ISSN 1066-369X, Russian Mathematics, 2021, Vol. 65, No. 9, pp. 33–40. © Allerton Press, Inc., 2021.
Russian Text © The Author(s), 2021, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2021, No. 9, pp. 40–48.

Generalized Lie-Type Derivations of Alternative Algebras

B. L. M. Ferreira[1*] and G. C. de Moraes[2]**

[1] *Federal University of Technology, 800 Professora Laura Pacheco Bastos Ave., Guarapuava, 85053-510 Brazil*

[2] *Federal University of ABC, 5001 dos Estados Ave., Santo André, 09210-580 Brazil*

Received September 16, 2020; revised November 16, 2020; accepted December 24, 2020

Abstract—In this paper we intend to describe generalized Lie-type derivations using, among other things, a generalization for alternative algebras of the result: "If $F : A \to A$ is a generalized Lie n-derivation associated with a Lie n-derivation D, then a linear map $H = F - D$ satisfies $H(p_n(x_1, x_2, \ldots, x_n)) = p_n(H(x_1), x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in A$". Thus, if A is a unital alternative algebra with a nontrivial idempotent e_1 satisfying certain conditions, then a generalized Lie-type derivation $F : A \to A$ is of the form $F(x) = \lambda x + \Xi(x)$ for all $x \in A$, where $\lambda \in Z(A)$ and $\Xi : A \to A$ is a Lie-type derivation.

DOI: 10.3103/S1066369X2109005X

Key words: *alternative algebra, generalized Lie derivation.*

1. INTRODUCTION

In this paper, we will deal with the structure named generalized Lie-type derivations of alternative algebras. For this, we will initially consider some definitions and conventions:

Let A be an algebra not necessarily associative or commutative and consider the following convention for its multiplication operation: $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in A$, in order to reduce the number of parentheses. We will denote the *associator* of A by $(x, y, z) = xy \cdot z - x \cdot yz$ for $x, y, z \in A$. An algebra A is said to be *flexible* if $(x, y, x) = 0$ for all $x, y \in A$. It is known that alternative algebras are flexible. By $[x_1, x_2] = x_1x_2 - x_2x_1$ we will denote the usual Lie product of x_1 and x_2.

According to [1], "It is a fascinating topic to study the connection between the associative, Lie and Jordan structures on A. In this field, two classes of mappings are of crucial importance. One of them consists of mappings, preserving a type of product, for example, Jordan homomorphisms and Lie homomorphisms. The other one is formed by differential operators, satisfying a type of Leibniz formulas, such as Jordan derivations and Lie derivations. In the AMS Hour Talk of 1961, Herstein proposed many problems concerning the structure of Jordan and Lie mappings in associative simple and prime rings [2]. Roughly speaking, he conjectured that these mappings are all of the proper or standard forms. The renowned Herstein's Lie-type mapping research program was formulated since then. Martindale gave a major force in this program under the assumption that the rings contain some nontrivial idempotents [3]. The first idempotent-free result on Lie-type mappings was obtained by Brešar in [4]. Recently, several new articles have also studied the additivity of maps that maintain new products and derivable maps about new products among them we can mention [5–9]. Also the structures of derivations, Jordan derivations and Lie derivations on (non-)associative rings were studied systematically by many people (cf. [1–4, 10–19]). It is obvious that every derivation is a Lie derivation. But the converse is in general not true. A basic question towards Lie derivations of the associative algebras is that whether they can be decomposed into the sum of a derivation and a central-valued mapping, see [3, 4, 10–14, 17] and references therein."

*E-mail: brunoferreira@utfpr.edu.br
**E-mail: gabriela.moraes@ufabc.edu.br

In [1] the authors proved that, under certain conditions on an alternative ring R, a multiplicative Lie-type derivation D from R into R can be written as a sum of a derivation and a center valued map vanishing for each $(n-1)$th commutator.

