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Oscillation Criteria for Solutions of Delay Differential Equations of the First Order

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Abstract—We establish some new effective oscillation conditions for solutions to linear delay differential equations of the first order. We develop a new approach to obtaining oscillation conditions in the form of the upper limit of a function of equation parameters. We apply the proposed approach to equations with one and several concentrated delays and to those with a distributed delay. We demonstrate the advantages of the obtained results over the well-known ones.

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INTRODUCTION

The main essential distinction of delay differential equations from ordinary ones consists in the fact that linear equations of the first order may have oscillating solutions [1]. Research techniques for distinguishing oscillating solutions to delay equations from sign-constant ones have been being systematically developed since the middle of the XXth century. First oscillation criteria were established for simplest delay equations. The more complicated are the considered equations, the more difficult is the task of establishing oscillation criteria for their solutions. It is possibly to generalize some criteria, provided that they satisfy certain requirements such as the effective verifiability, simplicity, and accuracy.

In this paper, we study the family of oscillation criteria that go back to results obtained in the 70th of the last century in papers [2, 3]. Let us give the result obtained in the monograph [4] (we generalize it below).

Consider the equation

$$\dot{x}(t) + p(t)x(h(t)) = 0, \quad t \geq 0, \tag{1}$$

where functions p and h are continuous, $p(t) \geq 0$ and $h(t) \leq t$ for all $t \geq 0$, and $\lim_{t\to+\infty} h(t) = +\infty$.

Theorem 1. *Let the function h be monotonically increasing, while*

$$\varlimsup_{t\to\infty} \int_{h(t)}^{t} p(s)\,ds > 1.$$

Then all solutions to Eq. (1) *are oscillating.*

Proof is rather simple. It consists in getting a rough integral estimate of the coefficient p, which ensures the oscillation property. This result was later refined and generalized. Here we consider only the ideas that seem to be most promising. The main result of this paper consists in strengthening

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the well-known sufficient oscillation conditions for all solutions to delay equations of various kinds by using the principal idea of [5].

In the first section of this paper, we consider the ways to refine Theorem 1. In the second section, we study an equation with several delays; we give some known results for it, get some new ones, and compare them to each other. In the third section, we establish oscillation conditions for solutions to distributed delay equations. We apply the obtained result to an integro-differential equation and compare it with well-known results.

1. THE WAYS TO REFINE THEOREM 1

Let us find out the ways to weaken the oscillation conditions indicated in Theorem 1.

In paper [3], one proves that the constant value in the right-hand side of the inequality is unimprovable. Moreover, one can easily modify considerations made in [3] so as to prove that the strict inequality cannot be replaced with a nonstrict one.

The requirement of the monotonicity of the function h mentioned in Theorem 1 is essential. Moreover, in the paper [6], one proves that if the function h is not monotonic, then no positive constant value A makes the inequality

$$\varlimsup_{t\to\infty}\int_{h(t)}^{t} p(s)\,ds > A$$

imply the oscillation of all solutions to Eq. (1). Nevertheless, one can weaken the requirement of the delay monotonicity by using functions constructed on the base of the function h.

Denote $g(t)=\sup_{s\le t} h(s)$. Evidently, the function g is monotonically increasing, while $g(t)\ge h(t)$ for all t.

Theorem 2. *Let*

$$\varlimsup_{t\to\infty}\int_{g(t)}^{t} p(s)\,ds > 1. \tag{2}$$

Then all solutions to Eq. (1) *are oscillating.*

Theorem 1 is a corollary of Theorem 2, because if the function h is monotonically increasing, then $g(t)\equiv h(t)$. Theorem 2, in turn, can be refined by increasing the integrand.

