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Discrete-Time Systems with Frequency Response of the Markov–Stieltjes Type

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Abstract—A class of discrete-time filters (systems) is selected, the frequency characteristics of which are functions of the Markov–Stieltjes type. A description of these filters is given in terms of their system function and impulse response. The properties of stationarity, causality, stability, and reversibility are investigated. A wide class of filters with rational transfer functions is indicated, which is subject to the main results of the work.

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1. INTRODUCTION

In this paper, we consider a class of filters (systems) with discrete time (DT systems) with a frequency response that is a function of the Markov–Stieltjes type, i.e., the Markov–Stieltjes transform of some bounded measure. We give a description of these filters in terms of their frequency response (or, equivalently, the transfer function), as well as in terms of their impulse response. In particular, it was shown that this class contains all filters with completely monotonic impulse responses. The latter case may arise, for example, in the discretization of filters (systems) with continuous time (CT systems; see Remark 3.2 below). The properties of stationarity, causality, stability, and reversibility of the corresponding systems are investigated. In the case of reversibility, the inverse operator is calculated. It is important to note that the class of Markov–Stieltjes type functions contains a wide subclass of rational functions, which are the frequency responses of linear time-invariant discrete-time filters (DT-LIT systems). In this connection, a wide class of filters with rational transfer functions (see Example 3.1) is indicated, which is a subject of the main theorem of this paper, Theorem 3.1.

Some of the results were announced in [1].

We note that earlier the authors investigated the properties of the Markov–Stieltjes transform of functions in the Hardy spaces $H^p(\mathbb{D})$ (here and below $\mathbb{D}$ denotes an open unit disk in the complex plane) and the Lebesgue spaces $L^p(0,1)$ (see [2]–[5]).

In what follows, we denote by $M^b([0,1],\mathbb{C})$ ($M^b([0,1],\mathbb{R})$) the space of all bounded complex (respectively, real) measures on $[0,1]$, and by $M^b_+([0,1])$ its subspace consisting of positive measures. The distribution function of the measure μ is denoted by $\mu(t)$.

Definition 1.1 ([6], Ch. 6)**.** The Markov–Stieltjes transform of the measure $\mu \in M^b([0,1],\mathbb{C})$ is a function given with $z \in \mathbb{C}\setminus[1,+\infty)$ by the formula

$$S\mu(z) = \int_0^1 \frac{d\mu(t)}{1-tz}. \tag{1}$$

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With $z \in [1, +\infty)$ the integral on the right-hand side of (1) is understood in the sense of the main value, i. e., as the limit

$$S\mu(z) = \lim_{\varepsilon \to +0} \int_{[0,1]\cap\{|t-1/z|>\varepsilon\}} \frac{d\mu(t)}{1-tz}. \qquad (1')$$

We note that rational functions of the form $\sum_{i=1}^{r} \mu_i/(1-t_i z)$ $(\mu_i \in \mathbb{C}, t_i \in [0,1])$ are the Markov–Stieltjes functions.

Theorem 1.1. *The Markov–Stieltjes transform of the measure $\mu \in M^b([0,1], \mathbb{C})$ is holomorphic in the domain $\mathbb{C} \setminus [1, +\infty)$, and also exists almost everywhere on the ray $[1, +\infty)$.*

Proof. Note that

$$S\mu(z) = \int_0^1 \frac{d\mu(t)}{1-tz} = \frac{1}{z}\int_0^1 \frac{d\mu(t)}{\frac{1}{z}-t} = \frac{\pi}{z} H\mu_1\left(\frac{1}{z}\right),$$

where $H\mu_1$ is the Hilbert transform of the measure μ_1, having the distribution function

$$\mu_1(t) := \begin{cases} \mu(t), t \in (0,1); \\ 0, t \notin (0,1). \end{cases}$$

Hence from the well-known property of the Hilbert transform, the assertion follows that the Markov–Stieltjes transform is holomorphic. An application of the Loomis theorem on the Hilbert transform (see, for example, [7], p. 239) completes the proof of the theorem.

In connection with the corollary below, we recall that the disk algebra $A(\overline{\mathbb{D}})$ consists of functions that are analytic in the open unit disk $\mathbb{D}$ and continuous in its closure.

Corollary 1.1. Let $\mu \in M^b([0,1], \mathbb{C})$, $F = S\mu$. Then

1) $F \in H^p(\mathbb{D})$ with all $p \in (0,1)$;

2) let $p \in [1, \infty)$, $1/p + 1/q = 1$; if $\int_0^1 d|\mu|(t)/(1-t)^{\varepsilon+1/q} < \infty$ with some $\varepsilon \in (0, 1/p)$, then $F \in H^p(\mathbb{D})$;

3) if $\int_0^1 d|\mu|(t)/(1-t) < \infty$, then F belongs to the disk algebra $A(\overline{\mathbb{D}})$; if, in addition, $\mu > 0$, then F does not vanish on the unit circle $\mathbb{T}$.

