conjugate.

3. Automorphism of a three-web W is a local diffeomorphism of a domain D, which translates the lines of the web W again into the line of this web. The proof of the following statement can be found in ([6], Theorem 6.8). *Local automorphisms of a three-web W are also automorphisms of the corresponding Chern connection. Conversely, let φ be an automorphism of the Chern connection of some three-web W, defined in the domain D, and there exists a point p in D such that the differential $d\varphi|_p$ translates tangents to lines of the web W passing through the point p into tangents to the corresponding lines passing through the point $\varphi(p)$. Then φ is an automorphism of the three-web W.*

(Recall that the automorphism of an affine connection Γ defined on a manifold X is a diffeomorphism of this manifold that preserves the law of parallel translation, that is, preserves the covariant differential with respect to this connection.)

Proof of Theorem 1. Suppose that the three-web W admits a one-parameter family of automorphisms defined by a vector field $\xi(\xi_1, \xi_2)$. Then the quantities ξ_1, ξ_2 satisfy the following equations from [5]:

$$d\xi_1 + \xi_1\omega = \xi\omega_1, \quad d\xi_2 + \xi_2\omega = \xi\omega_2,$$

or, by virtue of (2) and (3),

$$d\xi_1 + \xi_1\left(\frac{-f_{xy}}{f_y}dx - \frac{f_{xy}}{f_x}dy\right) = \xi f_x dx, \quad d\xi_2 + \xi_2\left(\frac{-f_{xy}}{f_y}dx - \frac{f_{xy}}{f_x}dy\right) = \xi f_y dy.$$

Hence,

$$\frac{\partial \xi_1}{\partial x} = \xi_1 \frac{f_{xy}}{f_y} + \xi f_x, \quad \frac{\partial \xi_1}{\partial y} = \xi_1 \frac{f_{xy}}{f_x},$$
$$\frac{\partial \xi_2}{\partial x} = \xi_2 \frac{f_{xy}}{f_y}, \quad \frac{\partial \xi_2}{\partial y} = \xi_2 \frac{f_{xy}}{f_x} + \xi f_y.$$

We put $\xi_1 = \alpha f_x$, $\xi_2 = \beta f_y$. Substituting into the previous equations, we get

$$\alpha_y = 0, \;\; \beta_x = 0, \;\; \alpha_x f_x + \alpha f_{xx} = \alpha f_x \frac{f_{xy}}{f_y} + \xi f_x, \;\; \beta_y f_y + \beta f_{yy} = \beta f_y \frac{f_{xy}}{f_x} + \xi f_y.$$

This implies $\alpha = \alpha(x)$, $\beta = \beta(y)$ and

$$f_y(\alpha f_x + \beta f_y)_x = f_x(\alpha f_x + \beta f_y)_y. \tag{8}$$

Lemma. *By variable replacements $x = x(\tilde{x})$, $y = y(\tilde{y})$ the functions $\alpha(x)$ and $\beta(y)$ can be reduced to unity.*

Proof. Let us denote $f(x(\tilde{x}), y(\tilde{y})) = \tilde{f}(\tilde{x}, \tilde{y})$, then

$$\tilde{f}_{\tilde{x}} = f_x \frac{dx}{d\tilde{x}}, \quad \tilde{f}_{\tilde{y}} = f_y \frac{dy}{d\tilde{y}}.$$

We put $dx/d\tilde{x} = \alpha(x)$, $dy/d\tilde{y} = \beta(y)$ and substitute it into (8). After transformations we arrive at the equation

$$\tilde{f}_{\tilde{y}}(\tilde{f}_{\tilde{x}} + \tilde{f}_{\tilde{y}})_{\tilde{x}} = \tilde{f}_{\tilde{x}}(\tilde{f}_{\tilde{x}} + \tilde{f}_{\tilde{y}})_{\tilde{y}}. \tag{9}$$

