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Spectral Estimates for the Fourth-Order Differential Operator with Periodic Coefficients

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Abstract—This article considers a self-adjoint fourth-order differential operator on the unit interval with real 1-periodic coefficients whose domain is defined by Neumann–Dirichlet boundary conditions. The asymptotics of eigenvalues at high energies is derived for the above operator.

Keywords: spectrum, fourth-order differential operator, asymptotics of eigenvalues, fundamental matrix

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Let us consider a self-adjoint fourth-order operator H acting in the Hilbert space $L^2(0, 1)$ and given by

$$Hy = y^{(4)} + (py')' + qy, \quad y'(0) = y'''(0) + p(0)y'(0) = y(1) = y''(1) = 0, \tag{1}$$

where the coefficients p and q are real 1-periodic functions from the space $L^1(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The domain of this operator is

$$\mathrm{Dom}(H) = \Big\{y \in L^2(0,1) : y', y'', y''' + py' \in W_1^1(0,1),$$
$$y^{(4)} + (py')' + qy \in L^2(0,1), \quad y'(0) = y'''(0) + p(0)y'(0) = y(1) = y''(1) = 0\Big\}.$$

The purpose of this article is to provide an asymptotic description of the eigenvalues of the differential operator H both in the general case and under additional assumptions about the smoothness of coefficients p and q.

Operator H appears in the analysis of the behavior of nanometer thin liquid polymer films. This behavior is described by the following thin film equation [1]:

$$\partial_t y = -\partial_x \left(y^3 \partial_x (\partial_{xx} y - \Pi(y))\right),$$

where Π is the potential function. This equation is considered on a unit interval [0, 1] with boundary conditions

$$\partial_{xxx} y = 0, \quad \partial_x y = 0 \quad \text{in} \quad x = 0, \quad x = 1.$$

The linearization of this equation [1, 2] leads to a spectral problem for operator H given by (1). It should be noted that, from a mechanical point of view, operator H describes the deflections of a beam with a hinged-sliding anchorage.

Many works have been devoted to the study of spectral properties of fourth-order differential operators with different boundary conditions. Isospectral potentials for such operators were described in [3]. The inverse spectral problem for a fourth-order differential operator with Neumann boundary conditions was studied in [4]. The asymptotics of eigenvalues and the trace formula for a self-adjoint fourth-order operator with Dirichlet boundary conditions were derived in [5, 6]. Spectral asymptotics, estimates of spectral projector deviations, and estimates of equiconvergence of spectral decompositions in the non-self-adjoint case were obtained in [7].

A study of the spectrum of a periodic self-adjoint operator H on the axis was carried out in [8]. The non-self-adjoint operator H with periodic boundary conditions was considered in [9]. In it, it was proved that a system of eigenfunctions and associated functions forms a basis in the space $L^p(0, 1)$, $1 < p < \infty$. In addition, the asymptotics of eigenvalues, estimates of spectral projector deviations, and estimates of equiconvergence of spectral decompositions for the non-self-adjoint operator H with periodic conditions were obtained in [10].

Let us now proceed to the description of the spectral properties of operator H. The spectrum $\sigma(H)$ of operator H is purely discrete (see [11], Chapter I, Sections 2 and 4). In order to describe it, we introduce the fundamental solutions φ_j, $j = 1,2,3,4$, of the equation

$$y^{(4)} + (py')' + qy = \lambda y, \quad \lambda \in \mathbb{C}. \tag{2}$$

These fundamental solutions satisfy the conditions $\varphi_j^{(k-1)}(0,\lambda) = \delta_{jk}$, $k = 1,2,3$, $(\varphi_j''' + p\varphi_j')(0,\lambda) = \delta_{j4}$, where δ_{jk} is the Kronecker symbol. Each $\varphi_j(x, \cdot)$, $j = 1,2,3,4$, $x \in [0, 1]$ are entire functions.

Spectrum $\sigma(H)$ consists of real eigenvalues, and there is the following relation

$$\sigma(H) = \{\lambda \in \mathbb{C} : D(\lambda) = 0\}, \tag{3}$$

where D is the entire function given by the following formula:

$$D(\lambda) = -\det\begin{pmatrix} \varphi_1(1,\lambda) & \varphi_3(1,\lambda) \\ \varphi_1''(1,\lambda) & \varphi_3''(1,\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

The spectrum of the operator H is real, semibounded from below, and consists of eigenvalues λ_n, $n \in \mathbb{Z}_+$, which will be numbered (taking into account multiplicity) as follows:

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \dots.$$

In the unperturbed case $p = q = 0$, all eigenvalues λ_n^0 are simple, real, and have the form:

$$\lambda_n^0 = \left(\frac{\pi}{2} + \pi n\right)^4, \quad n \in \mathbb{Z}_+.$$

Our main result will be devoted to the proof of the asymptotics of the eigenvalues of the considered operator at high energies. The spectral asymptotics will be given in terms of the Fourier coefficients of the function p in the case $p, q \in L^1(\mathbb{T})$. In addition, we will derive the asymptotics of the eigenvalues with greater precision (assuming smoother coefficients). This is necessary in order to further derive the regularized trace formula for operator H.