Proof. All assertions, except the last one, follow from Theorem 1.1 and estimates established with the proof of Theorem 1 in ([8], p. 4). Let us prove the last assertion. Evidently, μ does not concentrated at point 1. Therefore $\mu([0,1)) > 0$. Then with $e^{\imath\theta} \in \mathbb{T}$ we have

$$|F(e^{\imath\theta})|^2 = \int_0^1\int_0^1 \frac{d\mu(t)d\mu(s)}{(1-te^{\imath\theta})(1-se^{-\imath\theta})} = \int_0^1\int_0^1 \mathrm{Re}\frac{1}{(1-te^{\imath\theta})(1-se^{-\imath\theta})} d\mu(t)d\mu(s).$$

But

$$\mathrm{Re}\frac{1}{(1-te^{\imath\theta})(1-se^{-\imath\theta})} = \frac{1-(t+s)\cos\theta + ts}{|1-te^{\imath\theta}|^2|1-se^{-\imath\theta}|^2} \geq \frac{(1-t)(1-s)}{|1-te^{\imath\theta}|^2|1-se^{-\imath\theta}|^2} \geq \frac{(1-t)(1-s)}{(1+t)^2(1+s)^2}.$$

Hence

$$|F(e^{\imath\theta})|^2 \geq \int_0^1\int_0^1 \frac{(1-t)(1-s)}{(1+t)^2(1+s)^2} d\mu(t)d\mu(s) = \left(\int_{[0,1)} \frac{1-t}{(1+t)^2} d\mu(t)\right)^2 > 0.$$

2. DESCRIPTION OF MARKOV–STIELTJIES TRANSFORMS OF MEASURES

This section has an auxiliary sense. It contains the description of functions presentable in form (1) for complex and positive measures, which we need below.

Lemma 2.1. *For the function $F(z)$ the following assertions are equivalent.*

1) *There exists a measure $\mu \in M^b([0,1],\mathbb{C})$ such that $\|\mu\| \le c$ and $F = S\mu$.*

2) *$F(z)$ is analytic in $\mathbb{D}$, $F(z) = \sum_{k=0}^{\infty} a_k z^k$ and for any complex numbers λ_i and any natural m*

$$\left|\sum_{k=0}^{m} \lambda_k a_k\right| \le c \max\left\{\left|\sum_{k=0}^{m} \lambda_k t^k\right| : t \in [0,1]\right\}. \tag{2}$$

Proof. 1) $\Rightarrow$ 2). Let $F = S\mu$. Decomposing the integrand into a series and applying the Lebesgue theorem on termwise integration of the series, we have

$$F(z) = \int_0^1 \frac{d\mu(t)}{1-tz} = \int_0^1 \left(\sum_{n=0}^{\infty} t^n z^n\right) d\mu(t) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \int_0^1 t^n d\mu(t)$. Therefore

$$\left|\sum_{k=0}^{m} \lambda_k a_k\right| = \left|\sum_{k=0}^{m} \lambda_k \int_0^1 t^k d\mu(t)\right| = \left|\int_0^1 \left(\sum_{k=0}^{m} \lambda_k t^k\right) d\mu(t)\right| \le$$

$$\le \|\mu\| \max\left\{\left|\sum_{k=0}^{m} \lambda_k t^k\right| : t \in [0,1]\right\} \le c \max\left\{\left|\sum_{k=0}^{m} \lambda_k t^k\right| : t \in [0,1]\right\}.$$

2) $\Rightarrow$ 1). For an arbitrary polynomial $p(t) = \sum_{k=0}^{m} \lambda_k t^k$ we set

$$\Lambda(p) := \sum_{k=0}^{m} \lambda_k a_k.$$

Evidently, Λ is a linear bounded functional on the space of all polynomials in the metric $C[0,1]$ and $\|\Lambda\| \le c$. Applying the Hahn–Banach theorem, we extend it to $C[0,1]$ with preservation of the norm. By F. Riesz's theorem on the general form of functionals in $C[0,1]$ there exists a measure $\mu \in M^b([0,1])$ such that

$$\Lambda(x) = \int_{[0,1]} x(t) d\mu(t),$$

and $\|\mu\| = \|\Lambda\| \le c$. In addition, $a_k = \Lambda(t^k) = \int_0^1 t^k d\mu(t)$.