Throughout this paper, we will consider F a field of characteristic different from 2, 3 and A an unital alternative algebra over F. We also will assume that A has an idempotent element $e_1 \neq 0, 1$ and will denote the idempotent $1 - e_1$ by e_2. In this case, A can be represented in the Pierce decomposition form by $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$ where $A_{11} = e_1 A e_1$, $A_{12} = e_1 A(1 - e_1)$, $A_{21} = (1 - e_1)Ae_1$ e $A_{22} = (1 - e_1)A(1 - e_1)$, satisfying the following multiplicative relations, as we can see in [20, Proposition 3.4]:

(i) $A_{ij}A_{jl} \subseteq A_{il}$ $(i, j, l = 1, 2)$;

(ii) $A_{ij}A_{ij} \subseteq A_{ji}$ $(i, j = 1, 2)$;

(iii) $A_{ij}A_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l)$, $(i, j, k, l = 1, 2)$;

(iv) $x_{ij}^2 = 0$, for all $x_{ij} \in A_{ij}$ $(i, j = 1, 2;\ i \neq j)$.

We will denote by a_{ij} an element of A_{ij}, and will assume that A satisfies

$$aA \cdot e_i = 0 \quad \textit{implies} \quad a = 0 \ \ (i = 1, 2). \tag{1}$$

Now consider

$$\begin{aligned} a_{11}A_{12} = \{0\} = A_{21}a_{11} \quad &\textit{implies} \quad a_{11} = 0, \\ A_{12}a_{22} = \{0\} = a_{22}A_{21} \quad &\textit{implies} \quad a_{22} = 0. \end{aligned} \tag{2}$$

In fact we have (1) implies (2). Assuming (1), if $a_{11}A_{12} = \{0\} = A_{21}a_{11}$ so

$$a_{11}A \cdot e_2 = a_{11}(A_{11} + A_{12} + A_{21} + A_{22}) \cdot e_2 = (a_{11}A_{11} + a_{11}A_{12}) \cdot e_2 = a_{11}A_{12}.$$

Therefore, $a_{11}A \cdot e_2 = \{0\}$ and using (1) it will implies $a_{11} = 0$. The case where $aA \cdot e_1 = \{0\}$ is analogous.

Note that any unital prime alternative algebra over a field of characteristic $\neq 3$ with a nontrivial idempotent satisfies the condition (1), as we can see in Therorem 1 at [21]: "Let $\mathfrak{R}$ be a 3-torsion free alternative ring. So $\mathfrak{R}$ is a prime ring if and only if $a\mathfrak{R} \cdot b = 0$ (or $a \cdot \mathfrak{R}b = 0$) implies $a = 0$ or $b = 0$ for $a, b \in \mathfrak{R}$".

As at [1], let us define the following sequence of polynomials:

$$p_1(x) = x \text{ and } p_n(x_1, x_2, \ldots, x_n) = [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]$$

for all integers $n \geq 2$, then $p_2(x_1, x_2) = [x_1, x_2]$, $p_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$, etc. Let $n \geq 2$ be an integer. A linear map $D : A \longrightarrow A$ is called a *Lie n-derivation* if

$$D(p_n(x_1, x_2, \ldots, x_n)) = \sum_{i=1}^{n} p_n(x_1, x_2, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in A$. In particular, a Lie 2-derivation is a Lie derivation and a Lie 3-derivation is a Lie triple derivation. Lie 2-derivations, Lie 3-derivations and Lie n-derivations are collectively referred to as *Lie-type derivations*. A linear map $F : A \to A$ is said to be a generalized Lie n-derivation if there exists a Lie n-derivation $D : A \to A$ such that

$$\begin{aligned} F(p_n(x_1, x_2, \ldots, x_n)) \ &= \ p_n(F(x_1), x_2, \ldots, x_n) \\ &\quad + \ \sum_{i=2}^{n} p_n(x_1, x_2, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in A$. In particular, a generalized Lie 2-derivation is a linear map F that satisfies

$$F([x_1, x_2]) = [F(x_1), x_2] + [x_1, D(x_2)] \qquad \text{for all} \qquad x_1, x_2 \in A,$$

where D is a Lie derivation of A. A generalized Lie 2-derivation is a generalized derivation for the Lie product. Note, that any Lie n-derivation is an example of a generalized Lie n-derivation. Just like for Lie n-derivations we will also refer to generalized Lie 2-derivations, generalized Lie 3-derivations and generalized Lie n-derivations collectively as *generalized Lie-type derivations*.

