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Dedicated to the bright memory of my father, Abduragimov Elderkhan Israpilovich

Existence and Uniqueness of a Positive Solution to a Boundary Value Problem for a Second Order Functional-Differential Equation

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Abstract—In the paper, we consider a boundary value problem for a second order functional-differential equation with sufficiently general linear homogeneous boundary conditions. On the basis of the theory of semi-ordered spaces and with the help of special topological methods, we prove the existence of a unique positive solution to the problem.

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Key words: *positive solution, boundary value problem, cone, asymptotic derivative, spectral radius.*

STATEMENT OF THE PROBLEM

Sufficiently many works are devoted to the solvability of functional-differential equations (see, e. g., [1]–[10]). Almost in all of these works, the problems of existence and uniqueness of positive solutions of boundary value problems mo second order differential equations with delay are considered, and the natural tools of their study are the methods of functional analysis based on the use of semi-ordered spaces, the theory of which is connected with the names of F. Riezs, M. G. Krein, L. V. Kantorovich, G. Freidental, G. Birkhoff and others. Further the methods of investigation of positive solutions of operator equations we developed by M. A. Krasnosel'skii and its followers, such as L. A. Ladyzhenskaya, I. A. Bakhtin, V. Ya. Stetsenko, Yu. V. Pokornii, and others.

This paper is a continuation of the author's investigations devoted to the problem of existence and uniqueness of positive solutions to boundary value problems for second order functional-differential equations.

The aim of the paper is to obtain, on the base of the theory of semi-ordered spaces and with the help of special topological methods, sufficient conditions of existence and uniqueness of a positive solution for boundary value problem for a functional-differential equation.

Denote by C the space $C[0,1]$, by $\mathbb{L}_p$ $(1 < p < \infty)$ the space $\mathbb{L}_p(0,1)$ and let $\mathbb{W}^2$ be the space of real-valued functions on $[0,1]$ with absolutely continuous derivative.

Consider the following boundary value problem:

$$x''(t) + f\left(t, (Tx)(t)\right) = 0, \qquad 0 < t < 1, \tag{1}$$

$$\alpha_{11}x(0) + \alpha_{12}x(1) + \beta_{11}x'(0) + \beta_{12}x'(1) = 0, \tag{2}$$

$$\alpha_{21}x(0) + \alpha_{22}x(1) + \beta_{21}x'(0) + \beta_{22}x'(1) = 0; \tag{3}$$

here $\alpha_{ij}, \beta_{ij} \quad (i,j = 1,2)$ are real numbers, $T\colon C \to \mathbb{L}_p$ $(1 < p < \infty)$ is a positive linear operator [11, p. 59], the function $f(t,u)$ is non-negative, increases with respect to the second argument, satisfies the Carathéodory condition, and $f(\,\cdot\,,0) \equiv 0$.

By positive solution of the problem (1)–(3), we mean a function $x \in \mathbb{W}^2$, positive on $(0,1)$, satisfying almost everywhere equation (1) and the boundary conditions (2)–(3).

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1. MAIN RESULTS

Consider the integral equation, equivalent to the problem (1)–(3):

$$x(t) = \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds, \quad 0 \le t \le 1, \tag{4}$$

where $G(t,s)$ is the Green function of the operator $-\dfrac{d^2}{dt^2}$ with boundary conditions (2)–(3).

Now we will introduce some notation: $\alpha \equiv \dfrac{\alpha_{12}+\beta_{11}+\beta_{12}}{\alpha_{11}+\alpha_{12}}$, $\beta \equiv \dfrac{\alpha_{22}+\beta_{21}+\beta_{22}}{\alpha_{21}+\alpha_{22}}$. Under the fulfillment of the conditions

a) $\alpha \ne \beta$, $\alpha_{11}+\alpha_{12} \ne 0$, $\alpha_{21}+\alpha_{22} \ne 0$,

b) $\dfrac{1}{\beta-\alpha}\left[\dfrac{\beta_{21}}{\alpha_{21}+\alpha_{22}}(1-\alpha) - \dfrac{\beta_{11}}{\alpha_{11}+\alpha_{12}}(1-\beta)\right] \le 0,$

c) $\dfrac{1}{\beta-\alpha}\left[\dfrac{\alpha_{21}-\beta_{21}}{\alpha_{21}+\alpha_{22}}(1-\alpha) - \dfrac{\alpha_{11}-\beta_{11}}{\alpha_{11}+\alpha_{12}}(1-\beta)\right] > 0,$

