

The Euler Summation of Numerical Series

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Abstract—We prove a combinatorial identity containing k -order differences of sequences, as well as binomial coefficients. The Euler summation operation implies the calculation of all order differences of terms of the initial series. The regularity of the summing function means the coincidence with the “ordinary” sum of the series, provided that this sum exists. The proved combinatorial identity allows one to easily prove the regularity of the Euler summation operation.

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1. INTRODUCTION

For calculating the sum of a series $\gamma = \sum_{n=0}^{\infty} u_n$, it is necessary to define the summation rule. According to the standard and most widely used definition of the sum, $s_0(\gamma) = \lim_{n \rightarrow \infty} \sum_{k=0}^{k=n} u_k$. But this is not the only way to map a series γ to its sum $s(\gamma)$. Such techniques are called *summing functions*. A summing function s has a sense if, at least, it obeys the regularity and linearity rules.

Definition 1 ([1], Ch. 15, §2). A *summing function* is a transform s that maps some numerical series $\gamma = \sum_n u_n$ to the value $s(\gamma)$ satisfying the following rules.

Regularity: if the series γ converges in the usual sense and its sum equals $s_0(\gamma)$, then the value $s(\gamma)$ exists and coincides with $s_0(\gamma)$. In particular, if γ is a finite sum, then $s(\gamma)$ coincides with the usual arithmetic sum of nonzero terms of the series γ .

Linearity: if series γ, β are s -summable, then for any numbers a, b the series $a\gamma + b\beta$ is also s -summable, and $s(a\gamma + b\beta) = as(\gamma) + bs(\beta)$.

The standard summation operation is regular by definition, and it is also linear. One of significant summation procedures for numerical series was proposed by L. Euler (see, e.g., [1], Ch. 14, §5). One should first define the *sequence of the first differences* for the sequence $u_0, u_1, u_2, u_3, \dots$. This new sequence is $\Delta u_0, \Delta u_1, \Delta u_2, \dots$, where $\Delta u_n = u_n - u_{n+1}$ with $n = 1, 2, 3, \dots$. We understand the *sequence of the k th differences* for u_0, u_1, u_2, \dots as the sequence of the first differences of the sequence of $(k - 1)$ th differences, namely,

$$\Delta^k u_0, \Delta^k u_1, \Delta^k u_2, \dots,$$

where $\Delta^k u_n = \Delta(\Delta^{k-1} u_n) = \Delta^{k-1} u_n - \Delta^{k-1} u_{n+1}$.

It is convenient to write terms of sequences of the k th differences for various k in the triangular table

$$\begin{array}{cccccc}
 u_0 & u_1 & u_2 & u_3 & u_4 & \\
 & \Delta u_0 & \Delta u_1 & \Delta u_2 & \Delta u_3 & \dots \\
 & & \Delta^2 u_0 & \Delta^2 u_1 & \Delta^2 u_2 & \dots \dots \\
 & & & \Delta^3 u_0 & \Delta^3 u_1 & \dots \dots \dots
 \end{array}$$

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For example, for a sequence of powers of two, this triangle takes the form

$$\begin{array}{cccccc}
 1 & 2 & 4 & 8 & 16 & \\
 & -1 & -2 & -4 & -8 & \dots \\
 & & 1 & 2 & 4 & \dots \\
 & & & -1 & -2 & \dots
 \end{array} \tag{1}$$

Applying the induction method with respect to k and using the main recurrent correlation between binomial coefficients, we can prove the formula

$$\Delta^k u_n = u_n - C_k^1 u_{n+1} + C_k^2 u_{n+2} - \dots + (-1)^k u_{n+k}; \tag{2}$$

here $C_k^m = \frac{k!}{m!(k-m)!}$ are binomial coefficients. It suffices to verify this formula for $n = 0$. Let us apply the mathematical induction method. For $k = 1$ this formula is valid, because its right-hand side turns into $u_0 - C_1^1 u_1 = u_0 - u_1$, which coincides with $\Delta^1 u_0$. Let us now assume that formula (2) is valid for $n = 0$ and for some k . Then

