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The Nonlocal Problem for a Hyperbolic Equation with a Parabolic Degeneracy

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Abstract—In this paper, the nonlocal problem for a second-order partial differential equation is considered in the characteristic domain. The considered expression represents an equation of two independent variables x and y. It is a hyperbolic-type equation in the half-plane $y > 0$ with a parabolic degeneracy at $y = 0$. The line of the parabolic degeneracy $y = 0$ represents the cusp locus of characteristic curves. The novelty of the formulation of the problem consists in the fact that the boundary condition contains a linear combination of operators D_{0x}^{α} and D_{x1}^{α}. For $\alpha > 0$, these operators are fractional differentiation operators of order α, while for $\alpha < 0$ they coincide with the Riemann–Liouville fractional integration operator of order α. For various orders of the operators included in the boundary condition, the unique solvability of the formulated problem is proven. The properties of the operators of fractional integro-differentiation and the properties of the Gaussian hypergeometric function are widely used in the proof. The solution of the problem is given in the explicit form.

Keywords: boundary value problem, operator of fractional integration, operator of fractional differentiation, Euler–Darboux equation, hypergeometric function

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FORMULATION OF THE PROBLEM

Consider the equation

$$y^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+a\frac{\partial u}{\partial x}=0, \tag{1}$$

where a is a real constant in finite domain Ω bounded by characteristics

$$AC : x-\frac{y^2}{2}=0, \quad BC : x+\frac{y^2}{2}=1$$

of Eq. (1) and segment $I \equiv [0, 1]$ of straight line $y = 0$, $A(0, 0)$, $B(1, 0)$.

Problem. *In the domain* Ω, *it is necessary to find solution* $u(x, y)$ (*of Eq.* (1) *from class* $u(x, y) \in C(\bar{\Omega}) \cap C^2(\Omega)$) *satisfying conditions*

$$u(x,0)=\tau(x) \quad \forall x \in \bar{I}, \tag{2}$$

$$A(x)D_{0x}^{b_1}u[\Theta_0(x)]+B(x)D_{x1}^{b_2}u[\Theta_1(x)]=E(x) \quad \forall x \in I, \tag{3}$$

where $\tau(x)$, $A(x)$, $B(x)$, *and* $E(x)$ *are the given functions*; *in this case*,

$$A(x)\cdot B(x)\cdot 0, \quad A^2(x)+B^2(x)\neq 0 \quad \forall x \in \bar{I}, \tag{4}$$

while $\Theta_0(x)$ *and* $\Theta_1(x)$ *are the points of intersection of the characteristics of Eq.* (1) *coming from point* $(x, 0) \in I$, *with characteristics AC and BC*, *respectively*. *Here*, D_{0x}^{l} *and* D_{x1}^{l} *are fractional integration and fractional differentiation operators* [1, p. 9] *having the following form*:

$$D_{0x}^{l}f(x)=\begin{cases}\dfrac{1}{\Gamma(-l)}\displaystyle\int_0^x\dfrac{f(t)dt}{(x-t)^{1+l}}, & -1<l<0\\ \dfrac{d}{dx}D_{0x}^{l-1}f(x), & 0<l<1,\end{cases} \tag{5}$$

$$D_{x1}^{l}f(x)=\begin{cases}\dfrac{1}{\Gamma(-l)}\displaystyle\int_x^1\dfrac{f(t)dt}{(t-x)^{1+l}}, & -1<l<0\\ -\dfrac{d}{dx}D_{x1}^{l-1}f(x), & 0<l<1,\end{cases} \tag{6}$$

where b_1 and b_2 are real numbers, on which the necessary conditions will be further imposed.

In this case, (1) is the equation of two independent variables x and y, it is hyperbolic everywhere outside straight line $y = 0$, and straight line $y = 0$ is the line of parabolic degeneracy. Problem (1)–(3) is nonlocal and its study is associated with the applied nature of the problems that can have a meaningful biological interpretation, e.g., in the theory of microbial population [2].

Note that for Eq. (1), the study of the Cauchy problem when requiring an increase in the smoothness of the initial data is available in [3], and in one particular case this problem is investigated in [4].

For Eq. (1), in [5–8] we investigate nonlocal boundary value problems, the distinguishing feature of which lies in the fact that the boundary conditions contain various linear combinations of generalized operators of fractional integro-differentiation with the hypergeometric Gaussian function [9].

Other scientists also study Eq. (1). For example, Nakhusheva [10] considers some constructive properties of all solutions of Eq. (1), which make it possible to answer the question of the correct formulation of local and nonlocal boundary value problems for this equation. In [11] a mixed problem for the equation of the hyperbolic–parabolic type on the plane is considered; here, it is Eq. (1) that is given in the hyperbolic part of the domain. The unambiguous solvability of the problem is proven.

