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The Total Approximation Method for the Dirichlet Problem for Multidimensional Sobolev-Type Equations

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Abstract—We study the Dirichlet problem for a multidimensional Sobolev-type differential equation with variable coefficients. The considered equation is reduced to a parabolic integrodifferential equation with a small parameter. To solve the obtained problem approximately, we construct a locally one-dimensional difference scheme. Using the method of energy inequalities, we obtain an a priori estimate of the solution of the locally one-dimensional difference scheme, which implies its stability and convergence. For a two-dimensional problem, an algorithm for the numerical solution of the posed problem is constructed and numerical experiments are carried out on test examples. This illustrates the theoretical results obtained in this work.

Keywords: boundary value problems, a priori estimate, multidimensional Sobolev-type equation, Dirichlet problem, locally one-dimensional scheme, stability, convergence

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INTRODUCTION

High-order differential equations with the first time-derivative of the kind

$$\frac{\partial}{\partial t}(A(u)) + B(u) = 0,$$

where $A(u)$ and $B(u)$ are elliptic operators, are called Sobolev-type equations (see [1]).

Various boundary value problems for Sobolev-type equations arise in the investigation of many natural processes and phenomena, e.g., in the modeling of the fluid flow in fractured-porous media (see [2, 3]), two-phase flow in porous media with a dynamical capillary pressure (see [4]), moisture flow (see [5, 6]), motions of undersurface free-boundary water in multilayer media (see [7, 8]), heat-conductivity in two-temperature systems (see [9]), and flow of non-Newtonian fluids (see [10]).

The present paper is devoted to the constructing of a locally one-dimensional (economic) difference scheme for an approximate solution of the Dirichlet problem for Sobolev-type partial differential equations in the multidimensional case. The main idea is to reduce the transition from one layer to another to the sequential solving of a number of one-dimensional problems with respect to each coordinate direction. For each intermediate problem, we construct an unconditionally stable scheme such that the number of operations required to solve it is proportional to the number of mesh knots at each time layer. The main difficulty is caused by the necessity to split not only the principal operator of the problem, but the operator at the time-derivative as well. Therefore, the considered multidimensional differential equation is reduced to a parabolic integrodifferential equation with a small parameter. To solve the obtained problem approximately, we construct a locally one-dimensional scheme. An a priori estimate for solutions of the locally one-dimensional difference scheme is obtained by means of the method of energy inequalities. This implies its stability and convergence. For the two-dimensional problem, we construct an algorithm to solve it approximately and conduct numerical experiments on test examples. These experiments illustrate theoretical results obtained in this paper.

Papers [11–14] are devoted to the constructing of locally one-dimensional schemes for the numerical solving of various boundary value problems for second-order partial differential equations.

In [15–20], various boundary value problems for Sobolev-type equations are investigated in the one-dimensional case: in [15–18], the order of the time-derivative is integer; in [19, 20], it is fractional.

1. PROBLEM SETTING AND A PRIORI ESTIMATE: DIFFERENTIAL FORM

In the closed domain $\bar{Q}_T = \bar{G} \times [0 \le t \le 1]$ such that its base is a p-dimensional square $\bar{G} = \{x = (x_1, x_2, \ldots, x_{\alpha-1}, x_\alpha, x_{\alpha+1}, \ldots, x_p) : 0 \le x_\alpha \le 1, \alpha = 1, 2, \ldots, p\}$ with boundary Γ, $\bar{G} = G \cup \Gamma$, consider the problem

$$\frac{\partial u}{\partial t} = Lu + \mu \frac{\partial}{\partial t} Lu + f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$u|_\Gamma = 0, \quad 0 \le t \le T, \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in \bar{G}, \tag{3}$$

where

$$Lu = \sum_{\alpha=1}^{p} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) - q_\alpha(x,t)u(x,t),$$

$$0 < c_0 \le k_\alpha(x,t), \quad q_\alpha(x,t) \le c_1, \quad k_\alpha(x,t) \in C^{3,1}(\bar{Q}_T), \quad q_\alpha(x,t), f(x,t) \in C^{2,1}(\bar{Q}_T),$$

c_0, c_1, and c_2 are positive constants, $\alpha = 1, 2, \ldots, p$, and $\mu = \text{const} > 0$.

In the sequel, it is assumed that coefficients of Eq. (1) satisfy the corresponding assumptions guaranteeing the desired smoothness of the solution $u(x, t)$ in the cylinder $\bar{Q}_T$.

Transform Eq. (1). Multiplying both parts by $\frac{1}{\mu} e^{\frac{1}{\mu}t}$ and integrating the obtained expression from τ from 0 to t, we obtain that

$$Lu + \frac{1}{\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} u(x,\tau)d\tau - \frac{1}{\mu}u + \tilde{f}(x,t) = 0, \tag{4}$$

where $\tilde{f}(x, t) = \frac{1}{\mu}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} f(x, \tau)d\tau - e^{-\frac{1}{\mu}t}\left(Lu_0(x) - \frac{1}{\mu}u_0(x)\right)$.

In the same domain, instead of Eq. (4), consider the following equation with the small parameter ε:

$$\varepsilon u_t^\varepsilon = Lu^\varepsilon + \frac{1}{\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} u^\varepsilon d\tau - \frac{1}{\mu}u^\varepsilon + \tilde{f}(x,t), \quad (x,t) \in Q_T, \tag{5}$$

where $\varepsilon = \text{const} > 0$.

Since the initial-value conditions for Eqs. (4) and (5) coincide for $t = 0$, it follows that no singularities of boundary-layer type arise for the derivative u_t^ε in a neighborhood of the point $t = 0$ (see [21, 22]).

Let us show that there exists a norm such that $u^\varepsilon \to u$ as $\varepsilon \to 0$. Introduce the notation $\tilde{z} = u^\varepsilon - u$ and substitute $u^\varepsilon = \tilde{z} + u$ in Eq. (5). We obtain the problem

$$\varepsilon \tilde{z}_t = L\tilde{z} + \frac{1}{\mu^2}\int_0^1 e^{-\frac{1}{\mu}(t-\tau)} \tilde{z} d\tau - \frac{1}{\mu}\tilde{z} + \bar{f}(x,t), \quad (x,t) \in Q_T, \tag{6}$$

$$\tilde{z}|_\Gamma = 0, \quad 0 \le t \le T, \tag{7}$$

$$\tilde{z}(x,0) = 0, \quad z \in \bar{G}, \quad \bar{G} = G + \Gamma, \tag{8}$$

where $\bar{f}(x, t) = -\varepsilon\frac{\partial u}{\partial t}$.