Theorem 3 ([7], [6]). *Let*

$$\varlimsup_{t\to\infty}\int_{g(t)}^{t} p(s)\exp\left\{\int_{h(s)}^{g(t)} p(\tau)d\tau\right\} ds > 1. \tag{3}$$

Then all solutions to Eq. (1) *are oscillating.*

Let us now consider Eq. (1) under the following assumptions: the function p is locally summable, the function h is Lebesgue measurable, and $h(t)\le t$ for almost all (a. a.) $t\ge 0$. We understand a *solution* to Eq. (1) as a locally absolutely continuous function $x:[0,\infty)\to\mathbb{R}$, for which there exists an *initial function* $\varphi:(-\infty,0]\to\mathbb{R}$ such that for a. a. $t\ge 0$, equality (1) is valid with $x(\xi)=\varphi(\xi)$ for $\xi\le 0$. For any bounded Borel measurable function φ, Eq. (1) has a unique solution ([8], Ch. 5).

We call an equation, all whose solutions are oscillating, an *oscillating equation.*

All further results are obtained under the assumption that $p(t)\ge 0$ for a. a. $t\ge 0$ and $\operatorname{ess}\lim\limits_{t\to+\infty} h(t)=+\infty$. Put $g(t)=\operatorname{ess}\sup\limits_{s\le t} h(s)$.

One can further refine inequality (3). In the paper [9], one replaces the function $\exp\int\limits_{h(s)}^{g(t)} p(\tau)d\tau$ with majorizing ones that are obtained with the help of a recurrent correlation. In a more general case, this idea is realized in the paper [10] for an equation with several delays.

Let us describe the result obtained for Eq. (1). Denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Put

$$a_0(t,s) = 1; \quad a_{k+1}(t,s) = \exp\left\{\int_s^t p(\tau)a_k(\tau, h(\tau))d\tau\right\}, \quad k \in \mathbb{N}_0. \tag{4}$$

Theorem 4 ([10]). *Assume that for some $k \in \mathbb{N}_0$,*

$$\overline{\lim_{t\to\infty}} \int_{g(t)}^t p(s)a_k(g(t), h(s))\, ds > 1. \tag{5}$$

Then Eq. (1) *is an oscillating one.*

Consider one more recently proposed approach to refining Theorem 2.

Denote $H(t) = \{s \mid h(s) \le t \le s\}$.

Theorem 5 ([5]). *Let*

$$\overline{\lim_{t\to\infty}} \int_{H(t)} p(s)\, ds > 1.$$

Then Eq. (1) *is an oscillating one.*

The proof of Theorem 5 is nearly as simple as that of Theorem 2. In the case when the function h is monotone, these theorems are equivalent, otherwise Theorem 5 always gives a larger value of the integral in the left-hand side of the inequality than that in Theorem 2. Moreover, Theorem 5 is valid in certain cases, when Theorem 3 is false.

Example 1. Consider Eq. (1), where

$$p(t) = \begin{cases} 0, & t \in [2n, 2n+1); \\ \dfrac{5}{9}, & t \in [2n+1, 2n+2), \end{cases} \qquad h(t) = \begin{cases} 2n, & t \in [2n, 2n+1); \\ 2n-1, & t \in [2n+1, 2n+2), \end{cases} \quad n \in \mathbb{N}_0.$$

By Theorem 5, this equation is an oscillating one. Really, since $H(2n+1) = [2n+1, 2n+2) \cup [2n+3, 2n+4)$, we conclude that

$$\overline{\lim_{t\to\infty}} \int_{H(t)} p(s)\, ds \ge \int_{H(2n+1)} p(s)\, ds = \int_{2n+1}^{2n+2} \frac{5}{9}\, ds + \int_{2n+3}^{2n+4} \frac{5}{9}\, ds = \frac{10}{9} > 1.$$

However Theorem 3 does not allow one to prove the oscillation. Really, for $t \in [2n, 2n+2)$, $n \in \mathbb{N}_0$, we get the equality $g(t) = 2n$, whence

$$\overline{\lim_{t\to\infty}} \int_{g(t)}^t p(s) \exp\left\{\int_{h(s)}^{g(t)} p(\tau)\, d\tau\right\} ds = \int_{2n}^{2n+2} p(s) \exp\left\{\int_{h(s)}^{2n} p(\tau)\, d\tau\right\} ds =$$
$$= \int_{2n+1}^{2n+2} \frac{5}{9} \exp\left\{\int_{2n-1}^{2n} \frac{5}{9}\, d\tau\right\} ds = \frac{5\exp\frac{5}{9}}{9} < 1.$$

Therefore, Theorem 3, which is the first iteration of Theorem 4, is not stronger than Theorem 5. However, the further iterations can compensate this drawback. In the next section, we obtain a result that has all virtues of theorems 4 and 5.