d) $\dfrac{1}{\beta-\alpha}\left[\dfrac{\beta_{21}}{\alpha_{21}+\alpha_{22}}\alpha - \dfrac{\beta_{11}}{\alpha_{11}+\alpha_{12}}\beta\right] > 0,$

e) $\dfrac{1}{\beta-\alpha}\left[\dfrac{\alpha_{21}}{\alpha_{21}+\alpha_{22}}\alpha - \dfrac{\alpha_{11}}{\alpha_{11}+\alpha_{12}}\beta + 1\right] < 0,$

the Green function exists, it is positive and has the form [12]

$$G(t,s) = \begin{cases} a_1(s)(t-\alpha) + a_2(s)(t-\beta), & \text{if } 0 \le t \le s; \\ b_1(s)(t-\alpha) + b_2(s)(t-\beta), & \text{if } s \le t \le 1, \end{cases}$$

where

$$a_1(s) = \frac{1}{\beta-\alpha}\left[\beta - \frac{\alpha_{22}s+\beta_{21}}{\alpha_{21}+\alpha_{22}}\right], \quad a_2(s) = \frac{1}{\beta-\alpha}\left[\frac{\alpha_{12}s+\beta_{11}}{\alpha_{11}+\alpha_{12}} - \alpha\right], \quad s \in [0,1],$$

$$b_1(s) = \frac{\alpha_{21}s-\beta_{21}}{(\beta-\alpha)(\alpha_{21}+\alpha_{22})}, \qquad b_2(s) = \frac{\alpha_{11}s-\beta_{11}}{(\alpha-\beta)(\alpha_{11}+\alpha_{12})}, \qquad s \in [0,1].$$

Assume that $f(t,u)$ satisfies the condition

$$f(t,u) \le a(t) + bu, \tag{5}$$

where $a(t) \in \mathbb{L}_p$, $b > 0$.

We can write equation (4) in the operator form:

$$x = GNTx,$$

where $N\colon \mathbb{L}_p \to \mathbb{L}_p$ is the Nemytskii operator, $G\colon \mathbb{L}_p \to C$ is the Green operator.

The operator A, defined by the equality

$$(Ax)(t) = \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds, \quad 0 \le t \le 1,$$

acts on the space of nonnegative continuous functions. It is completely continuous [13, p. 161] and keeps the cone $\hat{K}$ of nonnegative functions $x(t)$ of the space C, satisfying the condition

$$\min_{t\in[0,1]} x(t) \geq \frac{m}{M} \max_{t\in[0,1]} x(t) = \frac{m}{M} \|x\|_C ,$$

where m and M are the lower and the upper estimations of the Green functions.

Theorem 1. *Assume that $T\colon C \to \mathbb{L}_p$ is an operator, positive on the cone $\hat{K}$, inequality* (5) *holds and*

1. $\gamma < \dfrac{m}{bM^2}$, *where γ is the norm of operator T;*

2. $m \int\limits_0^1 f\left(s, (T1)(s)\right) ds \geq 1.$

Then the boundary value problem (1)–(3) *has at least one solution.*

Proof. Further by semi-orderings $u \prec v$ and $u \,\overline{\prec}\, v$ in the cone $\hat{K}$ of the space C we mean that the inequalities $u(x) \leq v(x)$ and $u(x) > v(x)$ hold for all $x \in [0,1]$.

Because of (5), the operator A has a majorant on the cone $\hat{K}$ of the space C:

$$(Ax)(t) \leq (Bx)(t) = b \int\limits_0^1 G(t,s)\left(Tx\right)(s)\, ds + \int\limits_0^1 a(s)G(t,s)ds.$$

We will show that the strong asymptotical derivative $B'(\infty)$, by the cone $\hat{K}$ of the space C, of the majorant B (see, e.g., [11, p. 109]) has spectral radius which is less than 1.

Actually, for every $x(t) \in \hat{K}$ from the space C we have

$$B'(\infty)x(t) = b \int\limits_0^1 G(t,s)\left(Tx\right)(s)\, ds \leq bM\|Tx\|_{\mathbb{L}_p} \leq bM\gamma\|x\|_C \leq \frac{bM^2}{m}\gamma \cdot x(t).$$

Taking into account Condition 1 of the theorem, from the theorem [14, p. 461] we obtain that the spectrum of the completely continuous linear operator $B'(\infty)$ by the cone $\hat{K}$ is in the disk of radius less than 1.