Therefore,

$$F(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \left(\int_0^1 t^k d\mu(t)\right) z^k = \int_0^1 \left(\sum_{k=0}^{\infty} t^k z^k\right) d\mu(t) = \int_0^1 \frac{d\mu(t)}{1-tz} = S\mu(z),$$

that completes the proof of lemma.

Remind [9] that the function g belongs to the class $R[a,b]$, if g is holomorphic in an open upper half-plane, maps it to the closure of this half-plane, and is also holomorphic and positive on $(-\infty, a)$ and holomorphic and negative on (b, ∞). (From these conditions it follows that g maps an open upper half-plane into itself.) Here due to ([9], Theorem P.6) this function can be uniquely presented in the form

$$g(z) = \int_a^b \frac{d\tau(t)}{t-z},$$

where τ is a bounded positive regular Borel measure concentrated on the segment $[a, b]$ ("representing measure").

Lemma 2.2. *The function F has the form $S\mu$ for some measure $\mu \in M_+^b([0,1])$, if and only if the following conditions are fulfilled:*

1) *F is holomorphic in $\mathbb{C} \setminus [1, +\infty)$ and positive on the interval $(-\infty, 1)$,*

2) *the function $\zeta F(\zeta)$ maps an open lower half-plane to its closure.*

Proof. Necessity. The holomorphic property of F in $\mathbb{C} \setminus [1, +\infty)$ was already indicated above.

Further, if $\zeta < 1$, then $1 - t\zeta > 0$ with $t \in [0, 1)$, and therefore $F(\zeta) > 0$.

Finally, since $\mathrm{Im}(\zeta/(1 - t\zeta)) < 0$ with $\mathrm{Im}\zeta < 0$, $t \in (0, 1)$, we have

$$\mathrm{Im}(\zeta F(\zeta)) = \int_0^1 \mathrm{Im}\frac{\zeta}{1 - t\zeta} d\mu(t) \leq 0.$$

That completes the proof of necessity.

Sufficiency. Note that under conditions 1) and 2) the function $(\zeta = 1/z)$

$$F_1(z) := -\frac{1}{z}F\left(\frac{1}{z}\right) = -\zeta F(\zeta)$$

is holomorphic in the upper half-plane and $\mathrm{Im}F_1(z) \geq 0$ with $\mathrm{Im}z > 0$. In addition, from the condition 1 it follows that it is holomorphic and negative on the interval $(0, 1)$ and holomorpic and positive on the interval $(-\infty; 0)$. Hence, F_1 belongs to the class $R[0, 1]$ of the Markov type functions and has the integral presentation

$$F_1(z) = \int_0^1 \frac{d\mu(t)}{t - z}$$

with representing measure $\mu \in M_+^b([0, 1])$, and therefore

$$F(\zeta) = \int_0^1 \frac{d\mu(t)}{1 - t\zeta} = S\mu(\zeta),$$

as desired.

3. SYSTEMS WITH MOMENT IMPULSE RESPONSE

Further we mostly use the terminology of the theory of signal processing, applied in ([10], pp. 153–159 and [11]). In particular, below the filter Φ is the operator $f \mapsto \Phi f$. We will consider only the case of discrete time (DT-signals).

We call a filter Φ *stationary*, if:

1) Φ is a linear bounded operator in $\ell^2(\mathbb{Z})$;

2) Φ is time-invariant, i. e., permutable with shift: $\Phi(D^s f) = D^s(\Phi f)$ for each moment s and each signal f, where $D^s f(n) = f(n - s)$ is the shift operator.

We call a filter Φ *causal*, if the absence of input signal up to the moment $s \in \mathbb{Z}$, i. e., $f(t) = 0$ for $t < s$, implies the absence of output signal: $(\Phi f)(t) = 0$ for $t < s$. For a time-invariant filter, this is equivalent to the fulfillment of the last condition for $s = 0$.

We call a filter Φ *stable*, if it transforms bounded signals to bounded ones.

We call a filter Φ *invertible*, if the corresponding linear operator has a bounded inverse operator.

We call a sequence $W = \Phi\delta_0$, where $\delta_n := (\delta_{kn})_{k\in\mathbb{Z}}$, the *impulse response of filter* Φ, its inverse (discrete) Fourier transform

$$\varphi(z) = (\mathcal{F}^{-1}W)(z) := \sum_{n\in\mathbb{Z}} W(n)z^n \ (z \in \mathbb{T})$$

the *frequency response of filter* Φ, and the norm $\|\varphi\|_{L^\infty}$ its *amplitude distortion.* We denote by $\mathcal{F}$ the Fourier transform on a unit circle $\mathbb{T}$ given by the formula ($n \in \mathbb{Z}$)

$$(\mathcal{F}g)(n) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{\imath\lambda})e^{-\imath n\lambda}d\lambda.$$

