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Guiding Functional Families, Lyapunov Vector Functions, and the Existence of Poisson Bounded Solutions

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Abstract—In this paper, we use the method of guiding functional families and the method of Lyapunov vector functions for establishing a sufficient condition for the existence of Poisson bounded solutions to systems of differential equations and a sufficient condition for the existence of partially Poisson bounded solutions to such systems.

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In the paper [1], V.M. Matrosov proposes a method of Lyapunov vector functions which generalizes the classical method of Lyapunov functions [2] for studying the stability of the equilibrium state of a dynamic system. He also applies this method [1] for studying conditions which ensure the boundedness of all solutions to an arbitrary system of differential equations. Independently of the method of Lyapunov functions, M.A. Krasnosel'skii [3] (see also the paper [4] by V.G. Zvyagin and S.V. Kornev) applies the technique of operators of translation along trajectories and rotations of vector fields for developing the method of guiding and preguiding functional families. This method allows one to study conditions for the existence of at least one bounded solution to any nonlinear system.

On the other hand, in the papers [5]–[7], we commence to develop a new branch in the theory of the boundedness of solutions to systems of differential equations, namely, we develop the theory of the Poisson boundedness of solutions. The Poisson boundedness of a solution means that this solution does not entirely belong to a certain ball in the phase space, but it returns to this ball countably many times. The mentioned papers are devoted only to studying conditions, which ensure the Poisson boundedness of the totality of all solutions to a system; so it makes sense to study conditions for the existence of at least one Poisson bounded solution to an arbitrary nonlinear system. In this paper, we develop a technique for studying conditions for the existence of Poisson bounded solutions; it represents a synthesis of the method of Lyapunov vector functions and the method of guiding and preguiding functional families. With the help of this method, we establish sufficient conditions for the existence of Poisson bounded solutions and partially Poisson bounded solutions. Let us now give exact definitions and statements.

Consider an arbitrary system of differential equations of n variables

$$\frac{dx}{dt} = f(t,x), \quad f(t,x) = (f_1(t,x),\ldots,f_n(t,x))^T, \tag{1}$$

whose right-hand side is given and continuous in $\mathbb{R}^+ \times \mathbb{R}^n$; here $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$. We assume that the function $f(t,x)$ satisfies the Lipschitz condition with respect to the variable x and, in addition, solutions to system (1) are extendable to the whole semiaxis $\mathbb{R}^+$.

In what follows, the symbol $\|\cdot\|$ stands for the usual Euclidean norm in $\mathbb{R}^n$, $n \geqslant 1$. We denote the solution $x = x(t)$ to system (1) that starts at the point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ as $x = x(t, t_0, x_0)$.

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For any $t_0 \in \mathbb{R}^+$, we use the denotation $\mathbb{R}^+(t_0)$ for the set $\{t \in \mathbb{R} \mid t \geqslant t_0\}$. We understand a $\mathcal{P}$-sequence as a nonnegative increasing numerical sequence $\tau = \{\tau_i\}_{i\geqslant 1}$ such that $\lim\limits_{i\to\infty} \tau_i = +\infty$. For each $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geqslant 1}$, we use the symbol $M(\tau)$ for the set $\bigcup\limits_{i=1}^{\infty} [\tau_{2i-1}; \tau_{2i}]$.

Recall [8] that the solution $x = x(t, t_0, x_0)$ to system (1) is said to be bounded, if one can find a value $\beta > 0$ for this solution so as to fulfill the condition $\|x(t, t_0, x_0)\| \leqslant \beta$ for all $t \in \mathbb{R}^+(t_0)$.

Definition 1 ([5]). The solution $x = x(t, t_0, x_0)$ to system (1) is said to be Poisson bounded, if one can find a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geqslant 1}$, where $t_0 \in M(\tau)$, and a value $\beta > 0$ such that $\|x(t, t_0, x_0)\| \leqslant \beta$ for all $t \in R^+(t_0) \bigcap M(\tau)$.

In the geometric sense, Definition 1 means that the solution, which starts at a certain time moment in a ball of radius $\beta > 0$ centered at the origin of coordinates, returns to this ball countably many times. Clearly, if a solution to system (1) is bounded, then it is also Poisson bounded.