Let us introduce the Fourier coefficients for some function $f \in L^1(\mathbb{T})$ and any $n \in \mathbb{Z}$ have the form:

$$f_0 = \int_0^1 f(x)dx, \quad \hat{f}_n = \int_0^1 f(x)e^{-i\pi(2n+1)x}dx, \quad \hat{f}_{cn} = \int_0^1 f(x)\cos\pi(2n+1)xdx.$$

We are now ready to formulate our main result.

Theorem 1. *Let $p, q \in L^1(\mathbb{T})$ and the number $n \in \mathbb{N}$ be chosen sufficiently large. Then eigenvalues of λ_n are simple and satisfy the following asymptotics*:

$$\lambda_n = \lambda_n^0 + \left(\frac{\pi}{2} + \pi n\right)^2 (\hat{p}_{cn} - p_0) + \mathbb{O}(n), \quad n \to +\infty. \tag{4}$$

If we additionally assume that p''', $q' \in L^1(\mathbb{T})$,

$$\lambda_n = \lambda_n^0 + \left(\frac{\pi}{2} + \pi n\right)^2 (\hat{p}_{cn} - p_0) + \frac{p_0^2 - \|p\|^2}{8} + q_0 + \hat{q}_{cn} + \mathbb{O}(n^{-2}), \quad n \to +\infty. \tag{5}$$

We will now briefly describe a scheme for proving Theorem 1. It will be based on a matrix version of the Birkhoff method developed in [6, 8, 12]. Let $z = \lambda^{1/4}$, $z \in \mathcal{L}_+$, and $\lambda \in \mathbb{C}$, where

$$\mathcal{L}_+ = \left\{ z \in \mathbb{C} : \arg z \in \left(0, \frac{\pi}{4}\right) \right\}.$$

Introduce the parameters $\omega_1 = -\omega_4 = i$, $\omega_2 = -\omega_3 = 1$. Then the following estimates hold:

$$\mathrm{Re}(i\omega_1 z) \leqslant \mathrm{Re}(i\omega_2 z) \leqslant \mathrm{Re}(i\omega_3 z) \leqslant \mathrm{Re}(i\omega_4 z), \quad z \in \mathcal{L}_+. \tag{6}$$

In addition, we define the fundamental matrix $A(x, z)$, $x \in [0, 1]$, $z \in \mathcal{L}_+$, of Eq. (2) as follows:

$$A = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1' & \phi_2' & \phi_3' & \phi_4' \\ \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' \\ \phi_1''' + p\phi_1' & \phi_2''' + p\phi_2' & \phi_3''' + p\phi_3' & \phi_4''' + p\phi_4' \end{pmatrix}, \tag{7}$$

where ϕ_j, $j = 1, 2, 3, 4$, are the fundamental solutions of Eq. (2), which satisfy the following asymptotics:

$$\begin{gathered} \phi_j(x, z) = e^{izx\omega_j}(1 + \mathcal{O}(z^{-1})), \quad \phi_j'(x, z) = iz\omega_j e^{izx\omega_j}(1 + \mathcal{O}(z^{-1})), \\ \phi_j''(x, z) = -z^2\omega_j^2 e^{izx\omega_j}(1 + \mathcal{O}(z^{-1})), \\ \phi_j'''(x, z) + p(x)\phi_j'(x, z) = (-iz^3\omega_j^3 + iz\omega_j p(x))e^{izx\omega_j}(1 + \mathcal{O}(z^{-1})) \end{gathered} \tag{8}$$

for $|z| \to \infty$. The proof of the existence of such solutions can be found in ([11], Chapter II). The matrix-valued function A satisfies the following equation:

$$A' = \mathcal{P}A, \quad \text{where} \quad \mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -p & 0 & 1 \\ \lambda - q & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

Therefore, function A is the solution of Eq. (9) satisfying the following asymptotics

$$A(x, z) = \Omega(z)(\mathbb{I}_4 + \mathcal{O}(z^{-1}))e^{izx\mathcal{T}}, \quad x \in [0, 1], \tag{10}$$

for $|z| \to \infty$, $z \in \mathcal{L}_+$, where

$$\mathcal{T} = \mathrm{diag}(\omega_1, \omega_2, \omega_3, \omega_4) = \mathrm{diag}(i, 1, -1, -i), \tag{11}$$

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -z & iz & -iz & z \\ z^2 & -z^2 & -z^2 & z^2 \\ -z^3 & -iz^3 & iz^3 & z^3 \end{pmatrix}. \tag{12}$$

Here and everywhere below, $\mathbb{I}_4$ denotes the 4×4 identity matrix.