It is known (Wiener's lemma) that every stationary filter is exactly a convolution operator with the sequence W, i. e., it has the form

$$(\Phi_W f)(n) = (f * W)(n) = \sum_{k\in\mathbb{Z}} f(k)W(n-k),$$

where the function $\varphi = \mathcal{F}^{-1}W$ belongs to $L^\infty(\mathbb{T})$. Here for the causality of this filter, it is necessary and sufficient that the condition is fulfilled $\varphi \in H^\infty(\mathbb{D})$, i. e., the condition $W(n) = 0$ holds with $n < 0$ (see, for example, [10], pp. 155–156, Lemmas 7.2.1, 7.2.3).

Below we consider a case, when the impulse response $(W(n))_{n\in\mathbb{Z}_+}$ is the sequence $(\int_0^1 t^n d\mu(t))_{n\in\mathbb{Z}_+}$ of moments of some measure μ (and, in particular, is a completely monotone sequence). In this case, filters arise with the frequency response $F(z)$ and the transfer function $F(1/z)$, where $F(z)$ is a Markov–Stieltjes type function with representing measure μ. Such functions, as a rule, allow good rational approximations. For example, in [12] a wide class of Markov–Stieltjes type functions is distinguished whose uniform approximations (that is, the approximation errors calculated in the Chebyshev metric) by rational functions of degree at most n are of order $\exp(-c\sqrt{n})$ ($c = \text{const} > 0$) (see also [8], [13], [14] and the references therein). In this case, the filters discussed below can be well approximated by filters with rational transfer functions, which are of great practical interest (see, for example, [11], [15], [16]).

As we noted earlier, for the measure $\mu \in M^b([0,1],\mathbb{C})$ with $|z| < 1$

$$F(z) = S\mu(z) = \sum_{n=0}^{\infty} h(n)z^n, \text{ where } h(n) = \int_0^1 t^n d\mu(t).$$

Let $Dx(k) = x(k-1)$ be a shift operator in $\ell^2(\mathbb{Z})$. We consider an operator

$$F(D) := \sum_{k=0}^{\infty} h(k)D^k,$$

defined initially on left finite signals from $\ell^2(\mathbb{Z})$.

Remark 3.1. On the set of plus-signals (i. e., on the subspace $\ell^2(\mathbb{Z}_+)$ of the space $\ell^2(\mathbb{Z})$) the system $y = F(D)v$ allows a realization if the form of the following dynamic system:

$$P_n(t) = tP_{n-1}(t) + v(n) \ (n \in \mathbb{N}), \quad P_0(t) = v(0),$$
$$y(n) = \int_0^1 P_n(t)d\mu(t).$$

(We note that the state equation of this system is independent of the measure μ, i. e., the filter.)

In fact, for the input plus-signal $v(n)$ we consider the following sequence of polynomials: $P_n(t) = \sum_{k=0}^n t^k v(n-k)(n \in \mathbb{N})$, $P_0(t) := v(0)$. It is easy to verify that it satisfies the state equation of the system. On the other hand, if $h(n)$ is, as above, the sequence of moments of measure μ, then it is not difficult to verify that

$$y(n) = \sum_{k=0}^{n} h(k)D^k v(n) = \sum_{k=0}^{n} h(k)v(n-k) = h * v(n) = \int_0^1 P_n(t)d\mu(t).$$

Thus, part of the results obtained in this paper can be interpreted in terms of this dynamical system.

To state and prove the following theorem (concerning the calculation of the inverse operator), we need some information about the functional calculus constructed in [17]–[19].

We will say that a (generally speaking, closed densely defined) operator A in a Banach space X belongs to the class $V_1(X)$, if $[0,1)$ is contained in the resolvent set $\rho(A)$ of operator A, and for some $M>0$ the inequality is fulfilled

$$\|R(t,A)\| \le \frac{M}{1-t}, \quad t\in[0,1)$$

(hereinafter $R(t,A)=(tI-A)^{-1}$ is the resolvent of operator A, I is the unit operator). Further we need the fact that unitary operators in a Hilbert space H belong to $V_1(H)$. This easily follows from the spectral theorem for such operators.

We also say that a function g belongs to the class R_1, if it belongs to the class $R[0,1]$ (see the definition of this class in Section 2) and is continuous at point 1. If g is a function of class R_1 with representing measure μ, $A\in V_1(X)$, then the operator $g(A)$ is defined by formula $g(A)=\int_0^1 R(t,A)d\mu(t)$. We call arising functional calculus R_1-*calculus*.