2. AN EQUATION WITH SEVERAL DELAYS

2.1. *Some well-known results.* Consider the equation

$$\dot{x}(t) + \sum_{i=1}^{m} p_i(t)x(h_i(t)) = 0, \quad t \geq 0, \tag{6}$$

where for $i = \overline{1, m}$ functions p_i are locally summable, functions h_i are Lebesgue measurable, and $h_i(t) \leq t$ for a. a. $t \geq 0$. For any bounded Borel initial function, Eq. (6) has a unique solution, whose definition verbatim coincides with the definition of a solution to Eq. (1).

In what follows, we assume that $p_i(t) \geq 0$ and $\operatorname{ess}\lim\limits_{t\to+\infty} h_i(t) = +\infty$, $i = \overline{1, m}$. Put $g_i(t) = \operatorname{ess}\sup\limits_{s\leq t} h_i(s)$ and $g(t) = \max_i g_i(t)$.

One can easily generalize Theorem 2 as follows.

Theorem 6 ([11], P. 44)**.** *Let*

$$\varlimsup_{n\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s)\, ds > 1.$$

Then Eq. (6) *is an oscillating one.*

An evident weakness of this oscillation condition is the fact that the value of the integral does not increase as $h_i(t)$ become less and $g(t)$ remains constant, though the oscillation property of Eq. (6) mostly depends just on the terms with relatively large differences $t - h_i(t)$. In particular, by adding a term without a delay in the equation, one makes Theorem 6 false.

The following generalization of inequality (2) for Eq. (6) seems to be quite natural:

$$\varlimsup_{t\to\infty} \sum_{i=1}^{m} \int_{g_i(t)}^{t} p_i(s)\, ds > 1. \tag{7}$$

However, the example given in the paper [5] demonstrates that inequality (7) does not imply the oscillation property of Eq. (6). On the other hand, considering in the same way as in the paper [6], we get the following refined variant of Theorem 6.

Theorem 7. *Let*

$$\varlimsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp \int_{h_i(s)}^{g(t)} \sum_{l=1}^{m} p_l(u)\, du\, ds > 1. \tag{8}$$

Then Eq. (6) *is an oscillating one.*

Theorem 7 is a generalization of Theorem 3 for Eq. (6).

One can further increase the left-hand side of inequality (8) and thus refine the oscillation condition. Put

$$a_0(t,s) = 1; \quad a_{k+1}(t,s) = \exp\left\{\int_{s}^{t} \sum_{i=1}^{m} p_i(\tau)a_k(\tau, h_i(\tau))d\tau\right\}, \quad k \in \mathbb{N}_0. \tag{9}$$

Theorem 8 ([10])**.** *Assume that for some* $k \in \mathbb{N}_0$,

$$\varlimsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s)a_k(g(t), h_i(s))\, ds > 1. \tag{10}$$

Then Eq. (6) *is an oscillating one.*

Applying Theorem 8 to Eq. (1), we get Theorem 4, and for $k = 1$ Theorem 8 implies Theorem 7. In subsection 2.3, we describe some recent attempts to strengthen Theorem 8.