The main idea of the study is an asymptotic analysis of the fundamental matrix A for sufficiently large $|z|$. Such an analysis is standard in the study of the spectral properties of the Schrödinger operator. However, in the case of the fourth-order operator, there appear additional difficulties related to the fact that the fundamental matrix contains both exponentially increasing elements and exponentially decreasing elements for $|z| \to \infty$ in any direction.

Thus, let us investigate the asymptotic behavior of the fundamental matrix. Let $z \in \mathcal{L}_+$ and A be a matrix-valued solution of Eq. (9) given by (7). Let us introduce a matrix-valued function Y such that

$$A(x, z) = \Omega(z)Y(x, z), \quad x \in [0, 1], \quad z \in \mathcal{L}_+. \tag{13}$$

Substituting this expression into (9) and using the identity

$$\Omega^{-1}\mathcal{P}\Omega = iz\mathcal{T} + \frac{P}{z}, \quad P = -\frac{p}{4}\begin{pmatrix} -1 & i & -i & 1 \\ 1 & -i & i & -1 \\ 1 & -i & i & -1 \\ -1 & i & -i & 1 \end{pmatrix} - \frac{q}{4z^2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ i & i & i & i \\ -i & -i & -i & -i \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

we obtain the fact that Y satisfies the following equation:

$$Y' - iz\mathcal{T}Y = \frac{P}{z}Y. \tag{14}$$

It is not difficult to find an inverse operator to the operator on the left-hand side of the last equation (see [12] for details). Consider a matrix-valued function X such that

$$Y(x,z) = X(x,z)e^{izx\mathcal{T}}, \quad x \in [0,1],\ z \in \mathcal{Z}_+. \tag{15}$$

Then X is the solution of the following differential equation:

$$X' + iz(X\mathcal{T} - \mathcal{T}X) = \frac{P}{z}X.$$

This equation (as well as the equivalent equations (9) and (14)) has many solutions. We will choose a solution that satisfies the conditions $X_{jk}(0,z) = 0$, $j < k$, $X_{jk}(1,z) = \delta_{jk}$, $j \ge k$. This choice X will lead us to the sought solution A of Eq. (9) given by (7).

Let $|z|$ be a sufficiently large number. It was shown in [12] that X is the only solution of the following integral equation

$$X = \mathbb{I}_4 + \frac{1}{z}KX, \tag{16}$$

where

$$(KX)_{lj}(x,z) = \int_0^1 K_{lj}(x,s,z)(PX)_{lj}(s,z)ds, \quad l,j = 1,2,3,4,$$

$$K_{lj}(x,s,z) = \begin{cases} e^{iz(x-s)(\omega_l-\omega_j)}\chi(x-s), & l < j; \\ -e^{iz(x-s)(\omega_l-\omega_j)}\chi(s-x), & l \geqslant j, \end{cases} \qquad \chi(s) = \begin{cases} 1, & s \geqslant 0; \\ 0, & s < 0. \end{cases}$$

It follows from estimates (6) that the kernel of the integral operator K satisfies the inequality $|K_{lj}(x,s,z)| \le 1$ for all l, $j = 1,2,3,4$, and $(x,s,z) \in [0,1]\times[0,1]\times\mathcal{Z}_+$. Applying the method of simple iterations to integral equation (16), we obtain the asymptotics

$$X(x,z) = \mathbb{I}_4 + \mathcal{O}(z^{-1}) \tag{17}$$

for $|z| \to \infty$, $z \in \mathcal{Z}_+$, uniformly with respect to $x \in [0, 1]$. The above reasoning shows that Y admits representation (15), where X is the solution of integral equation (16) that satisfies asymptotics (17). Substituting (15) into (13), we obtain (10).

As a result, we established a factorization theorem for the matrix-valued solution of Eq. (9), which plays a key role in proving the asymptotics of the eigenvalues of the operator H.