We set $Q_1=\{\varphi|\varphi=1/g,\ g\in R_1\}$. It is known that the function φ of class Q_1 has the form $\varphi(z)=\alpha+\beta z-f(z)$, where $f\in R_1$. In this case, with $A\in V_1(X)$ the operator $\varphi(A)$ with domain $D(A)$ is defined by the formula $\varphi(A)=\alpha+\beta A-f(A)$, in which $f(A)$ is understood in the sense of R_1-calculus. We call the arising functional calculus Q_1-*calculus*.

In [19] it was shown (the inversion theorem) that for any function $g\in R_1$ and any $A\in V_1(X)$ the operator $g(A)$ has the left inverse one given by the formula

$$g(A)^{-1}=\varphi(A),$$

where $\varphi=1/g$ and the right–hand side is understood in the sense of Q_1-calculus. Here from the point of view of applications of this theorem, it is important to note that the coefficients α and β, appearing in the representation $\varphi(z)=\alpha+\beta z-f(z)$ of the function $\varphi=1/g$, can be calculated by the formulas

$$\beta=-\frac{1}{\mu([a,b])},\ \alpha=\frac{1}{\mu([a,b])^2}\int\limits_a^b t d\mu(t),$$

where μ is the representing measure of function g, and the representing measure of the function f can be found, for example, using the inversion formula for the Stieltjes transform [20] (see [19]).

Theorem 3.1. *Let $F(z)=S\mu(z)$, $\mu\in M^b([0,1],\mathbb{R})$. If*

$$\int_0^1 \frac{d|\mu|(t)}{1-t}<\infty, \tag{3}$$

then operator $F(D)$ uniquely extends to an operator in $\ell^2(\mathbb{Z})$, which is stationary, causal, and stable filter with frequency response F and amplitude distortion $\|F\|_{H^\infty}$; here $\|F(D)\|_{\ell^2\to\ell^2}=\|h\|_{\ell^1}$ (h is the sequence of coefficients of the Taylor series of the function F at zero) and

$$F(D)=\int\limits_0^1 (I-tD)^{-1}d\mu(t), \tag{4}$$

where the Bochner integral converges in the norm of the operator.

With $\mu\ge 0$ condition (3) is also necessary for the operator $F(D)$ to have an extension to all $\ell^2(\mathbb{Z})$ to a stationary and causal filter. In addition, in this case the filter $F(D)$ is invertible, and its inverse one has the form

$$F(D)^{-1}=-\left(\frac{1}{F_1}\right)(D^{-1})D,$$

where F_1 is a function from R_1 with representing measure μ, and $(1/F_1)(D^{-1})$ is understood in the sense of Q_1-calculation.

Proof. Further we assume that $h(n) = 0$ with $n < 0$. Let condition (3) be fulfilled. Then $h \in \ell^1(\mathbb{Z})$, because

$$\sum_{n=0}^{\infty} |h(n)| = \sum_{n=0}^{\infty} \left| \int_0^1 t^n d\mu(t) \right| \le \int_0^1 \sum_{n=0}^{\infty} t^n d|\mu|(t) = \int_0^1 \frac{d|\mu|(t)}{1-t} < \infty.$$

As we noted above, for left finite signals x from $\ell^2(\mathbb{Z})$

$$F(D)x = h * x.$$

Therefore, for such signals $\|F(D)x\|_{\ell^2} \le \|h\|_{\ell^1} \|x\|_{\ell^2}$, and hence the operator $F(D)$ is uniquely continued to a (bounded) convolution operator with h in $\ell^2(\mathbb{Z})$, i. e., to the stationary filter, and $\|F(D)\| = \|h\|_{\ell^1}$. And since h is concentrated on $\mathbb{Z}_+$, the filter $F(D)$ is causal.

Since $h \in \ell^1(\mathbb{Z})$, we have that F belongs to a disk-algebra $A(\overline{\mathbb{D}})$ and, in particular, $F(z) = \sum_{n=0}^{\infty} h(n) z^n$ with all $z \in \mathbb{T}$. Hence, $F = \mathcal{F}^{-1} h$, i. e., F is the frequency response of the filter $F(D)$. In addition, the condition $h \in \ell^1(\mathbb{Z})$ guarantees the stability of the filter $F(D)$.