$$\begin{aligned}
 \Delta^{k+1} u_0 &= \Delta^k u_0 - \Delta^k u_1 = \\
 &= (u_0 - C_k^1 u_1 + C_k^2 u_2 - \dots + (-1)^k u_k) - (u_1 - C_k^1 u_2 + C_k^2 u_3 - \dots + (-1)^{k+1} u_{1+k}) = \\
 &= u_0 - (C_k^1 + 1)u_1 + (C_k^2 + C_k^1)u_2 - \dots + ((-1)^k + (-1)^k C_k^k)u_k + (-1)^{k+1} u_{1+k} = \\
 &= u_0 - C_{k+1}^1 u_1 + C_{k+1}^2 u_2 - \dots + (-1)^k C_{k+1}^k u_k + (-1)^{k+1} u_{k+1}.
 \end{aligned}$$

Definition 2 ([1], Ch. 14, §5). *The Euler transform* of a series $\sum_{n=0}^{\infty} (-1)^n u_n$ is the transition from this series to that

$$\sum_{n=0}^{\infty} \frac{\Delta^n u_0}{2^{n+1}} = \frac{1}{2} u_0 + \frac{1}{4} \Delta u_0 + \frac{1}{8} \Delta^2 u_0 + \dots \tag{3}$$

For $n = 0$, by definition, $\Delta^n u_m = u_m$. We sum the series $\gamma = \sum_{n=0}^{\infty} (-1)^n u_n$ in the sense of Euler, provided that there exists the limit

$$s_E(\gamma) = \lim_{n \rightarrow \infty} \sum_{k=0}^{k=n} \frac{\Delta^k u_0}{2^{k+1}}; \tag{4}$$

this limit value is said to be the *Euler sum* of the series γ .

The linearity of the Euler summation operation follows from the linearity of the limit transition and the linearity of the operation of forming the sequence of the first and, more generally, any difference. In the next theorem, we study the regularity of the Euler summation operation.

Theorem. *If the series $\sum_{n=0}^{\infty} (-1)^n u_n$ converges, then so does its Euler transform, and sums of these two series coincide.*

Here one does not assume that $u_n \geq 0$; the sign $(-1)^n$ is used only for technical reasons. The proof of this theorem is described in ([1], Ch. 14, §5); it occupies three pages and essentially uses the Markov theorem on double series. In Section 3, we briefly describe the proof of the regularity of the Euler summation; it is based on the combinatorial identity (11) which is proved below and constitutes the original part of this paper.

Let us prove that in the case when the sum $u_0 - u_1 + \dots + (-1)^N u_N$ is finite, the theorem is valid (this consideration is not necessary for the short proof of the theorem and identity (11)). Denote $w_n = (-1)^n u_n$. By formula (2),

$$\Delta^n u_0 = w_0 + C_n^1 w_1 + C_n^2 w_2 + \dots + C_n^n w_n.$$

Since

$$\sum_{n=0}^{\infty} \frac{\Delta^n u_0}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{w_0 + C_n^1 w_1 + \dots + C_n^n w_n}{2^{n+1}} = \sum_{k=0}^N w_k \sum_{n \geq k} \frac{C_n^k}{2^{n+1}},$$

it remains to prove the equality

$$\sum_{n \geq k} \frac{C_n^k}{2^{n+1}} = 1$$

for any fixed $k = 0, 1, 2, \dots$. Let us prove it by induction with respect to k . The induction base, i. e., the case $k = 0$, follows from the formula for the sum of a geometric progression, namely,

$$\sum_{n \geq 0} \frac{C_n^0}{2^{n+1}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1.$$

Denote $X_k := \sum_{n \geq k} \frac{C_n^k}{2^{n+1}}$. Again using the main correlation for binomial coefficients ($C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$), we conclude that

$$\begin{aligned} X_k &= \frac{1}{2^{k+1}} + \sum_{n \geq k+1} \frac{C_{n-1}^k + C_{n-1}^{k-1}}{2^{n+1}} = \frac{1}{2^{k+1}} + \frac{1}{2} \sum_{n-1 \geq k} \frac{C_{n-1}^k}{2^{(n-1)+1}} + \\ &+ \frac{1}{2} \sum_{n-1 \geq k} \frac{C_{n-1}^{k-1}}{2^{(n-1)+1}} = \frac{1}{2^{k+1}} + \frac{1}{2} X_k + \frac{1}{2} \left(X_{k-1} - \frac{1}{2^k} \right) = \frac{1}{2} X_k + \frac{1}{2} X_{k-1}, \end{aligned}$$

whence we get the equality $X_k = X_{k-1}$. Consequently, all X_k are equal to one.