Theorem 2. *Let* $p, q \in L^1(\mathbb{T})$. *Then there exists matrix-valued solution* A *of* Eq. (9) *such that every function* $A(x, \cdot)$, $x \in [0, 1]$, *is analytic in* $\mathcal{Z}_+$ *for sufficiently large* $|z|$ *and satisfies the following equality*:

$$A(x,z) = \Omega(z)X(x,z)e^{izx\mathcal{T}}, \tag{18}$$

where X *is the solution of integral equation* (16); $\mathcal{T}$ *and* Ω *are given by* (11) *and* (12), *respectively.*

According to formula (18), fundamental matrix A can be represented as a product of the simple matrix Ω, the bounded matrix, and the diagonal matrix. Moreover, the diagonal matrix contains all exponentially increasing terms. Therefore, it becomes convenient to analyze the properties of fundamental matrix A.

Formula (18) for factorization of the fundamental matrix A yields the asymptotics of the characteristic function D. As noted above (see formula (3)), the zeros of the function D are the spectrum of operator H.

Let $(x,z)\in[0,1]\times\mathcal{Z}_+$ and $|z|$ be sufficiently large. It is not difficult to show that

$$D(\lambda)=\frac{\det\phi(z)}{\det A(0,z)}, \tag{19}$$

where

$$\phi(z)=\begin{pmatrix}\phi_1'(0,z) & \phi_2'(0,z) & \phi_3'(0,z) & \phi_4'(0,z)\\ (\phi_1'''+p\phi_1')(0,z) & (\phi_2'''+p\phi_2')(0,z) & (\phi_3'''+p\phi_3')(0,z) & (\phi_4'''+p\phi_4')(0,z)\\ \phi_1(1,z) & \phi_2(1,z) & \phi_3(1,z) & \phi_4(1,z)\\ \phi_1''(1,z) & \phi_2''(1,z) & \phi_3''(1,z) & \phi_4''(1,z)\end{pmatrix}. \tag{20}$$

It follows directly from representation (10) and formula (12) that

$$\det A(0,z)=-16iz^6(1+\mathbb{O}(z^{-1})) \tag{21}$$

for $|z|\to\infty$, $z\in\mathcal{Z}_+$. This asymptotics and equality (19) show that the large positive zeros of the function D coincide with those of $\det\phi(z)$. Therefore, it is necessary to transform the determinant of the matrix ϕ given by (20). Expanding the determinant, we obtain the following asymptotics:

$$\xi(z)=\xi_1(z)\xi_2(z)+\xi_3(z)\xi_4(z)+\mathbb{O}(z^4) \tag{22}$$

in sector $\mathcal{Z}_+$, where

$$\xi_1(z)=\det\begin{pmatrix}\phi_3(1,z) & \phi_4(1,z)\\ \phi_3''(1,z) & \phi_4''(1,z)\end{pmatrix},\quad \xi_2(z)=\det\begin{pmatrix}\phi_1'(0,z) & \phi_2'(0,z)\\ (\phi_1'''+p\phi_1')(0,z) & (\phi_2'''+p\phi_2')(0,z)\end{pmatrix},$$

$$\xi_3(z)=\det\begin{pmatrix}\phi_2(1,z) & \phi_4(1,z)\\ \phi_2''(1,z) & \phi_4''(1,z)\end{pmatrix},\quad \xi_4(z)=\det\begin{pmatrix}\phi_3'(0,z) & \phi_1'(0,z)\\ (\phi_3'''+p\phi_3')(0,z) & (\phi_1'''+p\phi_1')(0,z)\end{pmatrix}.$$

Substituting asymptotics (21) and (22) into (19), we obtain

$$D(\lambda)=\frac{i}{16z^6}(\xi_1(z)\xi_2(z)+\xi_3(z)\xi_4(z)+\mathbb{O}(z^4)) \tag{23}$$

for $|z|\to\infty$. The last asymptotics shows that the asymptotic analysis of zeros of the 4×4 determinant $\det\phi(z)$ is reduced to the analysis for zeros of the sum of products of 2×2 determinants.

Note that asymptotics (8) is sufficiently rough and will not give the sharp asymptotics of zeros of function D. However, by applying the Rouché's theorem, we calculate the number of zeros in a circle of large radius as well as their localization. By taking the following iteration terms as an approximation of the solution of integral equation (16), we improve the asymptotics of fundamental matrix A and, hence, asymptotics ϕ_j, $j=1,2,3,4$. Substituting this sharp asymptotic ϕ_j into (23), we obtain the spectral asymptotics (4) and (5) of the zeros of function D.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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