Consider one more approach. For $i = \overline{1, m}$ we denote

$$H_i(t) = \{s \mid h_i(s) \le t \le s\}\,. \tag{11}$$

Theorem 9 ([5])**.** *Let*

$$\varlimsup_{t\to\infty} \sum_{i=1}^{m} \int_{H_i(t)} p(s)\,ds > 1.$$

Then Eq. (6) *is an oscillating one.*

Note that condition (7) does not ensure the oscillation property of Eq. (6). However, by replacing the integration over segments $[g_i(t), t]$ with that over sets $H_i(t)$ we generalize Theorem 5 as Theorem 9, where each coefficient p_i is integrated over its set.

Example 1 demonstrates that Theorem 5 can be valid in some cases, when Theorem 3 is false. In view of the said above, the transition to several delays strengthens the effect. This suggests the possibility of combining the virtues of theorems 8 and 9 in one assertion.

2.2. *New results.* Let us use the sequence of functions $\{a_k\}$ defined by formulas (9) and the family of sets $\{H_i\}$ defined by formula (11).

Lemma 1 ([10])**.** *Let x be a positive solution to Eq.* (6)*. Then $x(t)a_k(t, s) \le x(s)$ for all $k \in \mathbb{N}_0$, $s, t \in \mathbb{R}$, $0 \le s \le t$.*

Theorem 10. *Assume that for some $k \in \mathbb{N}_0$,*

$$\varlimsup_{t\to\infty} \sum_{i=1}^{m} \int_{H_i(t)} p_i(s) a_k(t, h_i(s))\,ds > 1.$$

Then Eq. (6) *is an oscillating one.*

Proof. Assume that all assumptions of the theorem are fulfilled, but the solution x to Eq. (6) is not oscillating. Without loss of generality, we assume that there exists $t_0 \ge 0$ such that $x(t) > 0$ for all $t \ge t_0$. Since $\operatorname{ess}\lim\limits_{t\to+\infty} h_i(t) = +\infty$, there exists $t_1 \ge t_0$ such that $h_i(t) \ge t_0$ for $i = \overline{1, m}$ and a. a. $t \ge t_0$. Since $\dot{x}(t) \le 0$ for a. a. $t \ge t_1$, there exists $t_2 \ge t_1$ such that $x(h_i(t)) \ge x(t)$ for all $t \ge t_2$ and $i = \overline{1, m}$; in addition,

$$\sum_{i=1}^{m} \int_{H_i(t_2)} p_i(s) a_k(t_2, h_i(s))\,ds > 1.$$

Finally, there also exists $t_3 \ge t_2$ such that

$$\sum_{i=1}^{m} \int_{S_i} p_i(s) a_k(t_2, h_i(s))\,ds > 1,$$

where $S_i = H_i(t_2) \cap [t_2, t_3]$. Therefore, in view of formula (6) and Lemma 1,

$$\begin{aligned} x(t_3) = x(t_2) + \int_{t_2}^{t_3} \dot{x}(s)\,ds = x(t_2) - \int_{t_2}^{t_3} \sum_{i=1}^{m} p_i(s) x(h_i(s))\,ds \le \\ \le x(t_2) - \sum_{i=1}^{m} \int_{S_i} p_i(s) x(h_i(s))\,ds \le x(t_2) - \sum_{i=1}^{m} \int_{S_i} p_i(s) x(t_2) a_k(t_2, h_i(s))\,ds = \\ = x(t_2)\left(1 - \sum_{i=1}^{m} \int_{S_i} p_i(s) a_k(t_2, h_i(s))\,ds\right) < 0, \end{aligned}$$

which contradicts the assumption. □

Theorem 8 follows from Theorem 10. Really, $[g(t), t] \subset H_i(g(t))$ for $i = \overline{1, m}$ and a. a. $t \geq 0$. Consequently, if condition (10) is fulfilled, then so is the condition

$$\varlimsup_{t\to\infty} \sum_{i=1}^{m} \int_{H_i(g(t))} p_i(s) a_k(g(t), h_i(s))\, ds > 1.$$

It remains to note that $g(t) \to \infty$ as $t \to \infty$.

By fixing $k = 1$ we get the following assertion.