Finally, since μ does not have mass at point 1, we obtain

$$\int_0^1 (I - tD)^{-1} d\mu(t) = \int_{[0,1)} (I - tD)^{-1} d\mu(t) =$$

$$= \int_{[0,1)} \left(\sum_{n=0}^{\infty} t^n D^n \right) d\mu(t) = \sum_{n=0}^{\infty} \int_{[0,1)} t^n D^n d\mu(t) = \sum_{n=0}^{\infty} h(n) D^n = F(D).$$

Here the termwise integration of the series is legal by virtue of the Lebesgue theorem, because, taking into account the B. Levy theorem and condition (3), we have

$$\sum_{n=0}^{\infty} \int_{[0,1)} \|t^n D^n\| d|\mu|(t) = \sum_{n=0}^{\infty} \int_{[0,1)} t^n d|\mu|(t) = \int_{[0,1)} \frac{d|\mu|(t)}{1-t} < \infty.$$

Let $\mu \ge 0$ and the operator $F(D)$ is continued to all $\ell^2(\mathbb{Z})$, where the corresponding filter is stationary and causal. Then by the Wiener lemma ([10], p. 155, 7.2.1)

$$F(D)x = W * x, \tag{5}$$

and by the other Wiener lemma (see [10], p. 156, 7.2.3) $W = \mathcal{F}\varphi$ and $\varphi \in H^\infty(\mathbb{D})$. By setting in (5) $x = \delta_n$, we obtain with all integer n

$$\mathcal{F}F(n) = h(n) = W(n) = \mathcal{F}\varphi(n),$$

hence $F = \varphi$ and, therefore, $F \in H^\infty(\mathbb{D})$. If the number c is such that $|F(z)| \le c$, then with $z = r \in (0,1)$ we have

$$c \ge |F(r)| = \int_0^1 \frac{d\mu(t)}{1 - rt}.$$

Tending r to 1 and applying B. Levy's theorem, we conclude that condition (3) is satisfied. Let us prove the reversibility. Since the operator D is unitary and its spectrum is $\mathbb{T}$, under condition (3) the convolution operator $F(D)$ coincides with the value of the function F on the operator D in the sense of the classical functional calculus of unitary operators ($F \in A(\overline{\mathbb{D}})$, see Corollary 1.1). In particular, the spectrum of $F(D)$ equals $F(\mathbb{T})$. Since, by Corollary 1.1, the function F has no zeros on the unit circle, the operator $F(D)$ has a bounded inverse operator, i. e., the corresponding filter is reversible. To find the inverse operator, we recall that the proof of Lemma 2.2 shows that the function $F_1(z) := -1/zF(1/z)$ belongs to $R[0,1]$ and has a representing measure μ. Moreover, condition (3) implies $F_1 \in R_1$. Since the unitary operator D^{-1} belongs to $V_1(\ell^2(\mathbb{Z}))$, by the definition of R_1-calculus and formula (4)

$$D^{-1}F_1(D^{-1}) = \int_0^1 D^{-1}R(t, D^{-1})d\mu(t) = -\int_0^1 (I - tD^{-1})^{-1}d\mu(t) = -F(D)$$

(we used the identity $D^{-1}R(t, D^{-1}) = -(I - tD^{-1})^{-1}$). Hence $F(D)^{-1} = -(F_1(D^{-1}))^{-1}D = -(1/F_1)(D^{-1})D$. □

Corollary 3.1. Under conditions of Theorem 3.1 the amplitude-frequency response (AFR) of the filter Φ on the unit circle has the form

$$A(\omega)^2 = \int_{[0,1)^2} \frac{1 - (t+s)\cos\omega + ts}{(1 - 2t\cos\omega + t^2)(1 - 2s\cos\omega + s^2)} d\mu(t)d\mu(s) \quad (\omega \in \mathbb{R}).$$

In particular, if $\mu > 0$, then the estimate holds

$$A(\omega) \geq \int_{[0,1)} \frac{1-t}{1 - 2t\cos\omega + t^2} d\mu(t) \geq \int_{[0,1)} \frac{1-t}{(1+t)^2} d\mu(t) \quad (\omega \in \mathbb{R}).$$

Proof. In fact,

$$A(\omega)^2 = |F(e^{\imath\omega})|^2 = F(e^{\imath\omega})\overline{F(e^{\imath\omega})} = \int_{[0,1)} \frac{d\mu(t)}{e^{\imath\omega} - t} \int_{[0,1)} \frac{d\mu(s)}{e^{-\imath\omega} - s} = \int_{[0,1)^2} \frac{d\mu(t)d\mu(s)}{1 + st - se^{\imath\omega} - te^{-\imath\omega}} =$$

$$= \int_{[0,1)^2} \mathrm{Re}\frac{1}{1 + st - se^{\imath\omega} - te^{-\imath\omega}} d\mu(t)d\mu(s) = \int_{[0,1)^2} \frac{1 - (t+s)\cos\omega + ts}{(1 - 2t\cos\omega + t^2)(1 - 2s\cos\omega + s^2)} d\mu(t)d\mu(s).$$

It remains to use the fact that $1 - (t+s)\cos\omega + ts \geq (1-t)(1-s)$.