For introducing denotations, let us recall (see [9]) certain properties of Lyapunov vector functions. The derivative of a given continuously differentiable vector function

$$v(t, x) = (v_1(t, x), \ldots, v_k(t, x))^T, \quad k \geqslant 1, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

in view of system (1) obeys the equality $\dot{v}(t, x) = (\dot{v}_1(t, x), \ldots, \dot{v}_k(t, x))^T$, where $\dot{v}_i(t, x)$ is the derivative (in view of system (1)) of the function $v_i(t, x)$, $1 \leqslant i \leqslant k$. For vectors $\xi = (\xi_1, \ldots, \xi_k)^T, \eta = (\eta_1, \ldots, \eta_k)^T \in \mathbb{R}^k$ we write $\xi \leqslant \eta$, if $\xi_i \leqslant \eta_i$ for any $1 \leqslant i \leqslant k$. Let now the following continuous vector function be given:

$$g(t, \xi) = (g_1(t, \xi), \ldots, g_k(t, \xi))^T, \quad (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^k.$$

We write $g(t, \xi) \in W$, if $g(t, \xi)$ satisfies the Wazewski condition, i. e., for each $1 \leqslant s \leqslant k$ the function $g_s(t, \xi)$ is nondecreasing in variables $\xi_1, \ldots, \xi_{s-1}, \xi_{s+1}, \ldots, \xi_k$; in other words, the condition $\xi_i \leqslant \eta_i$, $1 \leqslant i \leqslant k$, $i \neq s$, $\xi_s = \eta_s$ implies that $g_s(t, \xi) \leqslant g_s(t, \eta)$.

The continuously differentiable vector function $v(t, x) \geqslant (0 \in \mathbb{R}^k)$ and the system

$$\frac{d\xi}{dt} = g(t, \xi), \quad g(t, \xi) \in W, \tag{2}$$

are called, respectively, the Lyapunov vector function and the comparison system for system (1), if

$$\dot{v}(t, x) \leqslant g(t, v(t, x)).$$

In what follows, we always assume that the right-hand side of system (2) satisfies the Lipschitz condition in the variable ξ. Since the Cauchy problem for system (2) has a unique solution, according to the Wazewski theorem (see, e. g., [9]), for any point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ the solution $x(t, t_0, x_0)$ to system (1), the Lyapunov vector function $v(t, x)$, and the solution $\xi(t, t_0, v(t_0, x_0))$ to the comparison system (2) for system (1) are interconnected with all $t \geqslant t_0$ by the following inequality:

$$v(t, x(t, t_0, x_0)) \leqslant \xi(t, t_0, v(t_0, x_0)). \tag{3}$$

Let us now recall some necessary notions and constructions connected with rotations of vector fields and operators of translation along trajectories [3] (see also [4]). Let the symbol B_r^n stand for an n-dimensional ball of radius r in $\mathbb{R}^n$ centered at the origin of coordinates. Following [3], we understand a continuous vector field or, for short, a vector field ψ on B_r^n as any continuous map $\psi : B_r^n \to \mathbb{R}^n$. Clearly, for any vector field ψ on B_r^n we can always consider its narrowing on the $(n-1)$-dimensional sphere $S_r^{n-1} = \partial B_r^n$ of radius r, i. e., the vector field $\psi : S_r^{n-1} \subset B_r^n \to \mathbb{R}^n$. A vector field ψ on B_r^n is said to be nondegenerate on S_r^{n-1}, if $\psi(x) \neq 0 \in \mathbb{R}^n$ for all $x \in S_r^{n-1}$. One can easily see that any vector field $\psi : B_r^n \to \mathbb{R}^n$, which is nondegenerate on S_r^{n-1}, defines a continuous map

$$\zeta : S_r^{n-1} \to S_r^{n-1}, \quad \zeta(x) = r\frac{\psi(x)}{\|\psi(x)\|}, \quad x \in S_r^{n-1}.$$

We understand the rotation $\gamma(\psi, S_r^{n-1})$ of a nondegenerate on S_r^{n-1} vector field $\psi : B_r^n \to \mathbb{R}^n$ as the degree $\deg(\zeta) \in \mathbb{Z}$ of the map $\zeta : S_r^{n-1} \to S_r^{n-1}$. One can easily find the integer value $\deg(\zeta)$, for example, with the help of the functor $H_{n-1}(-;\mathbb{Z})$ of singular $(n-1)$-dimensional homologies of topological spaces with integer coefficients [10]. Really, a continuous map $\zeta : S_r^{n-1} \to S_r^{n-1}$ induces a homomorphism of groups of singular homologies

$$H_{n-1}(\zeta;\mathbb{Z}) : H_{n-1}(S_r^{n-1};\mathbb{Z}) \to H_{n-1}(S_r^{n-1};\mathbb{Z}).$$