Corollary 1. Let

$$\varlimsup_{t\to\infty} \sum_{i=1}^{m} \int_{H_i(t)} p_i(s) \exp \int_{h_i(s)}^{t} \sum_{l=1}^{m} p_l(\tau)\, d\tau\, ds > 1.$$

Then Eq. (6) is an oscillating one.

Since Theorem 8 follows from Theorem 10, Theorem 7 follows from Corollary 1. On the other hand, example 1 demonstrates that even among equations in form (1) there exist those whose oscillations are proved in Corollary 1 but cannot be substantiated by Theorem 7. As for Eq. (6), the following example demonstrates that Corollary 1 is more effective than Theorem 7 even in the case of constant coefficients and delays.

Example 2. Consider the equation

$$\dot{x}(t) + \frac{1}{4}x(t-1) + \frac{1}{4}x(t-2) = 0.$$

Let us apply Theorem 7. It holds that $g(t) = t - 1$,

$$\int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp \int_{h_i(s)}^{g(t)} \sum_{l=1}^{m} p_l(u)\, du\, ds = \int_{t-1}^{t} \frac{1}{4} \left(\exp \int_{s-1}^{t-1} \frac{1}{2}\, du + \exp \int_{s-2}^{t-1} \frac{1}{2}\, du \right) ds =$$

$$= \frac{1}{4} \int_{t-1}^{t} \left(\exp \frac{t-s}{2} + \exp \frac{t-s+1}{2} \right) ds = \frac{1}{4} \int_{0}^{2} \exp \frac{s}{2}\, ds = \frac{e-1}{2} < 1.$$

Now by applying Corollary 1 we conclude that $H_1(t) = [t, t+1]$, $H_2(t) = [t, t+2]$,

$$\sum_{i=1}^{m} \int_{H_i(t)} p_i(s) \exp \int_{h_i(s)}^{t} \sum_{l=1}^{m} p_l(u)\, du\, ds = \int_{t}^{t+1} \frac{1}{4} \exp \int_{s-1}^{t} \frac{1}{2}\, du\, ds +$$

$$+ \int_{t}^{t+2} \frac{1}{4} \exp \int_{s-2}^{t} \frac{1}{2}\, du\, ds = \frac{1}{4} \left(\int_{0}^{1} \exp \frac{s}{2}\, ds + \int_{0}^{2} \exp \frac{s}{2}\, ds \right) = \frac{e^{\frac{1}{2}} - 1 + e - 1}{2} > 1.$$

Therefore, the number of iterations of Theorem 10 that are necessary for proving the oscillation property of Eq. (6) cannot be greater (but can be less) than the number of necessary iterations of Theorem 8.

2.3. *Some comparisons.* Before comparing the results obtained above with results of other recently published papers, let us apply theorems given in Section 1 to an autonomous equation, i.e., the simplest case of Eq. (1).

Let $p(t) \equiv c = \text{const} > 0$, $h(t) \equiv t - r$, and $r = \text{const} > 0$. Eq. (1) takes the form

$$\dot{x}(t) + cx(t-r) = 0, \quad t \geq 0. \tag{12}$$

As is known ([1], Ch. 4), all solutions to Eq. (12) are oscillating if and only if the characteristic equation $\lambda + ce^{-\lambda r} = 0$ has no real roots, which is equivalent to the condition $cr > \frac{1}{e}$.

By applying theorems 1 and 2 to Eq. (12) we get the oscillation condition $cr>1$, while the application of Theorem 3 leads to the condition $cr > \ln 2$. Let us prove that Theorem 4 implies the condition $cr > \frac{1}{e}$.