Let us continue the proof, assuming that the indicated change of variables has been made and omit the tilde over the variables. After some calculations, we reduce Eq. (9) to the form

$$\left(\frac{f_x}{f_y}\right)_x + \left(\frac{f_x}{f_y}\right)_y = 0,$$

which leads to

$$\frac{f_x}{f_y} = \varphi(x - y), \tag{10}$$

where $\varphi(x - y)$ is a smooth function of the variable $x - y$. Let us further put $f(x, y) = g(u, v)$, $u = x + y$, $v = x - y$, then Eq. (10) becomes

$$g_u + g_v = (g_u - g_v)\varphi(v),$$

or $g_v = g_u \nu(v)$. We have

$$dg = g_u du + g_v dv = g_u(du + \nu(v)dv) = g_u d(u + \lambda(v)).$$

As a result, the web equation $z = f(x, y)$ takes the form $z = g(u + \lambda(v)) = f(x + y + \lambda(x - y))$. After admissible parameter replacement $f^{-1}(z) \to z$ we arrive at Eq. (1).

□

Because $\alpha = \beta = 1$, it is true that $\xi = (\xi_1, \xi_2) = (f_x, f_y)$. Automorphisms of the AW-web have the form $x \to x + a$, $y \to y + a$. The trajectories of the automorphisms are the lines $x - y = \text{const}$.

Theorem 3. *Trajectories of automorphisms of the AW-web are geodesic and coincide with its 3-conjugate family only if the three-web is regular.*

Proof. From Eq. (1) we find $f_x = 1 + \lambda'(x - y)$, $f_y = 1 - \lambda'(x - y)$, so the geodesic equation (7) for a three-web AW becomes

$$(1 + \lambda')dx - C(1 - \lambda')dy = 0,$$

where prime means the derivative with respect to the variable $v = x - y$. On the trajectory of automorphisms $x - y = \text{const}$ we have $\lambda'(x - y) = \text{const}$. We can choose the constant C so that the geodesic equation takes the form $dx - dy = 0$ or $x - y = \text{const}$. The first part of Theorem 3 is proven.

The equation of the 3-conjugate family for the AW-web transforms to

$$(1+\lambda')dx-(1-\lambda')dy=0.$$

This equation coincides with the equation for the trajectories of automorphism $dx-dy=0$ only in case $\lambda'=0$ or $\lambda=\mathrm{const}$. But then Eq. (1) defines a regular three-web.

□

4. Let us find the covariant derivatives of the curvature of web (1), denoting it, as above, by AW. First, we calculate the curvature b. We have

$$f_x=1+\lambda',\quad f_y=1-\lambda',\quad f_{xy}=-\lambda'',\quad \Gamma=\frac{\lambda''}{1-(\lambda')^2}=\frac{1}{2}\left(\ln\frac{1+\lambda'}{1-\lambda'}\right)'.$$

Further,

$$\Gamma_x=-\Gamma_y=\frac{1}{2}\left(\ln\frac{1+\lambda'}{1-\lambda'}\right),$$

$$b=\left(\ln\frac{1+\lambda'}{1-\lambda'}\right)(1-(\lambda')^2)^{-1}. \tag{11}$$

We find the covariant derivatives of the curvature b from the formula

$$db-2b\omega=b_1\omega_1+b_2\omega_2,$$

which is a differential continuation of the structure equation (5). We have

$$db=b_xdx+b_ydy=b'dx-b'dy=b'\left(\frac{\omega_1}{f_x}-\frac{\omega_2}{f_y}\right),\qquad 2b\omega=2b\Gamma(\omega_1+\omega_2).$$

Consequently,

$$b_1=\frac{b'}{f_x}-2b\Gamma=\frac{b'}{1+\lambda'}-b\left(\ln\frac{1+\lambda'}{1-\lambda'}\right),\quad b_2=-\frac{b'}{f_y}-2b\Gamma=-\frac{b'}{1-\lambda'}-b\left(\ln\frac{1+\lambda'}{1-\lambda'}\right). \tag{12}$$

5. Let us find three-webs AW for which $b_1=0$. According to ([7], p. 69) this condition distinguishes three-webs B_1 formed by a family of parallel lines and integral curves of two Riccati equations of a special form. Let us show that such a class exists, and the solution can be found in quadratures. Note that, to describe this class, we cannot use the results from [7], because the moving frame there was normalized by the condition that one of the covariant derivatives of the curvature is equal to unity, while in this article the normalization is different—the trajectories of automorphisms are written in the form $x-y=\mathrm{const}$.