The Euler summation is much stronger. Namely, let us calculate the Euler sum of the series $1 - 2 + 4 - 8 + \dots$; according to formula (1), it equals the standard sum

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots = \frac{1/2}{1 + 1/2} = \frac{1}{3}.$$

However, we do not calculate the Euler sum of the series formed by powers of two, i. e., $1 + 2 + 4 + 8 + 16 + \dots$, because for this series $u_n = (-1)^n 2^n$, $n \geq 0$, and

$$\Delta^n u_0 = 1 + C_n^1 \cdot 2 + C_n^2 \cdot 4 + \dots + C_n^n \cdot 2^n = (1 + 2)^n = 3^n.$$

We get the following divergent geometric progression: $\sum_{n \geq 0} \frac{3^n}{2^{n+1}}$. Note that in the 3-adic topology of rational numbers, this geometric progression converges to the value -1 , which remarkably coincides with the sum $\sum_{n \geq 0} 2^n = \frac{1}{1-2} = -1$ in the 2-adic topology.

2. THE COMBINATORIAL IDENTITY

For the sequence $a_0, a_1, a_2, a_3, \dots$, we denote partial sums

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad s_2 = a_0 + a_1 + a_2, \dots, \quad s_{n+1} = s_n + a_{n+1}$$

and differences of the alternate sequence $a_1, -a_2, a_3, -a_4, \dots$

$$\Delta_0 = a_0, \quad \Delta_k = a_0 + C_k^1 a_1 + C_k^2 a_2 + \dots + C_k^{k-1} a_{k-1} + a_k = \sum_{j=0}^k C_k^j a_j. \quad (5)$$

Let us find coefficients $x_0^k, x_1^k, x_2^k, \dots, x_k^k$ such that the correlation

$$\frac{1}{2}\Delta_0 + \frac{1}{4}\Delta_1 + \frac{1}{8}\Delta_2 + \dots + \frac{1}{2^{k+1}}\Delta_k = x_0^k s_0 + x_1^k s_1 + \dots + x_k^k s_k \tag{6}$$

takes place for any a_j .

First, let us substitute expressions $a_0 = 1$ and $a_j = 0$ for all $j > 0$ in formula (6). Calculate $s_j = 1 = \Delta_j$, which is true for any j , whence we get the correlation

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k+1}} = x_0^k + x_1^k + \dots + x_k^k.$$

Consequently,

$$x_0^k + x_1^k + \dots + x_k^k = 1 - \frac{1}{2^{k+1}}. \tag{7}$$

Second, let us substitute expressions $a_j = \lambda^j$ with $j = 0, 1, \dots, k$ in formula (6). Then by the binomial formula, $\Delta_t = (1 + \lambda)^t$. Let us transform the left-hand side of the desired identity (6) by using the formula for the sum of a geometric progression, namely,

$$\frac{1}{2} + \frac{1}{4}(1 + \lambda) + \frac{1}{8}(1 + \lambda)^2 + \dots + \frac{1}{2^{k+1}}(1 + \lambda)^k = \frac{1}{2} \frac{1 - (1 + \lambda)^{k+1} + 1}{1/2(1 + \lambda) - 1} = \frac{1 - (1 + \lambda)^{k+1} + 1}{\lambda - 1}.$$