CONDITIONS OF UNAMBIGUOUS SOLVABILITY OF THE PROBLEM

Theorem. *Suppose* $|a| < 1$, $\alpha = \dfrac{1-a}{4}$, $\beta = \dfrac{1+a}{4}$, $b_1 = 1-\alpha$, $b_2 = 1-\beta$, $\tau \in C(\bar{I}) \cap C^2(I)$, $A(x) \in C(\bar{I})$, $B(x) \in C(\bar{I})$, $E(x) \in C(\bar{I})$, *and*

$$A(x)\frac{\Gamma(1+\alpha+\beta)\Gamma(1-\beta)}{\Gamma(2\alpha+\beta)\Gamma(\alpha+2\beta)}x^{-\alpha}+B(x)\frac{\Gamma(1+\alpha+\beta)\Gamma(1-\alpha)}{\Gamma(2\alpha+\beta)\Gamma(\alpha+2\beta)}(1-x)^{-\beta}\neq 0. \tag{7}$$

Then, the solution of problem (1)–(3) *exists and it is unique.*

Proof. In characteristic variables

$$\xi = x+\frac{y^2}{2},\quad \eta = x-\frac{y^2}{2},$$

Eq. (1) becomes Euler–Darboux equation

$$\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{a-1}{4(\xi-\eta)}\frac{\partial u}{\partial\xi}+\frac{a+1}{4(\xi-\eta)}\frac{\partial u}{\partial\eta}=0,$$

whose elementary solution is found by Darboux [12]. We find a solution that satisfies conditions (2) and (3).

We introduce notation

$$\alpha=\frac{1-a}{4},\quad \beta=\frac{1+a}{4} \tag{8}$$

and obtain for a, conditions $-1 < a < 1$.

When $|a| < 1$ solution $u(x, y)$ of Eq. (1) in domain Ω satisfying conditions

$$\lim_{y\to+0}u(x,y)=\tau(x),\quad \lim_{y\to+0}\frac{\partial u(x,y)}{\partial y}=\nu(x)$$

has form [13, p. 267]

$$u(x,y) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-a}{4}\right)\Gamma\left(\frac{1+a}{4}\right)}\int_0^1 \tau\left[x + \frac{y^2}{2}(1-2t)\right](1-t)^{\frac{a-3}{4}} t^{-\frac{a+3}{4}} dt$$
$$+ y\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3-a}{4}\right)\Gamma\left(\frac{3+a}{4}\right)}\int_0^1 \nu\left[x + \frac{y^2}{2}(1-2t)\right](1-t)^{\frac{a-1}{4}} t^{-\frac{a+1}{4}} dt. \quad (9)$$

With allowance for notation (8), solution (9) takes form

$$u(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \tau\left[x + \frac{y^2}{2}(1-2t)\right](1-t)^{-2\alpha-\beta}t^{-\alpha-2\beta}dt$$
$$+ y\frac{\Gamma(1+\alpha+\beta)}{\Gamma(2\alpha+\beta)\Gamma(\alpha+2\beta)}\int_0^1 \nu\left[x + \frac{y^2}{2}(1-2t)\right](1-t)^{-\alpha}t^{-\beta}dt. \quad (10)$$

Using formula (10), we find

$$u[\Theta_0(x)] = k\Gamma(1-\alpha-2\beta)x^{\alpha+\beta}D_{0x}^{\alpha+2\beta-1}x^{-2\alpha-\beta}\tau(x) + m\Gamma(1-\beta)D_{0x}^{\beta-1}x^{-\alpha}\nu(x), \quad (11)$$

$$u[\Theta_1(x)] = k\Gamma(1-2\alpha-\beta)(1-x)^{\alpha+\beta}D_{x1}^{2\alpha+\beta-1}(1-x)^{-\alpha-2\beta}\tau(x)$$
$$+ m\Gamma(1-\alpha)D_{x1}^{\alpha-1}(1-x)^{-\beta}\nu(x), \quad (12)$$

where

$$k = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \quad m = \frac{\Gamma(1+\alpha+\beta)}{\Gamma(2\alpha+\beta)\Gamma(\alpha+2\beta)}.$$