Formula (4) gives the correlation

$$a_{k+1}(t,s) = \exp\left\{ c\int_s^t a_k(\tau,\tau-r)\,d\tau \right\},$$

whence

$$a_1(t,s) = e^{c\int_s^t d\tau} = e^{c(t-s)},\ a_2(t,s) = e^{c\int_s^t e^{cr}\,d\tau} = e^{(t-s)ce^{cr}}, \dots,\ a_k(t,s) = e^{(t-s)ce^{cre^{cr\dots e^{cr}}}}.$$

Denote the left-hand side of inequality (5) by $F(k) = \overline{\lim}_{t\to\infty} \int_{g(t)}^t p(s)a_k(g(t),h(s))\,ds$. It holds that

$$F(k) = c\lim_{t\to\infty}\int_{t-r}^t a_k(t,s)\,ds = c\int_0^r a_k(r,s)\,ds = \frac{f(k+1)-1}{f(k)}, \quad k\in\mathbb{N}_0,$$

where the recurrent sequence $\{f(k)\}$ is a solution to the problem

$$f(k+1) = (e^{cr})^{f(k)}, \quad f(0)=1.$$

Put $u(x) = e^{crx} - x$. If $cr > \frac{1}{e}$, then $\min\limits_{x\geq 0} u(x) > 0$, consequently, $\inf\limits_{k\in\mathbb{N}_0} f(k+1)-f(k) > 0$, and therefore $f(k)\to+\infty$ as $k\to\infty$. Hence, since $\frac{e^{crx}}{x}\to+\infty$ as $x\to+\infty$, we conclude that

$$F(k) = \frac{(e^{cr})^{f(k)}-1}{f(k)} \to +\infty \text{ with } k\to\infty.$$

By Theorem 4 Eq. (12) is an oscillating one.

Moreover, Theorem 4 demonstrates that the oscillation condition $cr > \frac{1}{e}$ is exact. Really, consider the case when $cr \leq \frac{1}{e}$. Then there exists $x_1 \in (1,e]$ such that $u(x_1) = 0$. It holds that $f(0) = 1 < x_1$, and if $f(k) < x_1$, then $f(k+1) = e^{crf(k)} < e^{crx_1} = x_1$, whence in view of the induction principle, $f(k) < x_1 \leq e$ for all $k\in\mathbb{N}_0$. One can easily see that the function $v(x) = \frac{e^{crx}-1}{x}$ is increasing on the segment $[1,e]$, consequently, for all $k\in\mathbb{N}_0$,

$$F(k) = v(f(k)) \leq v(e) \leq \frac{e-1}{e} < 1.$$

After publishing the paper [10], in international press there have appeared several papers whose authors claimed to having strengthened Theorem 8. Moreover, they believed that each new set of conditions ensuring the oscillation of Eqs. (1) and (6) "improves all known results".

Theorem 11 ([12])**.** *Let $p_0(t) = p(t)$ and for all $k = 0,1,2,\dots,$*

$$p_{k+1}(t) = p_0(t)\left[1 + \int_{h(t)}^t p_0(s)\exp\left(\int_{h(s)}^{g(t)} p_k(\tau)\,d\tau\right)ds\right]. \tag{13}$$

If for some $k\in\mathbb{N}_0$,

$$\overline{\lim_{t\to\infty}}\int_{g(t)}^t p_0(s)\exp\left(\int_{h(s)}^{g(t)} p_k(\tau)\,d\tau\right)ds > 1, \tag{14}$$

then Eq. (1) *is oscillating.*

The following example demonstrates that Theorem 11 does not allow one to get the oscillation criterion $cr > \frac{1}{e}$ for the autonomous equation (12).

Example 3. Consider the equation

$$\dot{x}(t) + \frac{1}{5}x(t-2) = 0, \quad t \geq 0. \tag{15}$$

Since $cr = \frac{1}{5} \cdot 2 > \frac{1}{e}$, Eq. (15) is oscillating.