Let us now transform the right-hand side, taking into account formula (7). We get the expression

$$\begin{aligned} x_0^k + x_1^k(1 + \lambda) + x_2^k(1 + \lambda + \lambda^2) + \dots + x_k^k(1 + \lambda + \lambda^2 + \dots + \lambda^k) &= \\ &= x_0^k \frac{1 - \lambda}{1 - \lambda} + x_1^k \frac{1 - \lambda^2}{1 - \lambda} + x_2^k \frac{1 - \lambda^3}{1 - \lambda} + \dots + x_k^k \frac{1 - \lambda^{k+1}}{1 - \lambda} = \\ &= \frac{1}{1 - \lambda} \left(1 - \frac{1}{2^{k+1}} \right) - \frac{\lambda}{1 - \lambda} \left(x_0^k + x_1^k \lambda + x_2^k \lambda^2 + \dots + x_k^k \lambda^k \right). \end{aligned}$$

Equating the left- and right-hand sides, concurrently multiplying them by $\lambda - 1$ and dividing by -1 , we get the correlation

$$\frac{1}{2^{k+1}}(1 + \lambda)^{k+1} = \frac{1}{2^{k+1}} + \lambda \left(x_0^k + x_1^k \lambda + x_2^k \lambda^2 + \dots + x_k^k \lambda^k \right).$$

Consequently,

$$\frac{1}{2^{k+1}} \frac{(1 + \lambda)^{k+1} - 1}{\lambda} = x_0^k + x_1^k \lambda + x_2^k \lambda^2 + \dots + x_k^k \lambda^k. \tag{8}$$

For calculating x_t^k let us use the Cramer's rule, replacing λ in formula (8) in rotation with pairwise distinct nonzero values $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_k$. As a result, we get the linear system of $k + 1$ equations with $k + 1$ unknowns and the Vandermonde determinant

$$W = (\lambda_1 - \lambda_0) \dots (\lambda_k - \lambda_0)(\lambda_2 - \lambda_1) \dots (\lambda_k - \lambda_1) \dots (\lambda_k - \lambda_{k-1}) \neq 0.$$

Let us now calculate the determinant D_t of the matrix obtained from W by replacing its t th column with that

$$S := \left(\frac{1}{2^{k+1}} \frac{(1 + \lambda_0)^{k+1} - 1}{\lambda_0}, \dots, \frac{1}{2^{k+1}} \frac{(1 + \lambda_k)^{k+1} - 1}{\lambda_k} \right)^\top,$$

i. e., with the column of free terms in the linear system mentioned above. Expand

$$\frac{1}{2^{k+1}} \frac{(1 + \lambda)^{k+1} - 1}{\lambda} = \frac{1}{2^{k+1}} ((k + 1) + C_{k+1}^2 \lambda + \dots + C_{k+1}^{t+1} \lambda^t + \dots + \lambda^k). \tag{9}$$

Substituting the column S at the t th place in the matrix W and expand it into the sum of k determinants in accordance with formula (9), we make all terms except that with the column

$$\frac{C_{k+1}^{t+1}}{2^{k+1}}(\lambda_0^t, \lambda_1^t, \dots, \lambda_k^t)^\top,$$

vanish. The reason is that each of them contains two proportional columns (for example, for the term, where the determinant contains the column $(k+1)/2^{k+1}(1, 1, 1, \dots, 1)^\top$ at the t th place, we get the proportionality with the first column, the column $C_{k+1}^2/2^{k+1}(\lambda_0, \lambda_1, \dots, \lambda_k)^\top$ appears to be proportional to the second one and so on). The unique nonzero term containing the column mentioned above gives the value $D_t = \frac{C_{k+1}^{t+1}}{2^{k+1}}W$. Consequently, by applying the Cramer formula $x_t^k = D_t/W$ we obtain the correlation

$$x_t^k = \frac{C_{k+1}^{t+1}}{2^{k+1}} = \frac{(k+1)k \dots (k-t+1)}{2^{k+1}(t+1)!}. \quad (10)$$

We have proved that coefficients (10) are appropriate for $a_t = \lambda^t$ and turn formula (6) into the equality. Note that rows $(1, \lambda_j, \lambda_j^2, \dots, \lambda_j^k)$ for pairwise distinct $\lambda_0, \lambda_1, \dots, \lambda_k$ form a basis in the space of $k+1$ -dimensional vectors, because the determinant of the matrix composed of these rows, i. e., W , differs from zero. The summation operation s_t , as well as operations Δ_t are linear. Consequently, if numbers (10) are appropriate for some basis, then they also are appropriate for any row (a_0, a_1, \dots, a_k) .