As is well known (see, for example, [10]), the group $H_{n-1}(S_r^{n-1};\mathbb{Z})$ is isomorphic to the group $\mathbb{Z}$, and the generator of this group is the fundamental class $[S_r^{n-1}]$ of the closed oriented smooth manifold S_r^{n-1}. In terms of the group homomorphism $H_{n-1}(\zeta;\mathbb{Z})$ and the fundamental class $[S_r^{n-1}]$, the map degree $\deg(\zeta) \in \mathbb{Z}$ obeys the formula

$$H_{n-1}(\zeta;\mathbb{Z})([S_r^{n-1}]) = \deg(\zeta)[S_r^{n-1}].$$

We treat the subset $Tr(x_0) = \{x \in \mathbb{R}^n \mid x = x(t,0,x_0),\ t \geqslant 0\} \subset \mathbb{R}^n$, where $x(t,0,x_0)$ is the solution to system (1), while x_0 is an arbitrary point in $\mathbb{R}^n$, as the trajectory of system (1) that starts at the point x_0. For arbitrarily fixed $\tau > 0$, let us consider a continuous map

$$u(\tau) : B_r^n \to \mathbb{R}^n, \quad u(\tau)(x_0) = x(\tau, 0, x_0),$$

where $x(t,0,x_0)$ is the solution to system (1) and x_0 is an arbitrary point in B_r^n. The map $u(\tau)$ is said to be [3] the operator of translation along trajectories of system (1) within time $0 \leqslant t \leqslant \tau$. We understand a τ-irrevocability point of a trajectory of system (1) [3] as a point $x_0 \in \mathbb{R}^n$ such that the solution $x(t,0,x_0)$ to system (1) satisfies the condition $x(t,0,x_0) \neq x_0$ with all $0 < t \leqslant \tau$. Let us now consider the vector field

$$\psi_0 : B_r^n \to \mathbb{R}^n, \quad \psi_0(x) = -f(0,x),$$

where $f(t,x)$ is the right-hand side of system (1). The rotation $\gamma(\psi_0, S_r^{n-1})$ of this vector field is closely connected with the problem of the existence of fixed points of the operator $u(\tau)$ of translation along trajectories of system (1). Really, in [3], M.A. Krasnosel'skii proves that if a nondegenerate on S_r^{n-1} vector field $\psi_0 : B_r^n \to \mathbb{R}^n$ has the rotation $\gamma(\psi_0, S_r^{n-1}) \neq 0$ and all points in S_r^{n-1} are τ-irrevocability points of trajectories of system (1), then inside B_r^n there exists at least one fixed point $x \in B_r^n \setminus S_r^{n-1}$ of the operator $u(\tau)$ of translation along trajectories of system (1) (i. e., such a point that $u(\tau)(x) = x$). Below we use the following notions. We treat subsets

$$\mathrm{Tr}^+(x_0, t_0) = \{x \in \mathbb{R}^n \mid x = x(t, t_0, x_0),\ t > t_0\} \subset \mathbb{R}^n,$$

$$\mathrm{Tr}^-(x_0, t_0) = \{x \in \mathbb{R}^n \mid x = x(t, t_0, x_0),\ 0 \leqslant t \leqslant t_0\} \subset \mathbb{R}^n,$$

where $x(t,t_0,x_0)$ is the solution to system (1), while (t_0,x_0) is an arbitrary point in $\mathbb{R}^+ \times \mathbb{R}^n$, correspondingly, as the right- and left-hand part of the trajectory $\mathrm{Tr}(x(0,t_0,x_0))$ of system (1).

Let us now formulate and prove the following sufficient condition for the existence of Poisson bounded solutions to system (1).