As can be seen from (12), the condition $b_1=0$ leads to the equation

$$\frac{b'}{1+\lambda'}-b\left(\ln\frac{1+\lambda'}{1-\lambda'}\right)=0\Rightarrow\frac{b'}{b}=\frac{2\lambda''}{1-\lambda'},$$

or $(\ln b)'=(\ln(1-\lambda')^{-2})'$. Hence,

$$b=\frac{\alpha}{2(1-\lambda')^2},\quad \alpha=\mathrm{const}.$$

If $\alpha=0$, then $b=0$, and we get a regular web. Let further $\alpha\neq 0$. Comparing the last equality with (11), we arrive at the relation

$$\frac{\alpha}{2}\frac{1+\lambda'}{1-\lambda'}=\left(\ln\frac{1+\lambda'}{1-\lambda'}\right),$$

or

$$t''=\frac{1}{2}\alpha e^t, \tag{13}$$

where

$$t = \ln\frac{1+\lambda'}{1-\lambda'}. \tag{14}$$

In (13) we put

$$t'(v) = \Theta(t(v)) \Rightarrow t''(v) = \Theta'(t)t'(v)) = \Theta'(t)\Theta.$$

As a result, (13) takes the form $2\Theta d\Theta = \alpha e^t dt$. Therefore, $\Theta^2 = \alpha e^t + \beta$, $\beta = \text{const}$, and

$$\Theta = \frac{dt}{dv} = \sqrt{\alpha e^t + \beta} \Rightarrow v = \int \frac{dt}{\sqrt{\alpha e^t + \beta}}.$$

Replacement $\alpha e^t + \beta = u^2$ reduces the integral to the form

$$v = 2\int \frac{du}{u^2 - \beta}. \tag{15}$$

Case 1. $\beta = \delta^2 \neq 0$, then

$$v = \frac{1}{\delta}\ln\left|\frac{\sqrt{\alpha e^t + \delta^2} - \delta}{\sqrt{\alpha e^t + \delta^2} + \delta}\right| - \frac{\gamma}{\delta}, \quad \gamma = \text{const}.$$

After simple transformations, $\alpha > 0$ get for

$$e^t = \frac{4\delta^2}{\alpha}\frac{e^{\delta v + \gamma}}{(e^{\delta v + \gamma} - 1)^2},$$

for $\alpha < 0$

$$e^t = -\frac{4\delta^2}{\alpha}\frac{e^{\delta v + \gamma}}{(e^{\delta v + \gamma} - 1)^2}.$$

As we can see, these cases are identical, so we consider only the first option below. It follows from (14) that

$$\lambda' = \frac{e^t - 1}{e^t + 1}.$$

Substituting here the previous expression for e^t, we find

$$\lambda' = \frac{\varepsilon - \sinh^2\left(\dfrac{\delta v + \gamma}{2}\right)}{\varepsilon + \sinh^2\left(\dfrac{\delta v + \gamma}{2}\right)}, \quad \varepsilon = \frac{\delta^2}{\alpha}. \tag{16}$$