Substituting (11) and (12) into boundary condition (3), according to the conditions of theorem $b_1 = 1-\alpha$ and $b_2 = 1-\beta$ with allowance for the properties of fractional integro-differentiation operators [14, pp. 50, 51]

$$D_{0x}^{l}D_{0x}^{-l}f(x) = D_{x1}^{l}D_{x1}^{-l}f(x) = f(x),$$

we obtain

$$A(x)m\Gamma(1-\beta)x^{-\alpha}\nu(x) + B(x)m\Gamma(1-\alpha)(1-x)^{-\beta}\nu(x) = \gamma(x), \quad (13)$$

where

$$\gamma(x) = E(x) - A(x)k\Gamma(1-\alpha-2\beta)D_{0x}^{1-\beta}x^{\alpha+\beta}D_{0x}^{\alpha+2\beta-1}x^{-2\alpha-\beta}\tau(x)$$
$$- B(x)k\Gamma(1-2\alpha-\beta)D_{x1}^{1-\alpha}(1-x)^{\alpha+\beta}D_{x1}^{2\alpha+\beta-1}(1-x)^{-\alpha-2\beta}\tau(x).$$

Consider right-hand side $\gamma(x)$ of Eq. (13). We show that

$$\tau_1(x) = D_{0x}^{1-\beta}x^{\alpha+\beta}D_{0x}^{\alpha+2\beta-1}x^{-2\alpha-\beta}\tau(x) = x^{-\alpha}D_{0x}^{\alpha+\beta}\tau(x) = x^{-\alpha}D_{0x}^{\frac{1}{2}}\tau(x),$$
$$\tau_1 \in C(\bar{I}) \cap C^2(I), \quad (14)$$

$$\tau_2(x) = D_{x1}^{1-\alpha}(1-x)^{\alpha+\beta}D_{x1}^{2\alpha+\beta-1}(1-x)^{-\alpha-2\beta}\tau(x)$$
$$= (1-x)^{-\beta}D_{x1}^{\alpha+\beta}\tau(x) = (1-x)^{-\beta}D_{x1}^{\frac{1}{2}}\tau(x), \quad (15)$$
$$\tau_2 \in C(\bar{I}) \cap C^2(I).$$

Using the formula of fractional integration operators and fractional differentiation operators (5), we have

$$\tau_1(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{d}{dx}\int_0^x \frac{t^{\alpha+\beta}dt}{(x-t)^{1-\beta}}\int_0^t \frac{\xi^{-2\alpha-\beta}\tau(\xi)d\xi}{(t-\xi)^{1-\alpha}}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{d}{dx} \int_0^x \xi^{-2\alpha-\beta} \tau(\xi) d\xi \int_\xi^x \frac{t^{\alpha+\beta} dt}{(x-t)^{1-\beta}(t-\xi)^{1-\alpha}}.$$

Making the change of variable $t = x - (x - \xi)z$, we have

$$\tau_1(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{d}{dx} \int_0^x \frac{\tau(\xi) d\xi}{\xi^\alpha (x-\xi)^{1-\alpha-\beta}} \int_0^1 z^{\beta-1}(1-z)^{\alpha-1} \left(1 - \frac{x-\xi}{x} z\right)^{\alpha+\beta} dz. \tag{16}$$

Using the formula for integral representation of the hypergeometric Gaussian function [15, p. 72]

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,$$

$$0 < \operatorname{Re} b < \operatorname{Re} c, \quad |\arg(1-z)| < \pi,$$

we obtain the fact that relation (16) can be presented as follows:

$$\tau_1(x) = \frac{1}{\Gamma(\alpha+\beta)} \frac{d}{dx} \int_0^x \xi^{-\alpha}(x-\xi)^{\alpha+\beta-1} F\left(-\alpha-\beta, \beta, \alpha+\beta; \frac{\xi - x}{\xi}\right) \tau(\xi) d\xi.$$

Due to formula [16, p. 317]

$$F(\alpha,\beta,\gamma,z) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{z}{z-1}\right),$$

with allowance for $\alpha + \beta - 1 = -\alpha - \beta$, we get

$$\tau_1(x) = \frac{1}{\Gamma(\alpha+\beta)} \frac{d}{dx} \int_0^x \xi^{-2\alpha-\beta} \left(\frac{x-\xi}{x}\right)^{-\alpha-\beta} F\left(-\alpha-\beta, \alpha, \alpha+\beta; \frac{x-\xi}{x}\right) \tau(\xi) d\xi.$$

We introduce function

$$\tau_\delta(x) = \frac{1}{\Gamma(\alpha+\beta)} \frac{d}{dx} \int_0^{x-\delta} \xi^{-2\alpha-\beta} \left(\frac{x-\xi}{x}\right)^{-\alpha-\beta} F\left(-\alpha-\beta, \alpha, \alpha+\beta; \frac{x-\xi}{x}\right) \tau(\xi) d\xi$$

and show that

$$\tau_1(x) = \lim_{\delta \to 0} \tau_\delta(x) = \frac{x^{-\alpha}}{\Gamma(\alpha+\beta)} \frac{d}{dx} \int_0^x \frac{\tau(\xi) d\xi}{(x-\xi)^{\alpha+\beta}}. \tag{17}$$