The main purpose of the paper is to describe generalized Lie n-derivation of unital alternative algebras with idempotents, which satisfy (1).

In a recent work [22], Benkovič proved a result about generalized Lie derivations of unital algebras with idempotents.

The hypotheses in Benkovič's theorem [22] allowed the author to make its proof based on calculus using the Pierce decomposition notion for associative algebras.

The notion of the Pierce decomposition for alternative algebras is similar to that one for associative algebras. However, this similarity is restricted to its written form, not including its theoretical structure since the Pierce decomposition for alternative algebras is a generalization of that classical one for associative algebras.

Taking this fact into account, in the present paper many results can be seen as generalizations of Benkovič's results [22] to the class of alternative algebras.

2. PRELIMINARIES AND THE MAIN THEOREM

Let us start with a result that appears in [21] that will be very useful for us what characterize the commutative center of an alternative ring A:

Lemma 1.

$$\begin{aligned}\mathcal{Z}(A) = &\{z_{11}+z_{22} : z_{11}\in A_{11}, z_{22}\in A_{22},\\ &[z_{11}+z_{22}, A_{12}] = [z_{11}+z_{22}, A_{21}] = \{0\}\}.\end{aligned}$$

Proof. On the one hand assume that $z = z_{11}+z_{12}+z_{21}+z_{22}\in\mathcal{Z}(A)$.

Then $ze_1 = e_1z$ implies $z_{12}=z_{21}=0$. Furthermore, for any $x_{12}\in A_{12}$ and $x_{21}\in A_{21}$, it follows that $zx_{12} = x_{12}z$ and $zx_{21} = x_{21}z$ that

$$[z_{11}+z_{22}, A_{12}] = [z_{11}+z_{22}, A_{21}] = \{0\}.$$

On the other hand, assume that $z_{11}\in A_{11}, z_{22}\in A_{22}$, and

$$[z_{11}+z_{22}, A_{12}] = [z_{11}+z_{22}, A_{21}] = \{0\}.$$

To prove $z_{11}+z_{22}\in\mathcal{Z}(A)$, one only needs to check $z_{ii}\in\mathcal{Z}(A_{ii}), i=1,2$. In fact, for any $r_{11}\in A_{11}$ and any $r_{12}\in A_{12}$, we have

$$(z_{11}r_{11}-r_{11}z_{11})r_{12} = (z_{11}r_{11})r_{12}-(r_{11}z_{11})r_{12} = z_{11}(r_{11}r_{12})-r_{11}(z_{11}r_{12})$$

$$= (r_{11}r_{12})z_{22}-r_{11}(r_{12}z_{22}) = r_{11}(r_{12}z_{22})-r_{11}(r_{12}z_{22}) = 0.$$

Hence $(z_{11}r_{11}-r_{11}z_{11})A\cdot e_2 = 0$. Therefore $z_{11}\in\mathcal{Z}(A_{11})$ by condition of [21, Theorem 4]. Similarly, we can check $z_{22}\in\mathcal{Z}(A_{22})$.

□

We will follow with a result that is a generalization of Proposition 2.1 in [23].