In conclusion, we immediately verify equality (7), namely,

$$\begin{aligned} x_0^k + x_1^k + \dots + x_k^k &= \sum_{t=0}^k \frac{C_{k+1}^{t+1}}{2^{k+1}} = \frac{1}{2^{k+1}} \left(\sum_{t=-1}^k C_{k+1}^{t+1} - 1 \right) = \\ &= \frac{1}{2^{k+1}} \left(\sum_{m=0}^{k+1} C_{k+1}^m - 1 \right) = \frac{1}{2^{k+1}} (2^{k+1} - 1) = 1 - \frac{1}{2^{k+1}}. \end{aligned}$$

Thus, we deduce the main result of this paper, namely, the combinatorial identity

$$\frac{1}{2}\Delta_0 + \frac{1}{4}\Delta_1 + \frac{1}{8}\Delta_2 + \dots + \frac{1}{2^{k+1}}\Delta_k = \frac{1}{2^{k+1}} \left[C_{k+1}^1 s_0 + C_{k+1}^2 s_1 + \dots + C_{k+1}^{k+1} s_k \right] \quad (11)$$

or

$$\Delta_0 + \frac{1}{2}\Delta_1 + \frac{1}{2^2}\Delta_2 + \dots + \frac{1}{2^k}\Delta_k = \frac{1}{2^k} \left[C_{k+1}^1 s_0 + C_{k+1}^2 s_1 + \dots + C_{k+1}^{k+1} s_k \right].$$

Note that the value written at the left of the equality sign in formula (11) is the k th partial sum of the series that defines the Euler sum of the series $a_0 - a_1 + a_2 - a_3 + \dots$. Let us mention the following corollary.

Corollary. All values x_t^k ($t = 0, 1, 2, \dots, k$) are positive, and their sum for $t \in [0; k]_{\mathbb{N}}$ is less than one by the value $1/2^{k+1}$ (see formula (7)).

Equating coefficients at a_t in formula (11), we get the correlation

$$\frac{1}{2^{t+1}} + \frac{C_{t+1}^1}{2^{t+2}} + \dots + \frac{C_k^t}{2^{k+1}} = \frac{1}{2^{k+1}} \left[C_{k+1}^{t+1} + C_{k+1}^{t+2} + \dots + C_{k+1}^{k+1} \right]$$

or, multiplying it by 2^{k+1} ,

$$2^{k-t}C_t^0 + 2^{k-t-1}C_{t+1}^1 + \dots + 2^0C_k^t = C_{k+1}^{t+1} + C_{k+1}^{t+2} + \dots + C_{k+1}^{k+1}. \quad (12)$$

The identity

$$\sum_{j \leq t} C_{t+j}^j 2^{-j} = 2^t$$

(see [2], p. 193, formula (5.20)) represents a particular case of identity (12) with $k = 2t$. This fact follows from the equality

$$C_{2t+1}^{t+1} + C_{2t+1}^{t+2} + \dots + C_{2t+1}^{2t+1} = \frac{1}{2} \sum_{j=0}^{2t+1} C_{2t+1}^j = 2^{2t}$$

where we used the correlation $C_n^t = C_n^{n-t}$ between binomial coefficients.

3. THE REGULARITY OF THE EULER SUMMATION

Given a series

$$u_0 - u_1 + u_2 - u_3 + \dots, \tag{13}$$

denote $a_k = (-1)^k u_k$. Let $\Delta_k := \Delta^k u_0$ stand for the difference of the k th order. Then the value Δ_k obeys formula (5).

Proof of the theorem. If series (13) converges to some value s , then the series $\sum_{k=0}^{\infty} \frac{\Delta^k u_0}{2^{k+1}}$ also converges to s .