To obtain an a priori estimate, use the method of energy inequalities.

Scalarly multiply Eq. (6) by $\tilde{z}$ and obtain the energy identity

$$\left(\varepsilon\frac{\partial\tilde{z}}{\partial t},\tilde{z}\right)=\left(\sum_{\alpha=1}^{p}\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial\tilde{z}}{\partial x_\alpha}\right),\tilde{z}\right)-\left(\sum_{\alpha=1}^{p}q_\alpha(x,t)\tilde{z},\tilde{z}\right)$$
$$+\left(\frac{1}{\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)}\tilde{z}d\tau,\tilde{z}\right)-\left(\frac{1}{\mu}\tilde{z},\tilde{z}\right)+(\overline{f}(x,t),\tilde{z}). \tag{9}$$

Use the scalar product and norm

$$(u,v)=\int_G uv\,dx,\quad \|u\|_0^2=\int_G u^2dx,\quad \|u\|_{L_2(0,1)}^2=\int_0^1 u^2(x,t)dx_\alpha.$$

By virtue of the Cauchy ε-inequality and Cauchy–Bunyakovsky inequality (see [23, p. 142]), simple transformations of (9) yield the inequality

$$\frac{\varepsilon}{2}\frac{\partial}{\partial t}\|\tilde{z}\|_0^2+c_0\|\tilde{z}_x\|_0^2+\left(c_0+\frac{1}{\mu}-\frac{1}{4\mu^2}-\varepsilon_1\right)\|\tilde{z}\|_0^2+\frac{e^{-\frac{2}{\mu}t}}{2\mu}\int_0^t\|\tilde{z}\|_0^2d\tau\le\frac{1}{2\mu}\int_0^t\|\tilde{z}\|_0^2d\tau+\frac{1}{4\varepsilon_1}\|\overline{f}\|_0^2. \tag{10}$$

Assign $\varepsilon_1=\frac{c_0}{2}$. Then inequality (10) implies that

$$\frac{\varepsilon}{2}\frac{\partial}{\partial t}\|\tilde{z}\|_0^2+c_0\|\tilde{z}_x\|_0^2+\left(\frac{c_0}{2}+\frac{1}{\mu}-\frac{1}{4\mu^2}\right)\|\tilde{z}\|_0^2\le\frac{1}{2\mu}\int_0^t\|\tilde{z}\|_0^2d\tau+\frac{1}{2c_0}\|\overline{f}\|_0^2. \tag{11}$$

Integrate (11) with respect to ξ from 0 to t. This yields inequality

$$\frac{\varepsilon}{2}\|\tilde{z}\|_0^2+c_0\int_0^t\|\tilde{z}_x\|_0^2d\xi+\left(\frac{c_0}{2}+\frac{1}{\mu}-\frac{1}{4\mu^2}\right)\int_0^t\|\tilde{z}\|_0^2d\xi\le\frac{1}{2\mu}\int_0^t d\xi\int_0^\xi\|\tilde{z}\|_0^2d\tau+\frac{1}{2c_0}\int_0^t\|\overline{f}\|_0^2d\xi. \tag{12}$$

In (12), estimate the first term of the right-hand part:

$$\int_0^t d\xi\int_0^\xi\|\tilde{z}\|_0^2d\tau\le\int_0^t\|\tilde{z}\|_0^2d\xi,$$

for $\mu\ge\frac{1}{2}$, we obtain

$$\|\tilde{z}\|_1^2\le M\int_0^t\|\overline{f}\|_0^2d\xi=\varepsilon^2M\int_0^t\|u_\xi\|_0^2d\xi=O(\varepsilon^2), \tag{13}$$

where M depends only on the input data of problems (6)–(8).

From a priori estimate (13), it follows that u^ε tend to u as $\varepsilon\to 0$ in the norm $\|\tilde{z}\|_1^2=\varepsilon\|\tilde{z}\|_0^2+\|\tilde{z}\|_{2,Q_t}^2+\|\tilde{z}_x\|_{2,Q_t}^2$, where $\|\tilde{z}_x\|_{2,Q_t}^2=\int_0^t\|\tilde{z}_x\|_0^2d\tau$. Therefore, for small values of ε, the solution of problem (5), (2), (3) is treated as an approximate solution of the Dirichlet problem (1)–(3) for a multidimensional Sobolev-type differential equation with variable coefficients.

2. LOCALLY ONE-DIMENSIONAL SCHEME

On the segment [0, T], introduce the uniform mesh $\overline{\omega}_\tau=\{t_j=j\tau,\ j=0,1,\dots,j_0\}$ with step $\tau=T/j_0$. Each interval (t_j,t_{j+1}) decompose into p parts by points $t_{j+\frac{\alpha}{p}}=t_j+\tau\frac{\alpha}{p}$, $\alpha=1,2,\dots,p$ and denote by $\Delta_\alpha=(t_{j+\frac{\alpha-1}{p}},t_{j+\frac{\alpha}{p}}]$.

Select a spatial mesh uniform with respect to each direction Ox_α with step $h_\alpha = \frac{1}{N_\alpha}$, $\alpha = 1, 2, \ldots, p$, $\omega_{h_\alpha} = \{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha : i_\alpha = 1, \ldots, N_\alpha - 1, \alpha = 1, 2, \ldots, p\}$.