Proposition 1. *The commutative center of A is*

$$\begin{aligned}Z(A) = &\{a_{11}+a_{22}\in A_{11}+A_{22} \mid [a_{11}+a_{22}, x_{12}] = 0, [a_{11}+a_{22}, x_{21}] = 0\\ &\textit{for all } x_{12}\in A_{12}, x_{21}\in A_{21}\}.\end{aligned} \tag{3}$$

Furthermore, there exists a unique algebra isomorphism $\tau : Z(A)e_1 \to Z(A)e_2$, such that $ax_{12} = x_{12}\tau(a)$ and $x_{21}a = \tau(a)x_{21}$ for all $x_{12}\in A_{12}$, $x_{21}\in A_{21}$ and for any $a\in Z(A)e_1$.

Proof. In Lemma 1 we saw that $Z(A)$ is of the desired form.

It is easy to see that $Z(A) \subset \{a_{11}+a_{22} \in A_{11}+A_{22} \mid [a_{11}+a_{22}, x_{12}] = 0, [a_{11}+a_{22}, x_{21}] = 0$ for all $x_{12} \in A_{12}, x_{21} \in A_{21}\}$. Let $a_{11}+a_{22} \in A_{11}+A_{22}$ be such that $[a_{11}+a_{22}, x_{12}] = 0$, $[a_{11}+a_{22}, x_{21}] = 0$ for all $x_{12} \in A_{12}$, $x_{21} \in A_{21}$. To show that $a_{11}+a_{22} \in Z(A)$, we just prove that $[a_{11}+a_{22}, x_{11}] = 0$ and $[a_{11}+a_{22}, x_{22}] = 0$ for all $x_{11} \in A_{11}$, $x_{22} \in A_{22}$ respectively. Indeed, by flexible identity linearization we have

$$\begin{aligned}
[a_{11}+a_{22}, x_{11}]x_{12} &= (a_{11}x_{11})x_{12} - (x_{11}a_{11})x_{12} \\
&= a_{11}(x_{11}x_{12}) - x_{11}(a_{11}x_{12}) \\
&= (x_{11}x_{12})a_{22} - x_{11}(x_{12}a_{22}) \\
&\quad ([a_{11}+a_{22}, x_{12}] = 0 \text{ for all } x_{12} \in A_{12}) \\
&= x_{11}(x_{12}a_{22}) - x_{11}(x_{12}a_{22}) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
x_{21}[a_{11}+a_{22}, x_{11}] &= x_{21}(a_{11}x_{11}) - x_{21}(x_{11}a_{11}) \\
&= (x_{21}a_{11})x_{11} - (x_{21}x_{11})a_{11} \\
&= (a_{22}x_{21})x_{11} - a_{22}(x_{21}x_{11}) \\
&\quad ([a_{11}+a_{22}, x_{21}] = 0 \text{ for all } x_{21} \in A_{21}) \\
&= a_{22}(x_{21}x_{11}) - a_{22}(x_{21}x_{11}) \\
&= 0.
\end{aligned}$$

Since $[a_{11}+a_{22}, x_{11}]A_{12} = 0 = A_{21}[a_{11}+a_{22}, x_{11}]$, assumption (1) implies $[a_{11}+a_{22}, x_{11}] = 0$ for all $x_{11} \in A_{11}$. Similarly, we can prove that $[a_{11}+a_{22}, x_{22}] = 0$ for all $x_{22} \in A_{22}$, whence it follows that $Z(A) \supset \{a_{11}+a_{22} \in A_{11}+A_{22} \mid [a_{11}+a_{22}, x_{12}] = 0, [a_{11}+a_{22}, x_{21}] = 0$ for all $x_{12} \in A_{12}, x_{21} \in A_{21}\}$. Therefore commutative center is of the desired form.