Proposition 1. *Assume that for system* (1) *one can find a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\geqslant 1}$, *a nonincreasing function* $b : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $b(r) \to +\infty$ *as* $r \to +\infty$, *and a Lyapunov vector function* $v(t,x)$ *with comparison system* (2) *such that the inequality*

$$b(\|x\|) \leqslant \sum_{i=1}^{k} v_i(t,x) \tag{4}$$

is valid with any $(t,x) \in M(\tau) \times \mathbb{R}^n$. *Assume also that there exists a value* $r > 0$ *such that the following conditions are fulfilled:*

1) *for any* $\varrho_0 \in S_r^{k-1}$ *the right-hand part* $\mathrm{Tr}^+(\varrho_0, t_0)$ *of the trajectory* $\mathrm{Tr}(\varrho(0,t_0,\varrho_0))$ *of the system*

$$\frac{d\varrho}{dt} = p(t,\varrho), \quad (t,\varrho) \in \mathbb{R}^+ \times \mathbb{R}^k, \quad p(t,\varrho) = g(t, \varrho + \overline{r}), \quad \overline{r} = (r, \dots, r) \in \mathbb{R}^k, \tag{5}$$

where $g(t,\xi)$ is the right-hand side of system (2), *in B_r^k has no common point with the left-hand part $\mathrm{Tr}^-(\varrho_0,t_0)$ of this trajectory;*

2) *vector field $\psi_0 : B_r^k \to \mathbb{R}^k$, $\psi_0(\varrho) = -p(0,\varrho)$, is nondegenerate on S_r^{k-1} and $\gamma(\psi_0, S_r^{k-1}) \neq 0$;*

3) $B_r^k(\overline{r}) = \{\xi \in \mathbb{R}^k \mid \|\,\xi - \overline{r}\| \leqslant r\} \subset \mathrm{Im}\,(v : \{0\} \times \mathbb{R}^n \to \mathbb{R}^k)$.

Then system (1) *has at least one Poisson bounded solution.*

Proof. For each integer $m \geqslant 1$ let us consider the operator of translation $u(m) : B_r^k \to \mathbb{R}^k$ along trajectories of system (5) within time $0 \leqslant t \leqslant m$. Condition 1) of the theorem implies that for any $m \geqslant 1$ all points in S_r^{k-1} are m-irrevocability points of trajectories of system (5). In addition, according to condition 2) of the theorem, $\gamma(\psi_0, S_r^{k-1}) \neq 0$. As was mentioned above, this means that for each $m \geqslant 1$ the translation operator $u(m)$ has a fixed point $\vartheta_m \in B_r^k \smallsetminus S_r^{k-1}$. Consider the family of solutions $\{\varrho(t,0,\vartheta_m)\}_{m\geqslant 1}$ to system (5). Condition 2) of the mentioned theorem implies that $\varrho(t,0,\vartheta_m) \in B_r^k \smallsetminus S_r^{k-1}$ with any $0 \leqslant t \leqslant m$. Really, assuming the contrary, we conclude that for some point $\varrho_0 = \varrho(t_0,0,\vartheta_m) \in \mathrm{Tr}(\vartheta_m)$, where $0 < t_0 < m$ and $\varrho_0 \in S_r^{k-1}$,

$$\mathrm{Tr}^+(t_0,\varrho_0) \cap Tr^-(t_0,\varrho_0) \cap B_r^k = \{\vartheta_m\} \neq \varnothing,$$

but this contradicts condition 2) of the theorem. Consider in $B_r^k \smallsetminus S_r^{k-1}$ the sequence of points $(\vartheta_m)_{m\geqslant 1}$. Since the set B_r^k is compact, we can choose in the sequence $(\vartheta_m)_{m\geqslant 1}$ a subsequence $(\vartheta_{m_i})_{i\geqslant 1}$ which converges to some point $\mu \in B_r^k$. Let us prove that the solution $\varrho(t,0,\mu)$ to system (5) with all $t \geqslant 0$ satisfies the condition $\varrho(t,0,\mu) \in B_r^k$. Assume the contrary. Then there exists a number $\eta \in \mathbb{R}^+$ such that $\varrho(\eta,0,\mu) \notin B_r^k$. Since the comparison system (2) satisfies assumptions of the theorem about the continuous dependence on initial conditions (see, for example, [11]), so does system (5). Therefore, $\varrho(\eta,0,\vartheta_{m_i}) \notin B_r^k$ with sufficiently large i and $\eta \leqslant m_i$. This contradicts the fact that

$$\varrho(t,0,\vartheta_{m_i}) \in B_r^k \smallsetminus S_r^{k-1} \subset B_r^k \quad \text{with all} \quad 0 \leqslant t \leqslant m_i.$$

