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Kaczmarz Method for Fuzzy Linear Systems

L. Bian[1]*, S. Zhang[2], S. Wang[2]***, and K. Wang[1]******

[1]*Department of Mathematics, Shanghai University, Shanghai, 200444 P.R. China*

[2]*Nanyang Vocational College of Agriculture, Nanyang, 473000 P.R. China*

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Abstract—A Kaczmarz method is presented for solving a class of fuzzy linear systems of equations with crisp coefficient matrix and fuzzy right-hand side. The iterative scheme is established and the convergence theorem is provided. Numerical examples show that the method is effective.

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Key words: *fuzzy linear system, Kaczmarz method, iterative scheme.*

INTRODUCTION

Fuzzy linear systems (FLSs) occur in many fields, such as control problems, information, physics, statistics, engineering, economics, finance and even social sciences [1]. Thus, it is significant to develop models for solving FLSs theoretically and numerically [2]–[4].

A general model was suggested in [1] by Friedman et al. with embedding technique for solving a class of $n \times n$ FLSs

$$Ax = y, \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is a crisp matrix, $y = [y_1, y_2, \cdots, y_n]^{\mathrm{T}}$ is a fuzzy vector, and $x = [x_1, x_2, \cdots, x_n]^{\mathrm{T}}$ is unknown, by which, Large numbers of numerical methods [5]–[22] was established to solve FLS (1).

Kaczmarz algorithm is a popular iterative projection method [23] as it is simple to implement and suitable for parallel computing. Many authors studied Kaczmarz methods for solving linear systems [23]–[26]. In this paper, a Kaczmarz algorithm is proposed for fuzzy linear system (1).

The rest of the paper is organized as follows. Section 1 gives some basic definitions and results of FLS. In Section 2, the Kaczmarz method is presented with convergence theorem. Two numerical examples are discussed in Section 3 and the conclusion is in Section 4.

*E-mail: 2720976880@qq.com
**E-mail: 88182703@qq.com
***E-mail: 77917092@qq.com
****E-mail: kwang@math.ecnu.edu.cn

1. PRELIMINARIES

Generally, as defined in [1], a fuzzy number is a pair of $(\underline{u}(r), \overline{u}(r))$, $0 \leqslant r \leqslant 1$, satisfying

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$,
- $\underline{u}(r) \leqslant \overline{u}(r)$.

The arithmetic operations of arbitrary fuzzy numbers $x=(\underline{x}(r), \overline{x}(r))$, $y=(\underline{y}(r), \overline{y}(r))$, $0 \leqslant r \leqslant 1$, and real number k, are as follows,

(1) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(3) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \geqslant 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 1. [1] A fuzzy number vector $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), \quad 1 \leqslant i \leqslant n, \ \ 0 \leqslant r \leqslant 1,$$

is called a solution of fuzzy linear system (1) if

$$\begin{aligned} \underline{\sum_{j=1}^{n} a_{ij}x_j} &= \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} &= \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i. \end{aligned} \tag{2}$$

By (2), M. Friedman et al.[1] extend FLS (1) to a $2n \times 2n$ crisp linear system

$$SX = Y, \tag{3}$$

where $S = (s_{kl})$ determined as

$$\begin{aligned} a_{ij} \geqslant 0 \ \Rightarrow \ s_{ij} = a_{ij}, \qquad s_{n+i,\ n+j} = a_{ij}, \\ a_{ij} < 0 \ \Rightarrow \ s_{i,\ n+j} = a_{ij}, \qquad s_{n+i,\ j} = a_{ij}, \end{aligned} \qquad 1 \leqslant i, j \leqslant n,$$

and any s_{kl} not in the above is zero $(1 \leqslant k,\ l \leqslant 2n)$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \ Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ \overline{y}_1 \\ \vdots \\ \overline{y}_n \end{bmatrix}.$$

Furthermore, the matrix S has the structure $\begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$, $A = S_1 + S_2$, and (3) can be rewritten as

$$\begin{aligned} S_1\underline{X} + S_2\overline{X} &= \underline{Y}, \\ S_2\underline{X} + S_1\overline{X} &= \overline{Y}, \end{aligned} \tag{4}$$

where

$$\underline{X}=\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \vdots\\ \underline{x}_n\end{bmatrix},\ \overline{X}=\begin{bmatrix}\overline{x}_1\\ \overline{x}_2\\ \vdots\\ \overline{x}_n\end{bmatrix},\ \underline{Y}=\begin{bmatrix}\underline{y}_1\\ \underline{y}_2\\ \vdots\\ \underline{y}_n\end{bmatrix},\ \overline{Y}=\begin{bmatrix}\overline{y}_1\\ \overline{y}_2\\ \vdots\\ \overline{y}_n\end{bmatrix}.$$

The following theorem indicates when FLS (1) has a unique solution.

Theorem 1. *The matrix S is nonsingular if and only if the matrices $A = S_1 + S_2$ and $S_1 - S_2$ are both nonsingular.* See [1].

In the next section, a Kaczmarz iterative scheme is presented for nonsingular FLS (1).