Next, we put $\dfrac{\delta v + \gamma}{2} = X$ and in the right-hand side of (16) we express $\sinh^2 X$ through $\tanh X$. As a result, we get

$$\lambda' = \frac{\varepsilon - (\varepsilon + 1)\tanh^2 X}{\varepsilon - (\varepsilon - 1)\tanh^2 X},$$

$$\lambda = \int \frac{\varepsilon - (\varepsilon + 1)\tanh^2 X}{\varepsilon - (\varepsilon - 1)\tanh^2 X} dv = \frac{2}{\delta}\int \frac{\varepsilon - (\varepsilon + 1)\tanh^2 X}{\varepsilon - (\varepsilon - 1)\tanh^2 X} dX.$$

After standard replacement $\tanh X = z$ we arrive at the integral

$$\lambda = \frac{2}{\delta}\int \frac{\varepsilon - (\varepsilon + 1)z^2}{\varepsilon - (\varepsilon - 1)z^2}\frac{dz}{1 - z^2} = \frac{4\varepsilon}{\delta(\varepsilon - 1)}\int \frac{1}{\dfrac{\varepsilon}{\varepsilon - 1} - z^2} dz - \frac{2}{\delta}\int \frac{dz}{1 - z^2}. \tag{17}$$

Subcase 1a. $\frac{\varepsilon}{\varepsilon-1}=\eta^2$, and $\eta\neq 0$, because $\varepsilon=\delta^2/\alpha\neq 0$. Then we have

$$\begin{aligned}\lambda &= \frac{4\eta^2}{\delta}\int\frac{dz}{\eta^2-z^2}-\frac{2}{\delta}\int dX=-\frac{2\eta}{\delta}\ln\left|\frac{\tanh X-\eta}{\tanh X+\eta}\right|-\frac{2}{\delta}X+C_1\\ &= -\frac{2\eta}{\delta}\ln\left|\frac{\tanh\frac{\delta v+\gamma}{2}-\eta}{\tanh\frac{\delta v+\gamma}{2}+\eta}\right|-v+C.\end{aligned}\tag{18}$$

Subcase 1b. At $\frac{\varepsilon}{\varepsilon-1}=-\eta^2$ from (17) we obtain

$$\begin{aligned}\lambda &= -\frac{4\eta^2}{\delta}\int\frac{dz}{-\eta^2-z^2}-\frac{2}{\delta}\int dX=\frac{4\eta}{\delta}\arctan\frac{\tanh X}{\eta}-\frac{2}{\delta}X+C_1\\ &= \frac{4\eta}{\delta}\arctan\frac{\tanh\frac{\delta v+\gamma}{2}}{\eta}-v+C.\end{aligned}\tag{19}$$

Case 2. $\beta=-\delta^2\neq 0$. Then it follows from (15) that

$$v=\frac{2}{\delta}\arctan\frac{u}{\delta}-\frac{\gamma}{\delta},$$

which leads to

$$e^t=\frac{\delta^2}{\alpha}\cos^{-2}\left(\frac{\delta v+\gamma}{2}\right).$$

Putting $\frac{\delta v+\gamma}{2}=X$ and $\frac{\delta^2}{\alpha}=\eta$, we have

$$e^t=\eta(1+\tan^2X),\quad \lambda'=\frac{\tan^2X+\frac{\eta-1}{\eta}}{\tan^2X+\frac{\eta+1}{\eta}}.$$

Subcase 2a. $(\eta+1)/\eta=-\kappa^2\neq 0\Rightarrow$

$$\lambda'=\frac{\tan^2X+2+\kappa^2}{\tan^2X-\kappa^2}.$$

We put $\tan X=z$, then

$$\lambda=\int\frac{z^2+2+\kappa^2}{z^2-\kappa^2}dv=\frac{2}{\delta}\int\frac{z^2+2+\kappa^2}{z^2-\kappa^2}dX=\frac{2}{\delta}\int\frac{z^2+2+\kappa^2}{z^2-\kappa^2}\frac{dz}{z^2+1}.$$

However,

$$\frac{z^2+2+\kappa^2}{(z^2-\kappa^2)(z^2+1)}=\frac{2}{z^2-\kappa^2}-\frac{1}{z^2+1},$$