Thus, we have proved that $\varrho(t,0,\mu) \in B_r^k$, i. e., $\|\varrho(t,0,\mu)\| \leqslant r$ for any $t \geqslant 0$. Let us now prove that system (1) has a Poisson bounded solution $x(t,0,x_0)$ with certain $x_0 \in \mathbb{R}^n$. Since the change of variables $\xi = \varrho + \overline{r}$ in system (2) turns it into system (5), the solution $\xi(t,0,\mu+\overline{r}) = \varrho(t,0,\mu) + \overline{r}$ to system (2) satisfies the condition $\xi(t,0,\mu+\overline{r}) \in B_r^k(\overline{r})$ with all $t \geqslant 0$. Since $\mu + \overline{r} \in B_r^k(\overline{r})$, condition 3) of the theorem implies that there exists a point $(0,x_0) \in \{0\} \times \mathbb{R}^n$ such that $v(0,x_0) = \mu + \overline{r}$. Let us choose $R > r$ such that $B_r^k(\overline{r}) \subset B_R^k$. Clearly, $\|\xi(t,0,\mu+\overline{r})\| \leqslant R$ with all $t \geqslant 0$. Making use of inequalities (4) and (3), we get the following inequalities for the solution $x(t,0,x_0)$ of system (1) and the solution $\xi(t,0,v(0,x_0))$ of the comparison system (2):

$$b(\|x(t,0,x_0)\|) \leqslant \sum_{i=1}^{k} v_i(t, x(t,0,x_0)) \leqslant \sum_{i=1}^{k} \xi_i(t,0,v(0,x_0));$$

these inequalities are valid with all $t \in M(\tau)$. Moreover, for any $t \geqslant 0$ we get evident inequalities

$$\sum_{i=1}^{k} \xi_i(t,0,v(0,x_0)) \leqslant \sum_{i=1}^{k} |\,\xi_i(t,0,v(0,x_0))| \leqslant k\|\,\xi(t,0,v(0,x_0))\|.$$

Since $v(0,x_0) = \mu + \overline{r}$, we conclude that $\|\xi(t,0,v(0,x_0))\| \leqslant R$ with all $t \geqslant 0$. Hence and from above inequalities we conclude that $b(\|x(t,0,x_0)\|) \leqslant kR$ with all $t \in \mathbb{R}^+(0) \bigcap M(\tau)$. Making use of the fact that $b(r) \to +\infty$ as $r \to +\infty$, while the value kR is fixed, we can choose a number $\beta > 0$ such that $kR \leqslant b(\beta)$. Consequently, we conclude that $b(\|x(t,0,x_0)\|) \leqslant b(\beta)$ for all $t \in \mathbb{R}^+(0) \bigcap M(\tau)$. Since the function $b(r)$ is nonincreasing, the latter inequality implies that $\|x(t,0,x_0)\| \leqslant \beta$ with all $t \in \mathbb{R}^+(0) \bigcap M(\tau)$. Therefore, we have proved the Poisson boundedness of the solution $x(t,0,x_0)$ to system (1). □

In what follows, for each $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $n \geqslant 2$, and any fixed $1 \leqslant m < n$ we use the notation $x = (y, z)$, where $y = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$ and $z = (x_{m+1}, \ldots, x_n)^T \in \mathbb{R}^{n-m}$.

Recall [9] that the solution $x(t, t_0, x_0)$ to system (1) is said to be y-bounded, if for this solution there exists a number $\beta > 0$ such that $\|y(t, t_0, x_0)\| \leqslant \beta$ for all $t \in \mathbb{R}^+(t_0)$.

Definition 2 ([5])**.** The solution $x = x(t, t_0, x_0)$ to system (1) is said to be Poisson y-bounded, if for this solution one can find a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i\geqslant 1}$, where $t_0 \in M(\tau)$, and a number $\beta > 0$ such that $\|y(t, t_0, x_0)\| \leqslant \beta$ for all $t \in R^+(t_0) \bigcap M(\tau)$.

One can easily see that if a solution to system (1) is y-bounded, then it also is Poisson y-bounded.