Represent Eq. (1) in the form

$$\Re u = \varepsilon \frac{\partial u}{\partial t} - Lu - \frac{1}{\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} u d\tau + \frac{1}{\mu} u - \tilde{f} = 0 \tag{14}$$

or, which is the same,

$$\sum_{\alpha=1}^{p} \Re_\alpha u = 0, \quad \Re_\alpha u = \frac{\varepsilon}{p}\frac{\partial u}{\partial t} - L_\alpha u - \frac{1}{p\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} u d\tau + \frac{1}{p\mu} u - \tilde{f}_\alpha, \quad \sum_{\alpha=1}^{p} \tilde{f}_\alpha = \tilde{f}.$$

On each semi-interval Δ_α, $\alpha = 1, 2, \ldots, p$, sequentially solve problems

$$\Re_\alpha \vartheta_{(\alpha)} = \frac{\varepsilon}{p}\frac{\partial \vartheta_{(\alpha)}}{\partial t} - L_\alpha \vartheta_{(\alpha)} - \frac{1}{p\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} \vartheta_{(\alpha)} d\tau + \frac{1}{p\mu}\vartheta_{(\alpha)} - \tilde{f}_{(\alpha)} = 0, \quad x \in G, \quad t \in \Delta_\alpha, \tag{15}$$

$$\vartheta_{(\alpha)} = 0, \quad x_\alpha = 0, \quad \vartheta_{(\alpha)} = 0, \quad x_\alpha = 1,$$

assuming that

$$\vartheta_{(1)}(x,0) = u_0(x), \quad \vartheta_{(1)}(x,t_j) = \vartheta_{(p)}(x,t_j), \quad j = 1,2,\ldots,$$

$$\vartheta_{(\alpha)}\left(x, t_{j+\frac{\alpha-1}{p}}\right) = \vartheta_{(\alpha)}\left(x, t_{j+\frac{\alpha-1}{p}}\right), \quad \alpha = 2,3,\ldots,p.$$

(see [24, p. 522]).

Approximating each Eq. (15) of number α by a two-layer scheme on the semi-interval Δ_α, we obtain the following chain of p-one-dimensional difference equations

$$\varepsilon \frac{y^{j+\frac{\alpha}{p}} - y^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha y^{j+\frac{\alpha}{p}} + \frac{1}{p\mu^2}\sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} y(x, t^{j'+\frac{\alpha}{p}})\tau - \frac{1}{p\mu} y(x, t^{j+\frac{\alpha}{p}}) + \varphi_\alpha^{j+\frac{\alpha}{p}}, \tag{16}$$

$$y^{j+\frac{\alpha}{p}}|_{\gamma_{h,\alpha}} = 0, \quad y(x,0) = u_0(x), \tag{17}$$

where

$$\Lambda_\alpha y^{j+\frac{\alpha}{p}} = (a_\alpha u_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}})_{x_\alpha} - d_\alpha y(x, t^{j+\frac{\alpha}{p}}),$$

$$a_\alpha = k_\alpha(x^{-(0.5\alpha)}, \bar{t}), \quad x^{(-0.5\alpha)} = (x_1, \ldots, x_{\alpha-1}, x_\alpha - 0.5h_\alpha, x_{\alpha+1}, \ldots, x_p), \quad d_\alpha = q_\alpha(x, \bar{t}),$$

$\bar{t} = t^{j+1/2}$, and $\gamma_{h,\alpha}$ is the set of knots boundary with respect to the direction x_α.

3. APPROXIMATION ERRORS OF LOCALLY ONE-DIMENSIONAL SCHEMES

The accuracy of a solution of the locally one-dimensional scheme is the difference $z^{j+\frac{\alpha}{p}} = y^{j+\frac{\alpha}{p}} - u^{j+\frac{\alpha}{p}}$, where $u^{j+\frac{\alpha}{p}}$ is the solution of problem (5), (2), (3). Substituting $y^{j+\frac{\alpha}{p}} = z^{j+\frac{\alpha}{p}} + u^{j+\frac{\alpha}{p}}$ in the difference problem (16), (17), we obtain the following problem for the error $z^{j+\frac{\alpha}{p}}$:

$$\varepsilon \frac{z^{j+\frac{\alpha}{p}} - z^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha z^{j+\frac{\alpha}{p}} + \frac{1}{p\mu^2}\sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} z(x, t^{j'+\frac{\alpha}{p}})\tau - \frac{1}{p\mu} z(x, t^{j+\frac{\alpha}{p}}) + \psi_\alpha^{j+\frac{\alpha}{p}},$$

$$z^{j+\frac{\alpha}{p}}=0\quad\text{for}\quad x\in\gamma_{h,\alpha},\quad z(x,0)=0.$$

Considering (14), represent the error by the sum $\psi_\alpha^{j+\frac{\alpha}{p}}=\mathring{\psi}_\alpha+\psi_\alpha^*$ (see [24, p. 524]), where $\mathring{\psi}_\alpha=(\Re_\alpha u)^{j+\frac{1}{2}}O(1)$, $\psi_\alpha^*=O(h_\alpha^2+\tau)$. Then

$$\sum_{\alpha=1}^{p}\psi_\alpha^{j+\frac{\alpha}{p}}=\sum_{\alpha=1}^{p}\mathring{\psi}_\alpha+\sum_{\alpha=1}^{p}\psi_\alpha^*=O(|h|^2+\tau),\quad |h|^2=h_1^2+h_2^2+\ldots+h_p^2.$$

4. LOCALLY ONE-DIMENSIONAL SCHEME: STABILITY

Scalarly multiply Eq. (16) by $y^{(\alpha)}=y^{j+\frac{\alpha}{p}}$:

$$\frac{\varepsilon}{p}(y_{\bar t}^{(\alpha)},y^{(\alpha)})-(\Lambda_\alpha y^{(\alpha)},y^{(\alpha)})+\left(\frac{1}{p\mu}y^{(\alpha)},y^{(\alpha)}\right)=\left(\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}y^{(\alpha)}\tau,y^{(\alpha)}\right)+(\varphi_\alpha,y^{(\alpha)}),\tag{18}$$

where

$$y_{\bar t}^{(\alpha)}=\frac{y^{t+\frac{\alpha}{p}}-y^{j+\frac{\alpha-1}{p}}}{\frac{\tau}{p}},\quad (u,v)_\alpha=\sum_{i_\alpha=1}^{N_\alpha-1}u_{i_\alpha}v_{i_\alpha}h_\alpha,\quad \|y^{(\alpha)}\|_{L_2(\alpha)}^2=\sum_{i_\alpha=1}^{N_\alpha-1}y^2h_\alpha,$$

and

$$(u,v)_\alpha=\sum_{x\in\omega_h}uvH,\quad H=\prod_{\alpha=1}^{p}h_\alpha,\quad \|y^{(\alpha)}\|_{L_2(\omega_h)}^2=\sum_{i_\beta\neq i_\alpha}\|y^{(\alpha)}\|_{L_2(\alpha)}^2H/h_\alpha.$$