Hence,

$$\begin{aligned}\lambda &= \frac{2}{\delta}\int\left(\frac{2}{z^2-\kappa^2}-\frac{1}{z^2+1}\right)dz\frac{2}{\delta}\left(\frac{1}{\kappa}\ln\left|\frac{z-\kappa}{z+\kappa}\right|-X\right)+C\\ &= \frac{2}{\delta\kappa}\ln\left|\frac{\tan\frac{\delta v+\gamma}{2}-\kappa}{\tan\frac{\delta v+\gamma}{2}+\kappa}\right|-\frac{2}{\delta}\frac{\delta v+\gamma}{2}+C_1=\frac{2}{\delta\kappa}\ln\left|\frac{\tan\frac{\delta v+\gamma}{2}-\kappa}{\tan\frac{\delta v+\gamma}{2}+\kappa}\right|-v+C.\end{aligned}\tag{20}$$

Subcase 2b. $(\eta+1)/\eta = \kappa^2 \neq 0 \Rightarrow$

$$\lambda' = \frac{\tan^2 X + 2 - \kappa^2}{\tan^2 X + \kappa^2} = \frac{z^2 + 2 - \kappa^2}{z^2 + \kappa^2},$$

which results in

$$\lambda = \int \frac{z^2 + 2 - \kappa^2}{z^2 + \kappa^2} dv = \frac{2}{\delta}\int \frac{z^2 + 2 - \kappa^2}{z^2 + \kappa^2} dX = \frac{2}{\delta}\int \frac{z^2 + 2 - \kappa^2}{z^2 + \kappa^2}\frac{dz}{z^2+1}.$$

However,

$$\frac{z^2 + 2 - \kappa^2}{(z^2 + \kappa^2)(z^2+1)} = \frac{2}{z^2 + \kappa^2} - \frac{1}{z^2+1},$$

Therefore,

$$\lambda = \frac{2}{\delta}\int\left(\frac{2}{z^2 + \kappa^2} - \frac{1}{z^2+1}\right)dz = \frac{4}{\delta\kappa}\arctan\frac{z}{\kappa} - \frac{2}{\delta}X + C_1 = \frac{4}{\delta\kappa}\arctan\frac{\tan\frac{\delta v + \gamma}{2}}{\kappa} - v + C. \tag{21}$$

Subcase 2c. $\kappa = 0$. Then

$$\lambda' = \frac{\tan^2 X + 2}{\tan^2 X}$$

and

$$\lambda = \frac{2}{\delta}(-X - 2\cot X) + C_1 = -v - \frac{4}{\delta}\cot\frac{\delta v + \gamma}{2} + C. \tag{22}$$

Case 3. $\beta = 0$. From (15) we find

$$v = -\frac{2}{u} + \gamma = \mp\frac{2}{\sqrt{\alpha e^t}} + \gamma, \quad \gamma = \text{const}.$$

This leads to

$$e^t = \frac{4}{\alpha(v-\gamma)^2}, \quad \lambda' = \frac{e^t - 1}{e^t + 1} = \frac{4 - \alpha(v-\gamma)^2}{4 + \alpha(v-\gamma)^2}.$$

Integration yields

for $\alpha > 0$

$$\lambda = -v + \frac{4}{\sqrt{\alpha}}\arctan\left(\frac{\sqrt{\alpha}}{2}(v-\gamma)\right) + C; \tag{23}$$

for $\alpha < 0$

$$\lambda = -v - \frac{2}{\sqrt{-\alpha}}\ln\left|\frac{\sqrt{-\alpha}(v-\gamma) - 2}{\sqrt{-\alpha}(v-\gamma) + 2}\right| + C. \tag{24}$$

We have proved the following theorem.

Theorem 4. *There are seven types of curvilinear three-webs admitting a one-parameter family of infinitesimal automorphisms for which one of the covariant derivatives of the curvature is equal to zero. These are three-webs defined by equations of the form* (1), *where the function* λ *is calculated by formulas* (18)–(24).

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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