The next proposition represents a sufficient condition for the existence of Poisson y-bounded solutions to system (1). Its proof is analogous to the proof of Proposition 1,

Proposition 2. *Assume that all conditions of Theorem* 1 *with the inequality* $b(\|y\|) \leqslant \sum_{i=1}^{k} v_i(t, x)$ *in place of formula* (4) *are fulfilled. Then system* (1) *has at least one Poisson* y*-bounded solution.*

Let us now recall some necessary properties of guiding functions and their indices [3]. A continuously differentiable function $\eta : \mathbb{R}^n \to \mathbb{R}$ is said to be r_0-nondegenerate, if $\operatorname{grad} \eta(x) \neq 0 \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$, $\|x\| \geqslant r_0$. This condition implies that the vector field $\operatorname{grad} \eta : B^n_{r_0} \to \mathbb{R}^n$ is nondegenerate on $S^{n-1}_{r_0}$ and, consequently, defined is the rotation $\gamma(\operatorname{grad} \eta, S^{n-1}_{r_0})$ of this vector field. In [3], M.A. Krasnosel'skii proves that since for any $r > r_0$ the corresponding vector field $\operatorname{grad} \eta : B^n_r \to \mathbb{R}^n$, evidently, is nondegenerate on S^{n-1}_r, the equality $\gamma(\operatorname{grad} \eta, S^{n-1}_r) = \gamma(\operatorname{grad} \eta, S^{n-1}_{r_0})$, is valid. The index of an r_0-nondegenerate function η is the integer number $\operatorname{ind}(\eta)$ that obeys the formula

$$\operatorname{ind}(\eta) = \gamma(\operatorname{grad} \eta, S^{n-1}_{r_0}) = \gamma(\operatorname{grad} \eta, S^{n-1}_r), \quad r > r_0.$$

A continuously differentiable function $\eta : \mathbb{R}^n \to \mathbb{R}$ is said to be r_0-guiding for system (1), if the following condition is fulfilled:

$$(\operatorname{grad} \eta(x), F(t, x)) > 0, \quad t \geqslant 0, \quad \|x\| \geqslant r_0. \tag{6}$$

According to condition (6), any r_0-guiding function is an r_0-nondegenerate function and, consequently, for any r_0-guiding function η its index $\operatorname{ind}(\eta)$ is defined. In [3], M.A. Krasnosel'skii proves that if for system (1) there exists an r_0-guiding function η, then for any $r \geqslant r_0$ the rotation of the vector field $\psi_0 : B(r) \to \mathbb{R}^n$, $\psi_0(x) = -f(0, x)$, where $f(t, x)$ is the right-hand side of system (1), and the index of the r_0-guiding function η are interconnected by the following equality:

$$\gamma(\psi_0, S^{n-1}_r) = (-1)^n \operatorname{ind}(\eta). \tag{7}$$

In what follows, we understand an r_0-guiding functional family as any set of r_0-guiding functions $\eta_0, \eta_1, \ldots, \eta_q$ for system (1), where $q \geqslant 1$. Note that as distinct from the notion of a complete set of r_0-guiding functions [4], here we do not impose the following condition:

$$\lim_{\|x\| \to +\infty} (|\eta_0(x)| + |\eta_1(x)| + \cdots + |\eta_q(x)|) = +\infty.$$

Equality (7) implies that for any r_0-guiding functional family $\eta_0, \eta_1, \ldots, \eta_q$ and any $r \geqslant r_0$ the following equalities are valid:

$$\operatorname{ind}(\eta_0) = \operatorname{ind}(\eta_1) = \cdots = \operatorname{ind}(\eta_q) = (-1)^n \gamma(\psi_0, S^{n-1}_r). \tag{8}$$

Let us now formulate (in terms of Lyapunov vector functions and guiding functional families) and prove the following sufficient condition for the existence of Poisson bounded solutions to system (1).