By virtue of the first difference Green formula, the Cauchy–Bunyakovsky inequality, and the ε-inequality (see [24, p. 110]) transform each term of identity (18):

$$\left(\frac{\varepsilon}{p}y_{\bar t}^{(\alpha)},y^{(\alpha)}\right)_\alpha=\frac{\varepsilon}{2p}(\|y^{(\alpha)}\|_{L_2(\alpha)}^2)_{\bar t}+\frac{\varepsilon\tau}{2p}\|y_{\bar t}^{\alpha}\|_{L_2(\alpha)}^2,\tag{19}$$

$$\begin{aligned}&\left(\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}y(x,t^{j'+\frac{\alpha}{p}})\tau,y^{j+\frac{\alpha}{p}}\right)_\alpha\\
&\le\frac{1}{p\mu^2}\sum_{i_\alpha-1}^{N_\alpha-1}\left(\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}y(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\tau\right)^2h_\alpha+\frac{1}{4p\mu^2}\|y^{j+\frac{\alpha}{p}}\|_{L_2(\alpha)}^2\\
&\le\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{2}{\mu}(t_j-t_{j'})}\tau\sum_{j'=0}^{j}\|y(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\|_{L_2(\alpha)}^2\tau+\frac{1}{4p\mu^2}\|y^{j+\frac{\alpha}{p}}\|_{L_2(\alpha)}^2\\
&\le\frac{1-e^{-\frac{2}{\mu}t_j}}{2p\mu}\sum_{j'=0}^{j}\|y(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\|_{L_2(\alpha)}^2\tau+\frac{1}{4p\mu^2}\|y^{j+\frac{\alpha}{p}}\|_{L_2(\alpha)}^2.\end{aligned}\tag{20}$$

Sum with respect to $i_\beta\neq i_\alpha$, $\beta=1,2,\ldots,p$ and substitute (19)–(20) in identity (18).

We obtain

$$\begin{aligned}&\frac{\varepsilon}{2p}(\|y^{(\alpha)}\|_{L_2(\omega_h)}^2)_{\bar t}+\frac{\varepsilon\tau}{2p}\|y_{\bar t}\|_{L_2(\omega_h)}^2+c_0\|y_{\bar x_\alpha}^{(\alpha)}]|_{L_2(\omega_h)}^2+\left(\frac{c_0}{2}+\frac{1}{p\mu}-\frac{1}{4p\mu^2}\right)\|y^{j+\frac{\alpha}{p}}\|_{L_2(\omega_h)}^2\\
&+\frac{e^{-\frac{2}{\mu}t_j}}{2p\mu}\sum_{j'=0}^{j}\|y(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\|_{L_2(\omega_h)}^2\tau\le\frac{1}{2p\mu}\sum_{j'=0}^{j}\|y(x,t^{j'+\frac{\alpha}{p}})\|_{L_2(\omega_h)}^2\tau+\frac{1}{2c_0}\|\varphi^{j+\frac{\alpha}{p}}\|_{L_2(\omega_h)}^2.\end{aligned}\tag{21}$$

First, sum (21) with respect to α from 1 to p:

$$\frac{\varepsilon}{2p}(\|y^{j+1}\|^2_{L_2(\omega_h)})_{\bar{t}}+c_0\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+\left(\frac{c_0}{2}+\frac{1}{p\mu}-\frac{1}{4p\mu^2}\right)\sum_{\alpha=1}^{p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}$$
$$\leq\frac{1}{2p\mu}\sum_{\alpha=1}^{p}\sum_{j'=0}^{j}\|y(x,t^{j'+\frac{\alpha}{p}})\|^2_{L_2(\omega_h)}\tau+\frac{1}{2c_0}\sum_{\alpha=1}^{p}\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)},$$

Then sum the result with respect to j' from 0 to j:

$$\frac{\varepsilon}{2}\|y^{j+1}\|^2_{L_2(\omega_h)}+c_0\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+\left(\frac{c_0}{2}+\frac{1}{p\mu}-\frac{1}{4p\mu^2}\right)\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}$$
$$\leq\frac{1}{2p\mu}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{s=0}^{j'}\|y(x,t^{s+\frac{\alpha}{p}})\|^2_{L_2(\omega_h)}\tau+\frac{1}{2c_0}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|\varphi^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}+\frac{\varepsilon}{2}\|y^0\|^2_{L_2(\omega_h)}.$$

Estimate the first term at the right-hand as follows:

$$\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{s=0}^{j'}\|y(x,t^{s+\frac{\alpha}{p}})\|^2_{L_2(\omega_h)}\tau\leq\sum_{\alpha=1}^{p}\sum_{j'=0}^{j}\|y(x,t^{j'+\frac{\alpha}{p}})\|^2_{L_2(\omega_h)}\tau.$$

This yields that

$$\frac{\varepsilon}{2}\|y^{j+1}\|^2_{L_2(\omega_h)}+\left(\frac{c_0}{2}+\frac{1}{p\mu}-\frac{1}{4p\mu^2}-\frac{1}{2p\mu}\right)\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}$$
$$+c_0\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}\leq\frac{1}{4\varepsilon_1}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|\varphi^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}+\frac{\varepsilon}{2}\|y^0\|^2_{L_2(\omega_h)}. \tag{22}$$

Assigning $\mu\geq\frac{1}{2}$, we deduce the following a priori estimate from (22):

$$\varepsilon\parallel y^{j+1}\parallel^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}(\|y_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)})$$
$$\leq M\left(\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}(\|\varphi^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}+\|y^0\|^2_{L_2(\omega_h)})\right), \tag{23}$$

where $M=\text{const}>0$ depends nor on h_α neither on τ.

Thus, the following assertion is valid.

Theorem 1. *The locally one-dimensional scheme* (16), (17) *is stable with respect to the right-hand side and initial values, and, therefore, the solution of the difference problem* (16), (17) *obeys estimate* (23).