Let us apply Theorem 11 to this equation. Since $p(t) \equiv p_0 = \text{const}$ and $h(t) \equiv g(t) = t - 2$, in view of Definition (13) and the induction principle, $p_k(t) \equiv p_k = \text{const}$ for all $k \in \mathbb{N}_0$; here

$$p_{k+1} = \frac{1}{5}\left[1 + \int_{t-2}^{t} \frac{1}{5} \exp\left(\int_{s-2}^{t-2} p_k \, du\right) ds\right] = \frac{1}{5}\left(1 + \frac{1}{5}\int_0^2 e^{p_k s} \, ds\right) = \frac{1}{5}\left(1 + \frac{e^{2p_k} - 1}{5p_k}\right).$$

Denote $f(x) = \frac{e^{2x}-1}{x}$. One can easily see that $f(x)$ is increasing for $x > 0$. Consequently, if $p_k \in (0, \frac{1}{2})$, then

$$0 < p_{k+1} = \frac{1}{5} + \frac{1}{25}f(p_k) < \frac{1}{5} + \frac{1}{25} \cdot f\left(\frac{1}{2}\right) = \frac{1}{5} + \frac{1}{25} \cdot 2(e-1) < \frac{1}{2}.$$

Since $p_0 = \frac{1}{5} \in (0, \frac{1}{2})$, we conclude that $p_k \in (0, \frac{1}{2})$ for all $k \in \mathbb{N}_0$.

Let us verify inequality (14). It holds that

$$\overline{\lim_{t\to\infty}} \frac{1}{5}\int_{t-2}^{t} \exp\left(p_k \cdot (t-s)\right) ds = \frac{1}{5}\int_0^2 e^{p_k s} \, ds = \frac{e^{2p_k} - 1}{5p_k} = \frac{f(p_k)}{5} < \frac{f\left(\frac{1}{2}\right)}{5} = \frac{2(e-1)}{5} < 1.$$

Inequality (14) is violated with any $k \in \mathbb{N}_0$, consequently, Theorem 11 is inapplicable to Eq. (15).

In subsequent papers, Theorem 11 was refined by increasing the number of integrals in formulas (13) and (14) and by refining the function p_0. The result was applied both to Eq. (1) and to Eq. (6).

Theorem 12 ([13]). *Let*

- $h(t) = \max_i h_i(t)$, $i = \overline{1, m}$, *and* $P(t) = \sum_{j=1}^{m} p_j(t)$;
- $\alpha = \underline{\lim}_{t\to+\infty} \int_{h(t)}^{t} P(s) \, ds$ *and let* $\lambda = \lambda_0$ *be the least root of the equation* $\lambda = e^{\alpha\lambda}$;
- $P_0(t) = \lambda_0 P(t)$ *and for all* $k = 0, 1, 2, \dots$

$$P_{k+1}(t) = P(t)\left[1 + \int_{h(t)}^{t} P(s) \exp\left(\int_{h(s)}^{t} P(u) \exp\left(\int_{h(u)}^{u} P_k(v) \, dv\right) du\right) ds\right]. \tag{16}$$

If for some $k \in \mathbb{N}_0$ *it holds that*

$$\overline{\lim_{t\to\infty}} \int_{g(t)}^{t} P(s) \exp\left(\int_{h(s)}^{g(t)} P(u) \exp\left(\int_{h(u)}^{u} P_k(v) \, dv\right) du\right) ds > 1, \tag{17}$$

then Eq. (6) *is oscillating.*

The next example demonstrates that Theorem 12 generalizes neither Theorem 8, nor even Theorem 7. The reason is analogous to the mentioned above property of Theorem 6, namely, the left-hand side of inequality (17) does not grow when some functions among h_i diminish.

Example 4. Consider the equation

$$\dot{x}(t)+\frac{1}{10}(x(h_1(t))+x(h_2(t)))=0,\quad t\in\mathbb{R}_+, \tag{18}$$

where

$$h_1(t)=\begin{cases}t-1, & t\in[2n,2n+1);\\ t-12, & t\in[2n+1,2n+2),\end{cases}\qquad h_2(t)=\begin{cases}t-12, & t\in[2n,2n+1);\\ t-1, & t\in[2n+1,2n+2).\end{cases}$$