Theorem 1. *Assume that for system* (1) *one can find a* $\mathcal{P}$*-sequence* $\tau = \{\tau_i\}_{i\geqslant 1}$, *a nonincreasing function* $b : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $b(r) \to +\infty$ *as* $r \to +\infty$, *and a Lyapunov vector function* $v(t, x)$ *with the comparison system* (2) *such that inequality* (4) *is valid with any* $(t, x) \in M(\tau) \times \mathbb{R}^n$. *Assume* *also that there exist numbers $r_1 > r_0$ and an r_0-guiding functional family $\eta_0, \eta_1, \dots, \eta_q$ for system* (5) *with $r = r_1$, which satisfy the following conditions:*

1) $\operatorname{ind}(\eta_0) \neq 0$;

2) $\sum_{i=0}^{q} |\eta_i(\varrho)| > \sum_{i=0}^{q} (|m_i| + |M_i|)$ *with all $\varrho \in \mathbb{R}^k$, $\|\varrho\| = r_1$, where*

$$m_i = \min_{\|\varrho\| \leqslant r_0} \eta_i(\varrho), \quad M_i = \max_{\|\varrho\| \leqslant r_0} \eta_i(\varrho), \quad 0 \leqslant i \leqslant q;$$

3) $B^k_{r_1}(\overline{r}_1) = \{\xi \in \mathbb{R}^k \mid \| \xi - \overline{r}_1\| \leqslant r_1\} \subset \operatorname{Im}(v : \{0\} \times \mathbb{R}^n \to \mathbb{R}^k)$.

Then system (1) *has at least one Poisson bounded solution.*

Proof. Consider the vector field $\psi_0 : B^k_{r_1} \to \mathbb{R}^k$ that obeys the formula $\psi_0(\varrho) = -p(0, \varrho)$, where $p(t, \varrho)$ is the right-hand side of system (5). Condition 1) of the theorem and equality (9) imply that $\gamma(\psi_0, S^{k-1}_{r_1}) \neq 0$. Let us now prove that for any point $\varrho_0 \in S^{k-1}_{r_1}$ the right-hand part $\operatorname{Tr}^+(\varrho_0, t_0)$ of the trajectory $\operatorname{Tr}(\varrho(0, t_0, \varrho_0))$ of system (5) in $B^k_{r_1}$ has no common point with the left-hand part $\operatorname{Tr}^-(\varrho_0, t_0)$ of this trajectory. Condition 2) of the theorem implies that for each point $\varrho_0 \in S^{k-1}_{r_1}$ there exists a number $0 \leqslant i_0 \leqslant q$ such that $\eta_{i_0}(\varrho_0) < m_{i_0}$ or $\eta_{i_0}(\varrho_0) > M_{i_0}$. Let $\eta_{i_0}(\varrho_0) < m_{i_0}$. Consider the function $\varphi(t) = \eta_{i_0}(\varrho(t, t_0, \varrho_0))$, $t \geqslant 0$, and its derivative

$$\varphi'(t) = \frac{d(\eta_{i_0}(\varrho(t, t_0, \varrho_0))}{dt} = (\operatorname{grad} \eta_{i_0}(\varrho(t, t_0, \varrho_0)), g(t, \varrho(t, t_0, \varrho_0))), \quad t \geqslant 0.$$

Since the function η_{i_0} is an r_0-guiding function for system (5), the inequality $\varphi'(t) > 0$ is valid with all $t \geqslant 0$ such that $\|\varrho(t, t_0, \varrho_0)\| \geqslant r_0$. Evidently, $\varphi(t_0) < m_{i_0}$ and $\varphi(t) \geqslant m_{i_0}$ for all $t \geqslant 0$ such that $\|\varrho(t, t_0, \varrho_0)\| \leqslant r_0$. This means that $\operatorname{Tr}^-(\varrho_0, t_0) \cap B^k_{r_0} = \varnothing$, because the function $\varphi(t)$ is increasing for all $t \geqslant 0$ such that $\|\varrho(t, t_0, \varrho_0)\| \geqslant r_0$. Clearly, the condition $\operatorname{Tr}^-(\varrho_0, t_0) \cap B^k_{r_0} = \varnothing$ implies that

$$\operatorname{Tr}^+(\varrho_0, t_0) \cap \operatorname{Tr}^-(\varrho_0, t_0) \cap B^k_{r_0} = \varnothing.$$

Moreover, the fact that the function $\varphi(t)$ is increasing with all $t \geqslant 0$ such that $\|\varrho(t, t_0, \varrho_0)\| \geqslant r_0$ implies that

$$\operatorname{Tr}^+(\varrho_0, t_0) \cap \operatorname{Tr}^-(\varrho_0, t_0) \cap \Big(B^k(r_1) \setminus B^k(r_0)\Big) = \varnothing.$$