Further we call a Z-transform $\tilde{H}(z) := \sum_{n=-\infty}^{\infty} W(n)z^{-n}$ the *transfer function* of stationary filter $\Phi = \Phi_W$ (the term of the *system function* is also used, see, for example, [11], [15]).

The following example gives us a wide class of filters with rational transfer functions, which are subjects of Theorem 3.1.

Example 3.1. We consider DT-LTI-system (a recursive filter) Φ of finite order described by the difference equation $(a_k, b_l \in \mathbb{R})$

$$\sum_{k=0}^{p} a_k y[n+k] = \sum_{l=1}^{q} b_l x[n+l].$$

We assume that $q \leq p$ and the characteristic equation

$$\sum_{k=0}^{p} a_k z^k = 0$$

has p different roots $t_i \in [0,1)$ $(i = 1, \ldots, p)$. Using the decomposition into simplest fractions, we obtain for the transfer function of filter Φ the presentation

$$\tilde{H}_1(z) = \frac{\sum_{l=1}^{q} b_l z^l}{\sum_{k=0}^{p} a_k z^k} = \sum_{i=1}^{p} \frac{\mu_i z}{z - t_i} = \sum_{i=1}^{p} \frac{\mu_i}{1 - t_i z^{-1}}, \tag{6}$$

where $\mu_i = \tilde{H}_1(z)(1 - t_i z^{-1})|_{z=t_i}$ are real numbers. The measure $\mu := \sum_{i=1}^{p} \mu_i \delta_{t_i}$ (δ_{t_i} is the measure of unit mass concentrated at point t_i) satisfies condition (3). Let us consider a rational function of the Markov–Stieltjes type

$$F(z) := S\mu(z) = \sum_{i=1}^{p} \frac{\mu_i}{1 - t_i z}.$$

Then $\Phi = F(D)$. In fact, the impulse response of system $F(D)$, as we showed earlier, is $h(n) = \int_0^1 t^n d\mu(t) = \sum_{i=1}^{p} \mu_i t_i^n$ ($n \in \mathbb{Z}_+$). Therefore the transfer function of filter $F(D)$ is (with $|z| > \max_i |t_i|$)

$$\tilde{H}(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{p} \mu_i t_i^n \right) z^{-n} = \sum_{i=1}^{p} \sum_{n=0}^{\infty} \mu_i \left(\frac{t_i}{z} \right)^n = \sum_{i=1}^{p} \frac{\mu_i}{1 - t_i z^{-1}} = \tilde{H}_1(z),$$

i. e., it coincides with the transfer function of the filter Φ. Thus, by Theorem 3.1, Φ is stationary, causal, and stable filter with frequency response F and amplitude distortion $\|F\|_{H^\infty} = \sup_{z\in\mathbb{D}} |\sum_{i=1}^{p} \frac{\mu_i}{1-t_i z}|$. Here

$$\|\Phi\|_{\ell^2 \to \ell^2} = \|h\|_{\ell^1} = \sum_{n=0}^{\infty} \left| \sum_{i=1}^{p} \mu_i t_i^n \right|.$$

Due to the Corollary 3.1 for AFR of this filter we have

$$A(\omega)^2 = \sum_{i,j=1}^{p} \frac{1 - (t_i + s_j)\cos\omega + t_i s_j}{(1 - 2t_i \cos\omega + t_i^2)(1 - 2s_j \cos\omega + s_j^2)} \mu_i \mu_j.$$

Further, $\Phi = \sum_{i=1}^{p} \mu_i (I - t_i D)^{-1}$. As is known, the spectrum of shift operator D in the space $\ell^2(\mathbb{N})$ is the unit disk $\overline{\mathbb{D}}$, and therefore the operator $(I - tD)^{-1}$ exists with $|t| < 1$. It is easy to verify that for any left finite DT-signal $x \in \ell^2(\mathbb{N})$ and any t with $|t| < 1$

$$((I - tD)^{-1} x)[n] = t^{n-1} x[1] + t^{n-2} x[2] + \cdots + t x[n-1] + x[n] \ (n \in \mathbb{Z}).$$

Therefore, due to formula (4) under our assumptions the filter Φ has the form

$$y[n] = (\Phi x)[n] = \sum_{i=1}^{p} \mu_i (t_i^{n-1} x[1] + t_i^{n-2} x[2] + \cdots + t_i x[n-1] + x[n]) \ (n \in \mathbb{N}).$$

This formula also gives (under above imposed constraints) a solution to the considered difference equation in the space $\ell^2(\mathbb{N})$.