5. LOCALLY ONE-DIMENSIONAL SCHEME: CONVERGENCE

Similarly to [24], the solution $z_{(\alpha)}=z^{j+\frac{\alpha}{p}}$ of the error problem

$$\varepsilon\frac{z^{j+\frac{\alpha}{p}}-z^{j+\frac{\alpha-1}{p}}}{\tau}=\Lambda_\alpha z^{j+\frac{\alpha}{p}}+\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}z(x,t^{j'+\frac{\alpha}{p}})\tau-\frac{1}{p\mu}z(x,t^{j+\frac{\alpha}{p}})+\varphi_\alpha^{j+\frac{\alpha}{p}},$$

$$z^{j+\frac{\alpha}{p}}|_{\gamma_{h,\alpha}}=0,\quad z(x,0)=0,$$

where $\psi_\alpha^{j+\frac{\alpha}{p}}=\Lambda_\alpha u^{j+\frac{\alpha}{p}}+\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}u^{j+\frac{\alpha}{p}}\tau-\frac{1}{p\mu}u^{j+\frac{\alpha}{p}}+\varphi^{j+\frac{\alpha}{p}}-\varepsilon\frac{u^{j+\frac{\alpha}{p}}-u^{j+\frac{\alpha-1}{p}}}{\tau}$, is represented by the sum $z_{(\alpha)}=v_{(\alpha)}+\eta_{(\alpha)}$, where $\eta_{(\alpha)}$ is determined by the conditions

$$\varepsilon\frac{\eta_{(\alpha)}-\eta_{(\alpha-1)}}{\tau}=\overset{\circ}{\psi}_\alpha,\quad x\in\omega_h+\gamma_\alpha,\quad \alpha=1,2,\dots,p \tag{24}$$

and

$$\eta(x,0)=0.$$

From (24), it follows that $\varepsilon\eta^{j+1}=\varepsilon\eta_{(p)}=\varepsilon\eta^j+\tau(\overset{\circ}{\psi}_1+\overset{\circ}{\psi}_2+\dots+\overset{\circ}{\psi}_p)=\varepsilon\eta^j=\dots=\varepsilon\eta^0=0$. Then $\eta^\alpha=\frac{\tau}{\varepsilon}(\overset{\circ}{\psi}_1+\overset{\circ}{\psi}_2+\dots+\overset{\circ}{\psi}_\alpha)=-\frac{\tau}{\varepsilon}(\overset{\circ}{\psi}_{\alpha+1}+\dots+\overset{\circ}{\psi}_p)=O\left(\frac{\tau}{\varepsilon}\right)$.

The function $v_{(\alpha)}$ is defined by the conditions

$$\varepsilon\frac{v_{(\alpha)}-v_{(\alpha-1)}}{\tau}=\Lambda_\alpha v_{(\alpha)}-\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}v_{(\alpha)}\tau-\frac{1}{p\mu}v_{(\alpha)}+\tilde{\psi}_{(\alpha)},\quad x\in\omega_h,\quad \alpha=1,2,\dots,p, \tag{25}$$

$$v_{(\alpha)}=-\eta_\alpha,\quad x_\alpha\in\gamma_{h,\alpha},\quad v(x,0)=0,$$

and

$$\tilde{\psi}_\alpha=\psi^*_\alpha+\Lambda_\alpha\eta_{(\alpha)}+\frac{1}{p\mu^2}\sum_{j'=0}^{j}e^{-\frac{1}{\mu}(t_j-t_{j'})}\eta_{(\alpha)}\tau-\frac{1}{p\mu}\eta_{(\alpha)},\quad \psi^*_\alpha=O(h_\alpha^2+\tau).$$

If there exist derivatives $\bar{Q}_T$ continuous in the closed region $\frac{\partial^4 u}{\partial x_\alpha^2\partial x_\beta^2}$, $\alpha\neq\beta$, then $\Lambda_\alpha\eta_{(\alpha)}=-\frac{\tau}{\varepsilon}\Lambda_\alpha(\overset{\circ}{\psi}_{\alpha+1}+\dots+\overset{\circ}{\psi}_p)=O\left(\frac{\tau}{\varepsilon}\right)$.

Estimate the solution of problem (25) by means of Theorem 1:

$$\varepsilon\|v^{j+1}\|^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}(\|v_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}+]|v^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)})\leq M\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|\tilde{\psi}_\alpha^{j+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)}. \tag{26}$$

Since $\eta^j=0$, $\eta(\alpha)=O\left(\frac{\tau}{\varepsilon}\right)$, and $\|z^j\|\leq\|v^j\|$, estimate (26) yields the following assertion.

Theorem 2. *Let problem* (5), (2), (3) *have a unique solution* $u(x,t)$ *continuous in* $\bar{Q}_T$ *for all values of* ε *and there exist the following derivatives continuous in* $\bar{Q}_T$:

$$\frac{\partial^2 u}{\partial t^2},\frac{\partial^4 u}{\partial x_\alpha^2\partial x_\beta^2},\frac{\partial^3 u}{\partial x_\alpha^2\partial t},\frac{\partial^2 f}{\partial x_\alpha^2},\quad \alpha=1,2,\dots,p,\quad \alpha\neq\beta.$$

Then the locally one-dimensional scheme (16), (17) *converges to the solution of the differential problem* (1)–(3) *with rate* $O\left(|h|^2+\frac{\tau}{\varepsilon}+\varepsilon\right)$, $\tau=o(\varepsilon)$ *for all* $\mu\geq\frac{1}{2}$ *such that*

$$\|y^{j+1}-u^{j+1}\|_1\leq M\left(|h|^2+\frac{\tau}{\varepsilon}+\varepsilon\right),$$

where ε *is a small parameter*, $|h|^2=h_1^2+h_2^2+\dots+h_p^2$, *and*

$$\|z^{j+1}\|_1=\left(\varepsilon\|z^{j+1}\|^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}(\|z_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+\|z^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)})\right)^{1/2}.$$

It is obvious that the convergence rate is determined in the best way if we assign $\varepsilon=O(\tau^{\frac{1}{2}})$.

Corollary. If $\varepsilon=\tau^{\frac{1}{2}}$, then the solution of the difference problem (16), (17) converges to the solution of the differential problem (1)–(3) with rate $O(|h|^2+\sqrt{\tau})$.

Remark. The results obtained in the present paper are valid for the following fractional-order equation:

$$\partial_{0t}^{\delta} u = Lu + \partial_{0t}^{\delta} Lu - u + f(x,t), \quad (x,t) \in Q_T, \tag{27}$$

where $\partial_{0t}^{\delta} = \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{u_\tau d\tau}{(t-\tau)^{\delta}}$ is the Caputo fractional derivative of order δ, $0 < \delta < 1$.