To complete the proof, it is sufficient to prove the existence of a nonzero element $x_0 \in \hat{K}$ such that $Ax_0 \succ x_0$.

Putting $x_0 \equiv 1$, by Condition 2 of the theorem, we have

$$(A1)(t) = \int\limits_0^1 G(t,s) f\left(s, (T1)(s)\right)\, ds \geq m \int\limits_0^1 f\left(s, (T1)(s)\right)\, ds \geq 1.$$

From the theorem [11, p. 136], we obtain that the operator A has at least one nonzero fixed point in the cone $\hat{K}$ of the space C, what is equivalent to existence of at least one positive solution to the boundary value problem (1)–(3). □

Theorem 2. *Let $a(t) \equiv 0$ in* (5), *the conditions of Theorem* 1 *hold and*

$$f(t, \tau u) \geq \tau f(t,u), \quad t \in [0,1], \quad u \geq 0, \quad \forall \tau \in (0,1). \tag{6}$$

Then the boundary value problem (1)–(3) *has a unique positive solution.*

Proof. To establish the uniqueness of positive solution, we will show that the concave and monotone operator A, defined by inequality (4), is u_0-concave [11, p. 199] on the cone $\hat{K}$ of the space C.

Taking into account (5), where we put $a(t) \equiv 0$, we have

$$\int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds \le Mb \int_0^1 (Tx)(s)\, ds \le Mb\gamma \|x\|_C. \tag{7}$$

Further we will denote by σ the expression on the right-hand side of (7). It is evident that $\sigma > 0$.

For every positive $\lambda \in (0,1)$, from (6) and (7) it follows that

$$\int_0^1 G(t,s) f\left(s, (T\lambda x)(s)\right) ds - \lambda \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds$$
$$\ge \lambda \left(\frac{m^2}{M} \|x\|_C \int_0^1 f\left(s, (T1)(s)\right) ds - Mb\gamma \|x\|_C \right). \tag{8}$$

Denote by δ the difference containing in the parentheses on the right-hand side of (8). By the requirements of Theorem 1, the value of δ is strongly positive. In terms of σ and δ, we write inequality (8) in the form

$$\int_0^1 G(t,s) f\left(s, (T\lambda x)(s)\right) ds \ge \lambda \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds + \lambda\delta$$
$$\ge \lambda \left(1 + \frac{\delta}{\sigma}\right) \int_0^1 G(t,s) f\left(s, (Tx)(s)\right) ds.$$

The last inequality coincides with the condition of u_0-concavity ($u_0 \equiv 1$) and, according to the theorem [11, p. 200], the boundary value problem (1)–(3) has a unique positive solution. □

Remark. We note that in the case where $f(t,u)$ is linear with respect to the second argument, for existence of unique positive solution, it is necessary and sufficient that the spectral radius of the operator GT in the space C is less than 1. Under the assumptions of this paper, Condition 1 of Theorem 1 is sufficient for this purpose.

As an example, we consider the boundary value problem

$$x''(t) + b \cdot \ln\left((S_h x)(t) + 1\right) = 0, \qquad 0 < t < 1, \tag{9}$$
$$x(0) - 9x'(0) = 0, \tag{10}$$
$$10x(0) - 99x'(0) - x'(1) = 0, \tag{11}$$

where

$$(S_h x)(t) = \begin{cases} x(t-h), & t-h \in [0,1]; \\ 0, & t-h \notin [0,1]. \end{cases}$$

It is easy to see that the Green function of the operator $-\dfrac{d^2}{dt^2}$ with boundary conditions (10), (11) exists, it is positive, has the form

$$G(t,s) = \begin{cases} 0.1t + 0.9, & 0 \le t \le s; \\ -0.9t + s + 0.9, & s \le t \le 1, \end{cases}$$

and $0.9 \le G(t,s) \le 1$ $(t, s \in [0,1])$.

With the help of the above theorems, it is easy to verify that, under the fulfillment of the requirements

$$b \geq \frac{1}{0.9 \ln 2} \quad \text{and} \quad b \cdot (1-h)^{\frac{1}{p}} < 0.9,$$

the boundary value problem (9)–(11) has a unique positive solution.

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