Now by direct calculation using (3) we get $Z(A)e_1$ is a subalgebra of $Z(A_{11})$ and $Z(A)e_2$ is a subalgebra of $Z(A_{22})$. Clearly, for each $a \in Z(A)e_1$, there exists $b \in Z(A)e_2$ such that $a+b \in Z(A)$. We can write $\tau(a) = b$, because if $ax_{12} = x_{12}b = x_{12}b'$ and $x_{21}a = bx_{21} = b'x_{21}$ for all $x_{12} \in A_{12}$, $x_{21} \in A_{21}$ then $b = b'$ by assumption of (1). That means that there exists a unique $b = \tau(a) \in Z(A)e_2$ such that $a+b \in Z(A)$. For any $a, a' \in Z(A)e_1$ and $\lambda \in F$, we have

$$(\lambda a)x_{12} = \lambda(ax_{12}) = \lambda(x_{12}\tau(b)) = x_{12}(\lambda\tau(b)),$$

$$x_{21}(\lambda a) = \lambda(x_{21}a) = \lambda(\tau(b)x_{21}) = (\lambda\tau(b))x_{21},$$

$$(a+a')x_{12} = x_{12}(\tau(a)+\tau(a')),$$

$$x_{21}(a+a') = (\tau(a)+\tau(a'))x_{21},$$

and by linearization of flexible identity

$$\begin{aligned}
(aa')x_{12} &= a(a'x_{12}) = (a'x_{12})\tau(a) = a'(x_{12}\tau(a)) = (x_{12}\tau(a))\tau(a') \\
&= x_{12}(\tau(a)\tau(a')),
\end{aligned}$$

$$\begin{aligned}
x_{21}(aa') &= (x_{21}a)a' = \tau(a')(x_{21}a) = (\tau(a')x_{21})a = \tau(a)(\tau(a')x_{21}) \\
&= (\tau(a)\tau(a'))x_{21},
\end{aligned}$$

for all $x_{12} \in A_{12}$ and $x_{21} \in A_{21}$. Therefore $\tau(\lambda a) = \lambda\tau(a)$, $\tau(a+a') = \tau(a)+\tau(a')$ and $\tau(aa') = \tau(a)\tau(a')$, by assumption of (1) and the proof of the proposition is now complete. □

The following result, taken from [21, Lemma 8] plays a crucial role in this paper.

Proposition 2. *For $z_{ii} \in Z(A_{ii})$, $i = 1, 2$, there exists an element $z \in Z(A)$ such that $z_{ii} = ze_i$.*

Proof. Since A_{ii} is 3-torsion free alternative ring we get $\mathcal{Z}(A_{ii}) \subseteq \mathcal{N}(A_{ii})$. Let be $z_{ii} \in \mathcal{Z}(A_{ii})$, it is clear that $e_i x z_{ii} = z_{ii} x e_i$ holds for all $x \in A$ then, [24, Lemma 4], there is an element $z \in \mathcal{Z}(A)$ such that $z_{ii} = ze_i$. □

By a simple calculation we can easily see:

Remark 1. Let A be a unital alternative algebra with a nontrivial idempotent e_1 and $e_2 = 1 - e_1$. For any $x \in A$ with $x = x_{11} + x_{12} + x_{21} + x_{22}$ and for any integer $n \geq 2$ we have

$$p_n(x, e_1, \ldots, e_1) = (-1)^{n-1} x_{12} + x_{21}$$

and

$$p_n(x, e_2, \ldots, e_2) = x_{12} + (-1)^{n-1} x_{21}.$$

In particular, $[x, e_1] = -x_{12} + x_{21}$ and $[x, e_2] = x_{12} - x_{21}$.

The main result of the paper is as follows:

Theorem 1. *Let A be a unital alternative algebra with a nontrivial idempotent e_1 satisfying* (1).
Let us assume that

(i) $Z(A_{11}) = Z(A)e_1$,

(ii) $Z(A_{22}) = Z(A)e_2$.

Then any generalized Lie-type derivation $F : A \to A$ is of the form $F(x) = \lambda x + \Xi(x)$ for all $x \in A$, where $\lambda \in Z(A)$ and $\Xi : A \to A$ is a Lie-type derivation.

With the purpose of proving our main theorem, we will initially generalize the result that follows for alternative algebras.