Then, multiplying both sides of (27) by $\sum_{k=0}^{\infty} \frac{t^{k\delta}}{\Gamma(1+\delta k)}$ and acting by the fractional integration operator $D_{0t}^{-\delta} = \frac{1}{\Gamma(\delta)} \int_0^t \frac{u d\tau}{(t-\tau)^{1-\delta}}$, we obtain (after simple transformations)

$$Lu - u = -\tilde{f}(x,t), \tag{28}$$

where

$$\tilde{f}(x,t) = \frac{D_{0t}^{-\delta}\left(f(x,t) \sum_{k=0}^{\infty} \frac{t^{k\delta}}{\Gamma(1+\delta k)} \right) + Lu_0(x) - u_0(x)}{\sum_{k=0}^{\infty} \frac{t^{k\delta}}{\Gamma(1+\delta k)}},$$

while $D_{0t}^{-\delta} = \frac{1}{\Gamma(\delta)} \int_0^t \frac{u d\tau}{(t-\tau)^{1-\delta}}$ is the fractional Riemann–Liouville operator of order δ, $0 < \delta < 1$.

In the sequel, the following equation with a small parameter is considered instead of Eq. (28):

$$\varepsilon u_t = Lu - u + \tilde{f}(x,t).$$

6. NUMERICAL RESOLVING: ALGORITHM

To solve the differential problem (1)–(3) numerically, we use the following computational relations ($0 \le x_\alpha \le 1$, $\alpha = 1, 2$, $p = 2$):

$$\partial_{0t}^{\alpha} u = \frac{\partial}{\partial x_1}\left(k_1(x_1,x_2,t) \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2}\left(k_2(x_1,x_2,t) \frac{\partial u}{\partial x_2} \right) + \mu \frac{\partial^2}{\partial t \partial x_1}\left(k_1(x_1,x_2,t) \frac{\partial u}{\partial x_1} \right)$$

$$+ \mu \frac{\partial^2}{\partial t \partial x_2}\left(k_2(x_1,x_2,t) \frac{\partial u}{\partial x_2} \right) - q_1(x_1,x_2,t) u(x_1,x_2,t) - q_2(x_1,x_2,t) u(x_1,x_2,t) + f(x_1,x_2,t),$$

$$u(0,x_2,t) = 0, \quad u(l_1,x_2,t) = 0,$$

$$u(x_1,0,t) = 0, \quad u(x_1,l_2,t) = 0,$$

and

$$u(x_1,x_2,0) = u_0(x_1,x_2).$$

Consider the mesh $x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha$, $\alpha = 1, 2$, $t_j = j\tau$, where $i_\alpha = 0, 1, \ldots, N_\alpha$, $h_\alpha = 1/N_\alpha$, $j = 0, 1, \ldots, m$, $\tau = T/m$. Introduce one fractional step $t_{j+\frac{1}{2}} = t_j + 0.5\tau$. Denote the difference function by $y_{i_1,i_2}^{j+\frac{\alpha}{2}} = y^{j+\frac{\alpha}{2}} = y(i_1 h_1, i_2 h_2, (j + 0.5\alpha)\tau)$, $\alpha = 1, 2$.

Use the locally one-dimensional scheme

$$\begin{aligned} &\varepsilon \frac{y^{j+\frac{1}{2}} - y^j}{\tau} = \Lambda_1 y^{j+\frac{1}{2}} + \frac{1}{2\mu^2} \sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j+\frac{1}{2}} \tau - \frac{1}{2\mu} y^{j+\frac{1}{2}} + \varphi_1, \\ &\varepsilon \frac{y^{j+1} - y^{j+\frac{1}{2}}}{\tau} = \Lambda_2 y^{j+1} + \frac{1}{2\mu^2} \sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j+1} \tau - \frac{1}{2\mu} y^{j+1} + \varphi_2, \end{aligned} \tag{29}$$

$$y_{0,i_2}^{j+\frac{1}{2}} = 0, \quad y_{N_1,i_2}^{j+\frac{1}{2}} = 0,$$
$$y_{i_1,0}^{j+1} = 0, \quad y_{i_1,N_2}^{j+1} = 0, \tag{30}$$

$$y_{i_1,i_2}^{0} = u_0(i_1h_1, i_2, h_2), \tag{31}$$

$$\Lambda_\alpha y^{j+\frac{\alpha}{p}} = (a_\alpha y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}})_{x_\alpha} - d_\alpha y^{j+\frac{\alpha}{p}}, \quad \alpha = 1,2,$$

$$\varphi_\alpha = \frac{1}{2\mu}\sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} f^{j+\frac{\alpha}{2}}\tau - e^{-t_{j+\frac{\alpha}{2}}}\left[(y_{\bar{x}_1x_1}^{j+\frac{\alpha-1}{2}} + y_{\bar{x}_2x_2}^{j+\frac{\alpha-1}{2}}) - \frac{1}{\mu}y^{j+\frac{\alpha-1}{2}}\right].$$

The computational relations for the solving of problem (29)–(31) are as follows.

At the first stage, we find the solution $y_{i_1,i_2}^{j+\frac{\alpha}{2}}$. To this, we solve the following problem for each value $i_2 = \overline{1, N_2 - 1}$:

$$A_{1(i_1,i_2)} y_{i_1-1,i_2}^{j+\frac{1}{2}} - C_{1(i_1,i_2)} y_{i_1,i_2}^{j+\frac{1}{2}} + B_{1(i_1,i_2)} y_{i_1+1,i_2}^{j+\frac{1}{2}} = -F_{1(i_1,i_2)}^{j+\frac{1}{2}}, \quad 0 < i_1 < N_1,$$
$$y_{0,i_2}^{j+\frac{1}{2}} = 0, \quad y_{N_1,i_2}^{j+\frac{1}{2}} = 0, \tag{32}$$

where

$$A_{1(i_1,i_2)} = \frac{(a_1)_{i_1,i_2}}{h_1^2}, \quad B_{1(i_1,i_2)} = \frac{(a_1)_{i_1+1,i_2}}{h_1^2}, \quad C_{1(i_1,i_2)} = A_{1(i_1,i_2)} + B_{1(i_1,i_2)} + \frac{\varepsilon}{\tau} - \frac{\tau}{2\mu^2} + d_{1(i_1,i_2)} + \frac{1}{2\mu},$$

and

$$F_{1(i_1,i_2)}^{j+\frac{1}{2}} = \frac{\varepsilon}{\tau} y_{i_1,i_2}^{j} + \frac{1}{2\mu^2}\sum_{j'=0}^{j-1} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j'+\frac{1}{2}}\tau + \varphi_{1(i_1,i_2)}.$$