2. THE KACZMARZ METHOD

For nonsingular fuzzy linear system (3) or (4), a Kaczmarz method can be proposed as follows,

$$X_{k+1} = X_k + \frac{Y^{(i_k)} - S^{(i_k)}X_k}{\left\|S^{(i_k)}\right\|_2^2}\left(S^{(i_k)}\right)^{\mathrm{T}},$$

and can be implemented as the following algorithm.

Kaczmarz Algorithm. *Given initial vectors $\underline{X}_0, \overline{X}_0 \in \mathbb{R}^n$, for $k = 0, 1, 2, \cdots$, the following iterative scheme is taken,*

$$\begin{aligned}\underline{X}_{k+1} &= \underline{X}_k + \frac{(\underline{Y} - S_2\overline{X}_k)^{(i_k)} - S_1^{(i_k)}\underline{X}_k}{\left\|S_1^{(i_k)}\right\|_2^2}\left(S_1^{(i_k)}\right)^{\mathrm{T}},\\ \overline{X}_{k+1} &= \overline{X}_k + \frac{(\overline{Y} - S_2\underline{X}_{k+1})^{(i_k)} - S_1^{(i_k)}\overline{X}_k}{\left\|S_1^{(i_k)}\right\|_2^2}\left(S_1^{(i_k)}\right)^{\mathrm{T}},\end{aligned} \tag{5}$$

where $i_k = (k \bmod n) + 1$, and $(\cdot)^{(i_k)}$ denotes the i_kth row of a matrix.

The convergence result for method (5) is as the following theorem.

Theorem 2. *Suppose that fuzzy linear system (3) or (4) is consistent. Then the iterative sequence $\left\{\begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix}\right\}_{k=0}^{\infty}$, generated by the Kaczmarz method (5) starting from an initial guess $\begin{bmatrix}\underline{X}_0\\ \overline{X}_0\end{bmatrix}$ with $\underline{X}_0$ and $\overline{X}_0$ in the column space of S_2, converges to the unique solution $\begin{bmatrix}\underline{X}_*\\ \overline{X}_*\end{bmatrix} = \begin{bmatrix}S_1 & S_2\\ S_2 & S_1\end{bmatrix}^{-1}\begin{bmatrix}\underline{Y}\\ \overline{Y}\end{bmatrix}$ of (3). Moreover, the solution error for the iteration sequence is*

$$\|X_{k+1} - X_*\|_2^2 \leqslant \left(1 - \frac{\lambda_{\min}(S^{\mathrm{T}}S)}{\|S\|_F^2}\right)\|X_k - X_*\|_2^2,$$

where $\lambda_{\min}(\cdot)$ is the smallest nonzero eigenvalue of a matrix, $X_k = \begin{bmatrix}\underline{X}_k\\ \overline{X}_k\end{bmatrix}$ and $X_* = \begin{bmatrix}\underline{X}_*\\ \overline{X}_*\end{bmatrix}$.

Proof. Take $P^{(i_k)} = \left(\frac{S^{(i_k)}}{\left\|S^{(i_k)}\right\|_2}\right)^{\mathrm{T}}\left(\frac{S^{(i_k)}}{\left\|S^{(i_k)}\right\|_2}\right)$, thus

$$X_{k+1} - X_* = X_k - X_* + \frac{Y^{(i_k)} - S^{(i_k)}X_k}{\left\|S^{(i_k)}\right\|_2^2}\left(S^{(i_k)}\right)^{\mathrm{T}}$$

$$= X_k - X_* - \left(S^{(i_k)}\right)^{\mathrm{T}} \frac{S^{(i_k)}\left(X_k - X_*\right)}{\left\|S^{(i_k)}\right\|_2^2} = \left(I - P^{(i_k)}\right)\left(X_k - X_*\right),$$

then

$$\|X_{k+1} - X_*\|_2^2 = (X_k - X_*)^{\mathrm{T}}\left(I - P^{(i_k)}\right)^2 (X_k - X_*)$$

$$= (X_k - X_*)^{\mathrm{T}}\left(I - P^{(i_k)}\right)(X_k - X_*) = \|X_k - X_*\|_2^2 - (X_k - X_*)^{\mathrm{T}} P^{(i_k)} (X_k - X_*),$$

and

$$(X_k - X_*)^{\mathrm{T}} P^{(i_k)} (X_k - X_*) = \frac{(X_k - X_*)^{\mathrm{T}}\left(S^{(i_k)}\right)^{\mathrm{T}} S^{(i_k)} (X_k - X_*)}{\left\|S^{(i_k)}\right\|_2^2}$$

$$= \frac{\left|S^{(i_k)}\left(X_k - X_*\right)\right|^2}{\left\|S^{(i_k)}\right\|_2^2} \geqslant \frac{\|S\left(X_k - X_*\right)\|_2^2}{\|S\|_F^2}.$$