According to Benkovič [22], "If $F : A \to A$ is a generalized Lie n-derivation associated with a Lie n-derivation D, then a linear map $H = F - D$ satisfies

$$H(p_n(x_1, x_2, \ldots, x_n)) = p_n(H(x_1), x_2, \ldots, x_n) \tag{4}$$

for all $x_1, x_2, \ldots, x_n \in A$. So, it suffices to consider linear maps with property (4). Note that H is actually a generalized Lie n-derivation whose associated Lie n-derivation is the zero map."

Proposition 3. *Let A be a unital alternative algebra with a nontrivial idempotent e_1 satisfying* (1).
Let us assume that $Z(A_{11}) = Z(A)e_1$ and $Z(A_{22}) = Z(A)e_2$.
If a linear map $H : A \to A$ satisfies

$$H(p_n(x_1, x_2, \ldots, x_n)) = p_n(H(x_1), x_2, \ldots, x_n) \tag{5}$$

for all $x_1, x_2, \ldots, x_n \in A$, then $H(x) = \lambda x + \gamma(x)$ for all $x \in A$, where $\lambda \in Z(A)$ and $\gamma : A \to Z(A)$ is a linear map such that $\gamma(p_n(A, \ldots, A)) = 0$.

Remark 2. The next results are generalizations of the results obtained by Benkovič [22] for the case of alternative algebras. It is important to observe these results are held because the following properties the Pierce decomposition for alternative algebras are valid as we can see in [20, Proposition 3.4]:

(i) $(x_{ij}, y_{jk}, z_{ki}) = 0$ if $(i, j, k) \neq (i, i, i)$, for all $x_{ij} \in A_{ij}$, $y_{jk} \in A_{jk}$ and $z_{ki} \in A_{ki}$;

(ii) $(x_{ij} y_{ij}) z_{ij} = (y_{ij} z_{ij}) x_{ij} = (z_{ij} x_{ij}) y_{ij}$ if $i \neq j$ for all $x_{ij}, y_{ij}, z_{ij} \in A_{ij}$;

(iii) $x_{ij}(y_{ij} z_{jj}) = (x_{ij} z_{jj}) y_{ij} = z_{jj}(x_{ij} y_{ij})$ if $i \neq j$ for all $x_{ij}, y_{ij} \in A_{ij}$, $z_{jj} \in A_{jj}$;

(iv) $x_{ij}(z_{ii} y_{ij}) = (z_{ii} x_{ij}) y_{ij} = (x_{ij} y_{ij}) z_{ii}$ if $i \neq j$ for all $x_{ij}, y_{ij} \in A_{ij}$, $z_{ii} \in A_{ii}$.

To make it clearer, we divide the proof of Proposition 3 into some Lemmas. We will start with this one

Lemma 2. *We have* $H(e_1), H(e_2) \in A_{11} + A_{22}$.

Proof. Using Remark 1 and (5) we have

$$H(p_n(e_1, e_2, \ldots, e_2)) = p_n(H(e_1), e_2, \ldots, e_2) = H(e_1)_{12} + (-1)^{n-1}H(e_1)_{21}$$

and

$$H(p_n(e_2, e_1, \ldots, e_1)) = p_n(H(e_2), e_1, \ldots, e_1) = (-1)^{n-1}H(e_2)_{12} + H(e_2)_{21}.$$

Since $H(p_n(e_1, e_2, \ldots, e_2)) = H(p_n(e_2, e_1, \ldots, e_1)) = 0$, it follows that $H(e_1)_{12} = H(e_1)_{21} = H(e_2)_{12} = H(e_2)_{21} = 0$. Therefore, $H(e_1), H(e_2) \in A_{11} + A_{22}$. □

Lemma 3. *Let* $Z(A)$ *be the commutative center of* A, *hence we get* $H(e_i)_{jj} \in Z(A)e_j$ *where* $i, j \in \{1, 2\}$ *and* $i \neq j$.