According to assumptions of the theorem and the denotation $a_k = (-1)^k u_k$, the series $\sum_{k=0}^{\infty} a_k$ converges to the value s . We need to prove that the series $\sum_{k=0}^{\infty} \frac{\Delta^k}{2^{k+1}}$ also converges to the same value s . As above, $s_k = a_0 + \dots + a_k$ are partial sums.

Choose $\varepsilon > 0$ and find positive integer N such that $|s_n - s| < \varepsilon$ for any $n \geq N$. Estimate the value

$$\begin{aligned} \left| \frac{1}{2} \Delta_0 + \frac{1}{4} \Delta_1 + \frac{1}{8} \Delta_2 + \dots + \frac{1}{2^{k+1}} \Delta_k - s \right| &= \left| \frac{1}{2^{k+1}} \left[C_{k+1}^1 s_0 + C_{k+1}^2 s_1 + \dots + C_{k+1}^{k+1} s_k \right] - s \right| \leq \\ &\leq \left| \frac{C_{k+1}^1}{2^{k+1}} s_0 + \frac{C_{k+1}^2}{2^{k+1}} s_1 + \dots + \frac{C_{k+1}^N}{2^{k+1}} s_{N-1} \right| + \left| \sum_{j=N}^{j=k} \frac{C_{k+1}^{j+1}}{2^{k+1}} s_j - s \right|. \end{aligned}$$

Partial sums s_j are bounded. Assume that $|s_j| \leq C$ for any $j = 0, 1, 2, \dots$. Then it also holds that

$|s| \leq C$. There exists a number $K > N$ such that $\frac{C_{k+1}^j}{2^{k+1}} < \frac{\varepsilon}{N}$ for any $k \geq K$ and any $0 \leq j \leq N$.

This is possible, because 2^k grows faster than any polynomial of the variable k . More precisely, $\lim_{k \rightarrow \infty} \frac{P(k)}{2^k} = 0$ independently of the polynomial $P(k)$ (one can verify this fact with the help of the L'Hôpital rule for a continuous variable in place of the discrete one k). Then

$$\left| \frac{C_{k+1}^1}{2^{k+1}} s_0 + \frac{C_{k+1}^2}{2^{k+1}} s_1 + \dots + \frac{C_{k+1}^N}{2^{k+1}} s_{N-1} \right| \leq \sum_{t=0}^{N-1} |s_t| < C\varepsilon.$$

Furthermore,

$$\begin{aligned} \left| \sum_{j=N}^{j=k} \frac{C_{k+1}^{j+1}}{2^{k+1}} s_j - s \right| &= \left| \sum_{j=N}^{j=k} \frac{C_{k+1}^{j+1}}{2^{k+1}} (s_j - s) - \sum_{j=-1}^{N-1} \frac{C_{k+1}^{j+1}}{2^{k+1}} s \right| \leq \\ &\leq \left| \sum_{j=N}^{j=k} \frac{C_{k+1}^{j+1}}{2^{k+1}} (s_j - s) \right| + \left| \sum_{j=-1}^{N-1} \frac{C_{k+1}^{j+1}}{2^{k+1}} s \right| \leq \varepsilon + \frac{\varepsilon}{N} (N+1)C \leq \varepsilon + 2C\varepsilon \end{aligned}$$

because $\sum_{j=-1}^k \frac{C_{k+1}^{j+1}}{2^{k+1}} = 1$. Thus,

$$\left| \frac{1}{2}\Delta_0 + \frac{1}{4}\Delta_1 + \frac{1}{8}\Delta_2 + \cdots + \frac{1}{2^{k+1}}\Delta_k - s \right| \leq C\varepsilon + \varepsilon + 2C\varepsilon = (3C + 1)\varepsilon.$$

Since due to the choice of $\varepsilon > 0$ the value indicated above can be infinitesimally small, we have proved that

$$\lim_{k \rightarrow \infty} \left[\frac{1}{2}\Delta_0 + \frac{1}{4}\Delta_1 + \frac{1}{8}\Delta_2 + \cdots + \frac{1}{2^{k+1}}\Delta_k \right] = s.$$

□

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