As x_0 is in the column space of S, from [23], it holds that $\|S\left(X_k - X_*\right)\|_2^2 \geqslant \lambda_{\min}(S^{\mathrm{T}}S)\,\|(X_k - X_*)\|_2^2$. Therefore, the following is obtained

$$\|X_{k+1} - X_*\|_2^2 \leqslant \left(1 - \frac{\lambda_{\min}(S^{\mathrm{T}}S)}{\|S\|_F^2}\right)\|X_k - X_*\|_2^2.$$

The proof is completed. □

3. NUMERICAL EXAMPLES

This section gives two illustrative examples to show the effectiveness of the Kaczmarz method. All implements using Matlab 7 run in a Windows 7 DELL laptop with Intel 2.80GHz CPU and 8.00GB RAM. In the numerical experiments, the initial guess is zero and the stopping criterion is

$$\|R_k\|_2 < 10^{-6},$$

where R_k is the residual vector after k iterations, i. e., $R_k = Y - SX_k$.

In the actual calculations, $SX = Y$ is solved by two numeric systems

$$S\begin{bmatrix} x_{a1} \\ x_{a2} \\ \vdots \\ x_{a,2n} \end{bmatrix} = \begin{bmatrix} y_{a1} \\ y_{a2} \\ \vdots \\ y_{a,2n} \end{bmatrix} \text{ and } S\begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{b,2n} \end{bmatrix} = \begin{bmatrix} y_{b1} \\ y_{b2} \\ \vdots \\ y_{b,2n} \end{bmatrix},$$

not by one symbolic system

$$S\begin{bmatrix} x_{a1}+x_{b1}r \\ x_{a2}+x_{b2}r \\ \vdots \\ x_{a,2n}+x_{b,2n}r \end{bmatrix} = \begin{bmatrix} y_{a1}+y_{b1}r \\ y_{a2}+y_{b2}r \\ \vdots \\ y_{a,2n}+y_{b,2n}r \end{bmatrix}.$$

Example 1. Consider $n\times n$ fuzzy linear system $Ax=y$ with

$$A=\begin{bmatrix} 8 & -1 & -1 & -1 & & & & & \\ -1 & 8 & -1 & -1 & \ddots & & & & \\ -1 & -1 & 8 & -1 & \ddots & \ddots & & & \\ -1 & -1 & -1 & 8 & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 8 & -1 & -1 & -1 \\ & & & \ddots & \ddots & -1 & 8 & -1 & -1 \\ & & & & \ddots & -1 & -1 & 8 & -1 \\ & & & & & -1 & -1 & -1 & 8 \end{bmatrix}$$

and

$$y=\begin{bmatrix} (2+r,2+r) \\ (2+r,2+r) \\ \vdots \\ (2+r,2+r) \end{bmatrix}.$$

Example 2. Consider $n^2\times n^2$ fuzzy linear system $Ax=y$ with

$$A=\begin{bmatrix} D & B^{\mathrm{T}} & & & \\ B & D & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & D & B^{\mathrm{T}} \\ & & & B & D \end{bmatrix},$$

where

$$B=\begin{bmatrix} 0.5 & -0.25 & & & \\ -0.25 & 0.5 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0.5 & -0.25 \\ & & & -0.25 & 0.5 \end{bmatrix},\ D=\begin{bmatrix} 5 & -1 & & & \\ -1 & 5 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 5 & -1 \\ & & & -1 & 5 \end{bmatrix},$$

and

$$y = \begin{bmatrix} (1+r, 1+r) \\ (2+r, 2+r) \\ \vdots \\ (n^2+r, n^2+r) \end{bmatrix}.$$

Tables 1 and 2 give the number of iterations (IT) and the residual of the stopping step (RES). As n increases, the method requires more iterations and converges very slowly, thus, improvement should be made to change the convergence.

Table 1. Iterations (IT) and Residual (RES) for Example 1

n	IT_{x_a}	RES_{x_a}	IT_{x_b}	RES_{x_b}
16	387	9.8857e-007	371	9.9516e-007
32	937	9.8926e-007	903	9.7518e-007
64	2063	9.9769e-007	1959	9.9717e-007
128	4283	9.9940e-007	4119	9.9840e-007
256	8810	9.9827e-007	8420	9.9985e-007
512	18001	9.9982e-007	17298	9.9991e-007
1024	36721	9.6177e-007	35408	9.9968e-007

Table 2. Iterations (IT) and Residual (RES) for Example 2

n	IT_{x_a}	RES_{x_a}	IT_{x_b}	RES_{x_b}
10	1734	9.6080e-007	1283	9.3918e-007
15	4193	9.6755e-007	3023	9.9710e-007
20	7711	9.9794e-007	5543	9.9329e-007
25	12434	9.9822e-007	8704	9.9622e-007
30	18408	9.9942e-007	12563	9.9846e-007
35	25599	9.9671e-007	17283	9.9737e-007
40	33669	9.9956e-007	22833	9.9542e-007

4. CONCLUSION

A Kaczmarz method is proposed for solving $n \times n$ fuzzy linear systems. The numerical results show that the method is effective. Further work would be improving the method with stochastic approximation and comparing with other methods. And more interesting work would be exploring the real applications in natural science.

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