Proof. Let us start with $n = 2$, using Lemma 2 we have

$$0 = H([e_i, x_{jj}]) = [H(e_i), x_{jj}] = [H(e_i)_{jj}, x_{jj}]$$

for all $x_{jj} \in A_{jj}$ implying $H(e_i)_{jj} \in Z(A_{jj})$. By assumption $Z(A_{jj}) = Z(A)e_j$, therefore the Lemma holds to $n = 2$. Now consider $n \geq 3$, $x_{jj} \in A_{jj}$, $x_{ij} \in A_{ij}$ and $x_{ji} \in A_{ji}$ with $i \neq j$ arbitrary elements. We can write

$$H(p_n(e_i, x_{jj}, x_{ij}, e_j, \ldots, e_j)) = 0,$$

$$H(p_n(e_i, x_{jj}, x_{ji}, e_i, \ldots, e_i)) = 0,$$

$$p_n(H(e_i), x_{jj}, x_{ij}, e_j, \ldots, e_j) = -x_{ij}[H(e_i)_{jj}, x_{jj}],$$

and

$$p_n(H(e_i), x_{jj}, x_{ji}, e_i, \ldots, e_i) = [H(e_i)_{jj}, x_{jj}]x_{ji}.$$

By assumption of (1) we get $[H(e_i)_{jj}, x_{jj}] = 0$. Hence, $H(e_i)_{jj} \in Z(A_{jj}) = Z(A)e_j$ and the Lemma holds to $n \geq 3$. □

Lemma 4. *Let be* $\tau : Z(A)e_1 \to Z(A)e_2$ *the isomorphism of Proposition 1,* $\alpha = H(e_1)_{11} - \tau^{-1}(H(e_1)_{22}) \in A_{11}$ *and* $\beta = H(e_2)_{22} - \tau(H(e_2)_{11}) \in A_{22}$, *so we get* $\alpha + \beta \in Z(A)$.

Proof. Using the property of τ we have

$$[H(e_1), x_{12}] = H(e_1)_{11}x_{12} - x_{12}H(e_1)_{22} = (H(e_1)_{11} - \tau^{-1}(H(e_1)_{22}))x_{12} = \alpha x_{12},$$

$$[H(e_2), x_{12}] = H(e_2)_{11}x_{12} - x_{12}H(e_2)_{22} = x_{12}(\tau(H(e_2)_{11} - H(e_2)_{22}) = -x_{12}\beta,$$

for all $x_{12} \in A_{12}$. Now on the one hand we get,

$$\begin{aligned} H(x_{12}) &= H([e_1, x_{12}]) = H(p_n(e_1, x_{12}, e_2, \ldots, e_2)) \\ &= p_n(H(e_1), x_{12}, e_2, \ldots, e_2) = p_{n-1}([H(e_1), x_{12}], e_2, \ldots, e_2) \\ &= p_{n-1}(\alpha x_{12}, e_2, \ldots, e_2) = \alpha x_{12}. \end{aligned}$$

And on the other hand,

$$\begin{aligned} H(x_{12}) &= -H([e_2, x_{12}]) = -H(p_n(e_2, x_{12}, e_2, \ldots, e_2)) \\ &= -p_n(H(e_2), x_{12}, e_2, \ldots, e_2) = -p_{n-1}([H(e_2), x_{12}], e_2, \ldots, e_2) \\ &= -p_{n-1}(-x_{12}\beta, e_2, \ldots, e_2) = x_{12}\beta, \end{aligned}$$

for all $x_{12} \in A_{12}$. Hence $[\alpha + \beta, x_{12}] = 0$ for all $x_{12} \in A_{12}$. By a strictly analogous proof, we have $[\alpha + \beta, x_{21}] = 0$ for all $x_{21} \in A_{21}$. Therefore $\alpha + \beta \in Z(A)$. □

Lemma 5. *Let the linear map $\gamma : A \to A$ be defined by $\gamma(x) = H(x) - \lambda x$ for all $x \in A$ where $\lambda = \alpha_{11} + \alpha_{22} \in Z(A)$. Then γ satifies the properties of Proposition 3.*