Using the following relations for hypergeometric function [15, pp. 70, 71, 76]:

$$\frac{d}{dz} z^\alpha F(\alpha,\beta,\gamma,z) = \alpha \cdot z^{\alpha-1} F(\alpha+1,\beta,\gamma,z),$$

$$F(\alpha,\beta,\gamma,z) = F(\beta,\alpha,\gamma,z),$$

$$F(\alpha,\beta,\beta,z) = (1-z)^{-\alpha}, \quad |\arg(1-z)| < \pi,$$

we have

$$\tau_\delta(x) = \frac{(x-\delta)^{-2\alpha-\beta}}{\Gamma(\alpha+\beta)} \left(\frac{\delta}{x}\right)^{-\alpha-\beta} F\left(-\alpha-\beta, \alpha, \alpha+\beta; \frac{\delta}{x}\right) \tau(x-\delta) - \frac{(\alpha+\beta) x^{-\beta}}{\Gamma(\alpha+\beta)} \int_0^{x-\delta} \frac{\xi^{-\alpha+\beta} \tau(\xi) d\xi}{(x-\xi)^{\alpha+\beta+1}}.$$

Taking into account that

$$-\frac{(\alpha+\beta) x^{-\beta}}{\Gamma(\alpha+\beta)} \int_0^{x-\delta} \frac{\xi^{-\alpha+\beta} \tau(\xi) d\xi}{(x-\xi)^{\alpha+\beta+1}} = \frac{x^{-\alpha}}{\Gamma(\alpha+\beta)} \frac{d}{dx} \int_0^{x-\delta} \frac{\tau(\xi) d\xi}{(x-\xi)^{\alpha+\beta}} - \frac{x^{-\alpha}}{\Gamma(\alpha+\beta)} \delta^{-\alpha-\beta} \tau(x-\delta),$$

we obtain

$$\tau_\delta(x) = \frac{x^{-\alpha}}{\Gamma(\alpha+\beta)}\frac{d}{dx}\int_0^{x-\delta}\frac{\tau(\xi)d\xi}{(x-\xi)^{\alpha+\beta}}$$
$$+\frac{x^{-\alpha}}{\Gamma(\alpha+\beta)}\delta^{-\alpha-\beta}\tau(x-\delta)\left[\left(\frac{x}{x-\delta}\right)^{2\alpha+\beta}F\left(-\alpha-\beta,\alpha,\alpha+\beta;\frac{\delta}{x}\right)-1\right]. \quad (18)$$

Since $F\left(-\alpha-\beta,\alpha,\alpha+\beta;\frac{\delta}{x}\right) = 1 + 0\left(\frac{\delta}{x}\right)$, passing in (18) to the limit for $\delta \to 0$ and considering that $\alpha+\beta = \frac{1}{2}$, we check the validity of equality (17) or, equivalently, of equality (14).

By analogous reasoning, the validity of (15) is checked.

Using (14) and (15), we present right-hand side $\gamma(x)$ of Eq. (13) as follows:

$$\gamma(x) = E(x) - A(x)k\Gamma(1-\alpha-2\beta)x^{-\alpha}D_{0x}^{\frac{1}{2}}\tau(x) - B(x)k\Gamma(1-2\alpha-\beta)(1-x)^{-\beta}D_{x1}^{\frac{1}{2}}\tau(x). \quad (19)$$

Under condition

$$A(x)m\Gamma(1-\beta)x^{-\alpha} + B(x)m\Gamma(1-\alpha)(1-x)^{-\beta} \neq 0;$$

i.e., when condition (7) from (13) is met, we immediately find

$$\nu(x) = \frac{\gamma(x)}{A(x)m\Gamma(1-\beta)x^{-\alpha} + B(x)m\Gamma(1-\alpha)(1-x)^{-\beta}}, \quad \nu(x) \in C(\bar{I}) \cap C^2(I),$$

where $\gamma(x)$ has form (19), and solution $u(x, y)$ is presented by formula (9). The existence is proven.

Due to the extremum principle for hyperbolic equations [17], the positive maximum (negative minimum) of function $u(x, y)$ is attained in the closed domain $\bar{\Omega}$ at point $(x, 0) \in \bar{I}$. Taking advantage of the fact that fractional derivatives $D_{0x}^{\frac{1}{2}}\tau(x)$ and $D_{x1}^{\frac{1}{2}}\tau(x)$ at the point of the positive maximum are strictly positive (strictly negative at the point of negative minimum) [18, p. 123], when condition (4) is met, we obtain $\nu(x) > 0$. The latter contradicts the Zaremba–Giraud principle. From the extremum principle it follows that the formulated problem cannot have more than one solution. The uniqueness of the solution is proven.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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