To compute the sweep right-hand side $F_{i(i_1,i_2)}^{j+\frac{1}{2}}$, one has to use on the $\left(j + \frac{1}{2}\right)$th layer the values of the desired function y_{i_1,i_2}^{j} from all preceding (lower) layers. The reason is the term $\frac{1}{2\mu^2}\sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j+\frac{1}{2}}\tau$. This substantially increases the amount of computations even if the mesh partition is small. To avoid this, we propose a recurrent relation for the fast computing in the multidimensional case. This relation provides a possibility to keep the value of the specified sum at the previous layer; regarding the amount of operations, this is not worse than the two-layer scheme.

Once $\frac{1}{\mu^2}\int_0^t e^{-\frac{1}{\mu}(t-\tau)} u(x,\tau)d\tau$ is approximated by the sum $\frac{1}{\mu}\sum_{s=1}^{pj+\alpha}\left(e^{-\frac{1}{\mu}t_{j+\frac{\alpha-s}{p}}} - e^{-\frac{1}{\mu}t_{j+\frac{\alpha-s+1}{p}}}\right)u^{\frac{s}{p}}$, the fast-computing recurrent relation for $p = 2$ takes the following form at the $\left(j + \frac{1}{2}\right)$th layer:

$$\frac{1}{2}S^{j+\frac{1}{2}} = \frac{1}{2\mu^2}\sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j+\frac{1}{2}}\tau = \frac{1}{2\mu}\sum_{s=0}^{2j}(e^{-\frac{1}{\mu}t_{j-\frac{s}{2}}} - e^{-\frac{1}{\mu}t_{j+\frac{1-s}{2}}})y^{\frac{s}{p}}$$
$$= \frac{(1 - e^{-\frac{\tau}{2\mu}})}{\mu} y^{j+\frac{1}{2}} + e^{-\frac{\tau}{2\mu}}\frac{1}{2}S^{j},$$

where $S^0 = 0$.

At the second stage, we find the solution y_{i_1,i_2}^{j+1} as follows. As in the first case, for each value of $i_1 = \overline{1, N_1 - 1}$, we solve the problem

$$\begin{gathered} A_{2(i_1,i_2)} y_{i_1,i_2-1}^{j+1} - C_{2(i_1,i_2)} y_{i_1,i_2}^{j+1} + B_{2(i_1,i_2)} y_{i_1,i_2+1}^{j+1} = -F_{2(i_1,i_2)}^{j+1}, \quad 0 < i_2 < N_2, \\ y_{i_1,0}^{j+1} = 0, \quad y_{i_1,N_2}^{j+1} = 0, \end{gathered} \tag{33}$$

$$A_{2(i_1,i_2)} = \frac{(a_2)_{i_1,i_2}}{h_2^2}, \quad B_{2(i_1,i_2)} = \frac{(a_2)_{i_1,i_2+1}}{h_2^2}, \quad C_{2(i_1,i_2)} = A_{2(i_1,i_2)} + B_{2(i_1,i_2)} + \frac{\varepsilon}{\tau} - \frac{\tau}{2\mu^2} + d_{2(i_1,i_2)} + \frac{1}{2\mu},$$

$$F_{2(i_1,i_2)}^{j+1} = \frac{\varepsilon}{\tau} y_{i_1,i_2}^{j+\frac{1}{2}} + \frac{1}{2\mu^2} \sum_{j'=0}^{j-1} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j'+1} \tau + \varphi_{2(i_1,i_2)}.$$

On the $(j + 1)$th layer, the fast-computing recurrent relation is as follows:

$$\begin{aligned} \frac{1}{2} S^{j+1} &= \frac{1}{2\mu^2} \sum_{j'=0}^{j} e^{-\frac{1}{\mu}(t_j - t_{j'})} y^{j+\frac{1}{2}} \tau = \frac{1}{2\mu} \sum_{s=0}^{2j+1} \left(e^{-\frac{1}{\mu} t_{j-\frac{1-s}{2}}} - e^{-\frac{1}{\mu} t_{j+\frac{2-s}{2}}}\right) y^{\frac{s}{p}} \\ &= \frac{(1 - e^{-\frac{\tau}{2\mu}})}{\mu} y^{j+1} + e^{-\frac{\tau}{2\mu}} \frac{1}{2} S^{j+\frac{1}{2}}. \end{aligned}$$

Problems (32) and (33) are solved by the sweeping method (see [24]).

7. TEST PROBLEM AND NUMERICAL RESULTS

The coefficients of the equation of the original differential problem (1)–(3) are selected to ensure the function $u(x, t) = e^t \sin(x_1)\sin(x_2)$ to be the exact solution for $p = 2$.

Table 1. Variations of errors and convergence orders in the norms $\|\cdot\|_{C(\overline{w}_{h\tau})}$ and $\|[\cdot]\|_0$, $\tau = h^2$, $\mu = 0.01, 0.1, 1, 10, 100$

μ	h	$\|z\|_{C(\overline{w}_{h\tau})}$	CO in $\|\cdot\|_{C(\overline{w}_{h\tau})}$	$\max_{0<j<m} \|z^j\|_0$	CO in $\|\cdot\|_0$
0.01	1/16	0.250145310		0.107135926	
	1/32	44.433332627	−7.4727	0.562873801	−2.3934
	1/64	2621.538460058	−5.8826	26.142795852	−5.5375
	1/128	9778.462858191	−1.8992	156.781044210	−2.5843
0.1	1/16	0.107713533		0.043677506	
	1/32	0.099128894	0.1198	0.041012842	0.0908
	1/64	0.082095413	0.2720	0.036565200	0.1656
	1/128	0.060532319	0.4396	0.032440707	0.1727
1	1/16	0.077789885		0.029929599	
	1/32	0.070467707	0.1426	0.025367800	0.2386
	1/64	0.058965274	0.2571	0.019421653	0.3853
	1/128	0.044636825	0.4016	0.013344342	0.5414
10	1/16	0.077424177		0.029691895	
	1/32	0.070035823	0.1447	0.025146056	0.2397
	1/64	0.058719742	0.2542	0.019215040	0.3881
	1/128	0.044556322	0.3982	0.013158437	0.5462
100	1/16	0.077543902		0.029704129	
	1/32	0.070140075	0.1448	0.025153794	0.2399
	1/64	0.058803337	0.2543	0.019218674	0.3883
	1/128	0.044615784	0.3983	0.013159517	0.5464