Proof. Firstly observe

$$\gamma(p_n(x_1, x_2, \ldots, x_n)) = p_n(\gamma(x_1), x_2, \ldots, x_n).$$

For all $x_{ij} \in A_{ij}$ with $i \neq j$ we have $\gamma(x_{ij}) = 0$. Indeed,

$$\gamma(x_{ij}) = H(x_{ij}) - \lambda x_{ij} = \alpha_{ii}x_{ij} - (\alpha_{11} + \alpha_{22})x_{ij} = 0.$$

Let $x_{ii} \in A_{ii}$ be as $\gamma(p_n(x_{ii}, e_j, \ldots, e_j)) = p_n(\gamma(x_{ii}), e_j, \ldots, e_j)$, so $\gamma(x_{ii})_{ij} = 0$ for $i \neq j$. Now,

$$\begin{aligned} 0 = \gamma(x_{ii}x_{ij}) &= \gamma(p_n(x_{ii}, x_{ij}, e_j, \ldots, e_j)) \\ &= p_n(\gamma(x_{ii}), x_{ij}, e_j, \ldots, e_j) \\ &= [\gamma(x_{ii})_{ii} + \gamma(x_{ii})_{jj}, x_{ij}] \end{aligned}$$

and

$$\begin{aligned} 0 = \gamma(x_{ji}x_{ii}) &= \gamma(p_n(x_{ii}, x_{ji}, e_i, \ldots, e_i)) \\ &= p_n(\gamma(x_{ii}), x_{ji}, e_i, \ldots, e_i) \\ &= [\gamma(x_{ii})_{ii} + \gamma(x_{ii})_{jj}, x_{ji}]. \end{aligned}$$

Therefore, $\gamma(x_{ii}) = \gamma(x_{ii})_{11} + \gamma(x_{ii})_{22} \in Z(A)$ and γ maps into the center of A. By property $\gamma(p_n(x_1, x_2, \ldots, x_n)) = p_n(\gamma(x_1), x_2, \ldots, x_n)$ clearly we have $\gamma(p_n(A, A, \ldots, A)) = 0$. □

From the Lemmas proved above we have that Proposition 3 holds. Now we are ready to prove Theorem 1.

Proof of Theorem 1: Let $F : A \to A$ be a generalized Lie-type derivation with an associated Lie-type derivation D. According to Benkovič [22] we consider $H = F - D$ and clearly $H(p_n(x_1, x_2, \ldots, x_n)) = p_n(H(x_1), x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in A$. Now using Proposition 3 we have $F(x) = \lambda x + D(x) + \gamma(x)$ with $\lambda \in Z(A)$. We observe that $\Xi = D + \gamma$ is a Lie-type derivation and Theorem 1 has been proved.

Due to the proof of Theorem 1, we have the following

Theorem 2. *Let A be a unital alternative algebra with a nontrivial idempotent e_1 satisfying* (2).

Let us assume that

(i) $Z(A_{11}) = Z(A)e_1$,

(ii) $Z(A_{22}) = Z(A)e_2$.

Then any generalized Lie-type derivation $F : A \to A$ is of the form $F(x) = \lambda x + \Xi(x)$ for all $x \in A$, where $\lambda \in Z(A)$ and $\Xi : A \to A$ is a Lie-type derivation.

As a consequence we have the following result

Corollary 1 ([22], Theorem 2.3)**.** Let A be a unital associative algebra with a nontrivial idempotent e_1 satisfying (2).

Let us assume that

(i) $Z(A_{11}) = Z(A)e_1$,

(ii) $Z(A_{22}) = Z(A)e_2$.

Then any generalized Lie-type derivation $F : A \to A$ is of the form $F(x) = \lambda x + \Xi(x)$ for all $x \in A$, where $\lambda \in Z(A)$ and $\Xi : A \to A$ is a Lie-type derivation.

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