Therefore, in the case, when $\eta_{i_0}(\varrho_0) < m_{i_0}$, the right-hand part $\operatorname{Tr}^+(\varrho_0, t_0)$ of the trajectory $\operatorname{Tr}(\varrho(0, t_0, \varrho_0))$ of system (5) in $B^k_{r_1}$ has no common point with the left-hand part $\operatorname{Tr}^-(\varrho_0, t_0)$ of this trajectory. We can prove in the same way that in the case, when $\eta_{i_0}(\varrho_0) > M_{i_0}$, the right-hand part $\operatorname{Tr}^+(\varrho_0, t_0)$ of the trajectory $\operatorname{Tr}(\varrho(0, t_0, \varrho_0))$ of system (5) has in $B^k_{r_1}$ no common point with the left-hand part $\operatorname{Tr}^-(\varrho_0, t_0)$ of this trajectory. Therefore, we have proved that all conditions in Proposition 1 with $r = r_1$ are fulfilled and, consequently, system (1) has at least one Poisson bounded solution. □

The next theorem gives a sufficient condition for the existence of Poisson y-bounded solutions to system (1).

Theorem 2. *Assume that all conditions of Theorem* 1 *with the inequality $b(\|y\|) \leqslant \sum_{i=1}^{k} v_i(t, x)$ in place of formula* (4) *are fulfilled. Then system* (1) *has at least one Poisson y-bounded solution.*

Proof is analogous to the proof of Theorem 1.

Let us now consider (following [4]) a family of continuously differentiable functions $\nu_i : \mathbb{R}^n \to \mathbb{R}$, $0 \leqslant i \leqslant q$, $q \geqslant 1$, for system (1), which satisfies the following conditions:

1) the function ν_0 is r_0-nondegenerate and $\operatorname{ind}(\nu_0) \neq 0$;

2) $(\operatorname{grad} \nu_i(x), f(t, x)) \geqslant 0$, $0 \leqslant i \leqslant q$, $t \geqslant 0$, $\|x\| \geqslant r_0$;

3) $\left(\sum_{i=0}^{q} \operatorname{grad} \nu_i(x), f(t,x)\right) > 0$, $t \geqslant 0$, $\|x\| \geqslant r_0$.

In what follows, we treat any (mentioned above) family of functions $\nu_i : \mathbb{R}^n \to \mathbb{R}$, $0 \leqslant i \leqslant q$, as an r_0-prequiding functional family for system (1). Analogously to the paper [4], we can make sure that any r_0-prequiding functional family $\nu_0, \nu_1, \ldots, \nu_q$ for system (1) defines the r_0-guiding functional family $\eta_0, \eta_1, \ldots, \eta_q$ for system (1) that obeys formulas

$$\eta_0 = \sum_{i=0}^{q} \nu_i, \quad \eta_i = \eta_0 + \nu_i, \quad 1 \leqslant i \leqslant q. \tag{9}$$

Theorem 1 implies the following sufficient condition for the existence of Poisson bounded solutions to system (1) stated in terms of Lyapunov vector functions and preguiding functional families.

Corollary 1. Assume that for system (1) one can find a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geqslant 1}$, a nonincreasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $b(r) \to +\infty$ as $r \to +\infty$, and a Lyapunov vector function $v(t,x)$ with comparison system (2) such that inequality (4) is valid with any $(t,x) \in M(\tau) \times \mathbb{R}^n$. In addition, assume that there exist numbers $r_1 > r_0$ and an r_0-preguiding functional family $\nu_0, \nu_1, \ldots, \nu_q$ for system (5) with $r = r_1$ such that these numbers and the r_0-guiding functional family $\eta_0, \eta_1, \ldots, \eta_q$ defined by formulas (9) satisfy conditions 1) − 3) of Theorem 1. Then system (1) has at least one Poisson bounded solution.

Theorem 2 has the following corollary, which gives a sufficient condition for the existence of Poisson y-bounded solutions to system (1).

Corollary 2. Assume that all conditions of Corollary 1 with the inequality $b(\|y\|) \leqslant \sum_{i=1}^{k} v_i(t,x)$ in place of formula (4) are fulfilled. Then system (1) has at least one Poisson y-bounded solution.

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