In Table 1, under the decreasing of the mesh size, we provide the greatest value of the error ($z = y - u$) and the convergence order in the norms $\|[\cdot]\|_0$ and $\|\cdot\|_{C(\bar{w}_{h\tau})}$, where $\|y\|_{C(\bar{w}_{h\tau})} = \max\limits_{(x_i,t_j)\in\bar{w}_{h\tau}} |y|$ provided that $\tau = h^2$. The error decreases according to the approximation order. The convergence order (CO): $\mathrm{CO} = \log_{\frac{h_1}{h_2}} \frac{\|[z_1]\|_0}{\|[z_2]\|_0}$, where z_i is the error corresponding to h_i.

If we assign $\tau = h^2$, then the convergence rate is equal to $O(h)$.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. A. G. Sveshnikov, A. V. Al'shin, M. O. Korpusov, and Yu. D. Pletner, *Linear and Nonlinear Sobolev-Type Equations* (Fizmatlit, Moscow, 2007) [in Russian].
2. G. I. Barenblatt, V. M. Entov, and V. M. Ryzhik, *Motion of Liquids and Gases in Natural Rocks* (Nedra, Moscow, 1984) [in Russian].
3. M. Kh. Shkhanukov, "Some boundary value problems for a third-order equation that arise in the modeling of the filtration of a fluid in porous media," Differ. Equations **18** (4), 509–517 (1982).
4. C. Cuesta, C. J. van Duijn, and J. Hulshof, "Infiltration in porous media with dynamic capillary pressure: travelling waves," Eur. J. Appl. Math. **11** (4), 381–397 (2000). https://doi.org/10.1017/S0956792599004210
5. A. F. Chudnovskii, *Soil Thermophysics* (Nauka, Moscow, 1976) [in Russian].
6. M. Hallaire, "Le potentiel efficace de l'eau dans le sol en régime de dessèchement," L'Eau et la Production Végétale (INRA, Paris) **9**, 27–62 (1964).
7. D. Colton, "On the analytic theory of pseudoparabolic equations," Q. J. Math. **23** (2), 179–192 (1972). https://doi.org/10.1093/qmath/23.2.179
8. E. S. Dzektser, "Equations of motion of free-surface underground water in layered media," Dokl. Akad. Nauk SSSR **220** (3), 540–543 (1975).
9. P. J. Chen and M. E. Curtin, "On a theory of heat conduction involving two temperatures," J. Appl. Math. Phys. (ZAMP) **19**, 614–627 (1968). https://doi.org/10.1007/BF01594969
10. T. W. Ting, "Certain non-steady flows of second-order fluids," Arch. Rational Mech. Anal. **14**, 1–26 (1963). https://doi.org/10.1007/BF00250690
11. V. N. Abrashin and V. A. Asmolik, "Locally one-dimensional difference schemes for multidimensional quasilinear hyperbolic equations," Differ. Equations **18** (7), 767–774 (1982).
12. A. A. Samarskii, "On an economical difference method for the solution of a multidimensional parabolic equation in an arbitrary region," USSR Comput. Math. Math. Phys. **2** (5), 894–926 (1963). https://doi.org/10.1016/0041-5553(63)90504-4
13. A. A. Samarskii, "Local one dimensional difference schemes on non-uniform nets," USSR Comput. Math. Math. Phys. **3** (3), 572–619 (1963). https://doi.org/10.1016/0041-5553(63)90290-8
14. A. A. Samarskii, "Local one-dimensional difference schemes for multi-dimensional hyperbolic equations in an arbitrary region," USSR Comput. Math. Math. Phys. **4** (4), 21–35 (1964). https://doi.org/10.1016/0041-5553(64)90002-3
15. M. Kh. Beshtokov, "Finite-difference method for a nonlocal boundary value problem for a third-order pseudoparabolic equation," Differ. Equations **49** (9), 1134–1141 (2013). https://doi.org/10.1134/S0012266113090085
16. M. Kh. Beshtokov, "On the numerical solution of a nonlocal boundary value problem for a degenerating pseudoparabolic equation," Differ. Equations **52** (10), 1341–1354 (2016). https://doi.org/10.1134/S0012266116100104
17. M. Kh. Beshtokov, "Difference method for solving a nonlocal boundary value problem for a degenerating third-order pseudo-parabolic equation with variable coefficients," Comput. Math. Math. Phys. **56** (10), 1763–1777

(2016).
https://doi.org/10.1134/S0965542516100043

18. M. Kh. Beshtokov, " Boundary value problems for degenerating and nondegenerating Sobolev-type equations with a nonlocal source in differential and difference forms," Differ. Equations **54** (2), 250–267 (2018). https://doi.org/10.1134/S0012266118020118
19. M. Kh. Beshtokov, "Numerical analysis of initial-boundary value problem for a Sobolev-type equation with a fractional-order time derivative," Comput. Math. Math. Phys. **59** (2), 175–192 (2019). https://doi.org/10.1134/S0965542519020052
20. M. Kh. Beshtokov, "Boundary-value problems for loaded pseudoparabolic equations of fractional order and difference methods of their solving," Russ. Math. **63** (2), 1–10 (2019). https://doi.org/10.3103/S1066369X19020014
21. M. I. Vishik and L. A. Lyusternik, "Regular degeneration and boundary layer for linear differential equations with small parameter," Usp. Mat. Nauk **12** (5), 3–122 (1957).
22. S. K. Godunov and V. S. Ryaben'kii, *Difference Schemes. An Introduction to the Underlying Theory* (Nauka, Moscow, 1977; North-Holland, Amsterdam, 1987).
23. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Nauka, Moscow, 1968) [in Russsian].
24. A. A. Samarskii, *The Theory of Difference Schemes* (Nauka, Moscow, 1977; Marcel Dekker, New York, 2001).

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