If additionally all $\mu_i > 0$, then we have a case of positive measure. (The simplest situation, when this can happen: $p = q = 1, a_1 = 1, a_0 \in (-1, 0), b_1 > 0$. In the case $p = q = 2$ we are talking about systems of the form

$$y[n+2] + a_1 y[n+1] + a_0 y[n] = b_2 x[n+2] + b_1 x[n+1],$$

where $a_1 = -(t_1 + t_2), a_0 = t_1 t_2, \ b_1 = -(t_1 \mu_2 + t_2 \mu_1), b_2 = \mu_1 + \mu_2$, and $t_1, t_2 \in [0, 1), \ \mu_1, \mu_2 > 0$.) In this case, the filter is invertible and Theorem 3.1 gives an algorithm of its inversion.

Remark 3.2. The case of positive measure can also arise with the discretization of CT-systems of finite order. Really, if the transfer function H_I of such a system is decomposed into simplest fractions as follows:

$$H_I(s) = \sum_{i=1}^{m} \frac{A_i}{s - s_i},$$

then the transfer function arising with its discretization, with the step T of DT-system is

$$\tilde{H}(z) = \sum_{i=1}^{m} \frac{A_i T z}{z - e^{s_i T}} \quad (T > 0)$$

(see, for example, [15], p. 167, (5-133), (5-134)). With $A_i > 0$ and $s_i < 0$ we obtain a transfer function of form (6) with $\mu_i = A_i T > 0$, $t_i = e^{s_i T} \in [0, 1)$. As we showed in Example 3.1, the corresponding DT-system has the form $F(D)$ with positive representing measure $\mu = \sum_{i=1}^{m} \mu_i \delta_{t_i}$.

The following two theorems characterize the filters of the form $F(D)$ in terms of their transfer functions. To formulate the first of them (which also describes the impulse response) we consider iterated difference operators ($\Delta := I - D^{-1}$)

$$\Delta^n W(k) = \sum_{j=0}^{n} (-1)^j C_n^j W(j + k) \quad (k, n \in \mathbb{Z}_+).$$

Theorem 3.2. *Let Φ be a stationary and causal filter with transfer function $\tilde{H}$ and impulse response W. The following assertions are equivalent:*

1) *Φ has the form $F(D)$, where $F = S\mu$, $\mu \in M_+^b([0, 1])$;*

2) *the function $\tilde{H}(1/z)$ satisfies the conditions of Lemma 2.2;*

3) *$\Delta^n W(k) \geq 0$ $(k, n \in \mathbb{Z}_+)$, i. e., the sequence W is completely monotone.*

Here $F(z) = \tilde{H}(1/z)$, and W is the sequence of moments of the measure μ.

Proof. 1) ⇒ 2). In notation introduced above, if $\Phi = F(D) := S\mu(D)$, then $\forall n \ \ W(n) = h(n)$. And since

$$\tilde{H}(\zeta) = \sum_{n=0}^{\infty} W(n)\zeta^{-n} = \sum_{n=0}^{\infty} h(n) \left(\frac{1}{\zeta}\right)^n = F\left(\frac{1}{\zeta}\right),$$

we have that the function $\tilde{H}(1/z) = F(z)$ satisfies conditions of Lemma 2.2.

2) ⇒ 3). Let a function $\tilde{H}(1/z)$ satisfy conditions of Lemma 2.2. Then $\tilde{H}(1/z) = S\mu(z)$, $\mu \geq 0$, where $z \in \mathbb{D}$. Hence it follows that $W(n) = \int_0^1 t^n d\mu(t)$ $(n \in \mathbb{Z}_+)$, therefore due to the Hausdorff theorem, assertion 3) holds true (see, for example, [21]).

3) ⇒ 1). If 3) is fulfilled, then by the mentioned Hausdorff theorem $W(n) = \int_0^1 t^n d\mu(t)$ $(n \in \mathbb{Z}_+)$ for some measure $\mu \in M_+^b([0, 1])$. If we set $F(z) := S\mu(z)$, then in terms introduced above $h = W$. Here for any left finite subsequence from $\ell^2(\mathbb{Z})$ due to formula (5) the equality holds $\Phi x = W * x = F(D)x$. Therefore, the operator $F(D)$ is bounded, hence $\Phi = F(D)$.

Theorem 3.3. *A stationary causal filter Φ with transfer function $\tilde{H}$ has the form $F(D)$, where $F = S\mu$, $\mu \in M^b([0, 1], \mathbb{C})$ if and only if the function $\tilde{H}(1/z)$ satisfies conditions of Lemma 2.1. In addition, $F(z) = \tilde{H}(1/z)$.*

The proof of this theorem is similar to the proof of Theorem 3.2.

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