Let us apply Theorem 7. Since $g(t)\equiv t-1$, we conclude that

$$\varlimsup_{t\to\infty}\int_{g(t)}^{t}\sum_{i=1}^{m}p_i(s)\exp\int_{h_i(s)}^{g(t)}\sum_{l=1}^{m}p_l(u)\,du\,ds=\varlimsup_{t\to\infty}\frac{1}{10}\int_{t-1}^{t}\left(\sum_{l=1}^{2}\exp\frac{1}{5}\int_{h_l(s)}^{t-1}du\right)ds\geq$$
$$\geq\frac{1}{10}\int_{t-1}^{t}\left(\exp\frac{1}{5}(t-s+11)+\exp\frac{1}{5}(t-s)\right)ds=\frac{1}{2}\left(\exp\frac{12}{5}-\exp\frac{11}{5}+\exp\frac{1}{5}-1\right)>1.$$

By applying Theorem 9, we get the correlation

$$\varlimsup_{t\to\infty}\sum_{i=1}^{m}\int_{H_i(t)}p_i(s)\,ds=\varlimsup_{t\to\infty}\int_{t}^{t+1}\frac{1}{10}\,ds+\int_{t}^{t+12}\frac{1}{10}\,ds=\frac{13}{10}>1.$$

Therefore, Theorem 7, i. e., the very first iteration of Theorem 8, as well as Theorem 9 allow us to reveal the oscillation of Eq. (18).

Let us now apply Theorem 12. Since $p(t)\equiv p_0=\frac{\lambda_0}{5}=\text{const}$ and $h(t)=t-1$, in view of Definition (16) and the induction principle, the correlation $P_k(t)\equiv p_k=\text{const}$ is valid for all $k\in\mathbb{N}_0$, and

$$p_{k+1}=\frac{1}{5}\left[1+\int_{t-1}^{t}\frac{1}{5}\exp\left(\int_{s-1}^{t}\frac{1}{5}\exp\left(\int_{u-1}^{u}p_k\,d\xi\right)du\right)ds\right]=$$
$$=\frac{1}{5}\left[1+\frac{1}{5}\int_{t-1}^{t}\exp\left(\frac{1}{5}e^{p_k}(t-s+1)\right)ds\right]=\frac{1}{5}\left[1+\frac{e^{\frac{2}{5}e^{p_k}}-e^{\frac{1}{5}e^{p_k}}}{e^{p_k}}\right].$$

It holds that $\alpha=\varliminf\limits_{t\to\infty}\int\limits_{t-1}^{t}\frac{1}{5}\,ds=\frac{1}{5}$. The least root λ_0 of the equation $\lambda=e^{\frac{\lambda}{5}}$ satisfies the inequality $\lambda_0<5$, whence $p_0=\frac{\lambda_0}{5}<1$.

Denote $f(x)=\frac{e^{\frac{2x}{5}}-e^{\frac{x}{5}}}{x}$. One can easily see that $f(x)$ grows with $x>0$. If $p_k\in(0,1)$, then

$$0<p_{k+1}=\frac{1}{5}\left[1+f(e^{p_k})\right]<\frac{1+f(e)}{5}=\frac{1+e^{\frac{2e}{5}-1}-e^{\frac{e}{5}-1}}{5}<1.$$

According to the induction principle, we conclude that $p_k<1$ for all $k\in\mathbb{N}_0$.

Let us verify condition (17). It holds that

$$\varlimsup_{t\to\infty}\int_{t-1}^{t}\frac{1}{5}\exp\left(\int_{s-1}^{t-1}\frac{1}{5}\exp\left(\int_{u-1}^{u}p_k\,dv\right)du\right)ds=\frac{1}{5}\int_{0}^{1}\exp\frac{e^{p_k}s}{5}\,ds=\frac{e^{\frac{1}{5}e^{p_k}}-1}{e^{p_k}}.$$

The function $f_1(x)=\frac{e^{\frac{x}{5}}-1}{x}$ is increasing on the interval $(1,e)$, consequently, for $p_k<1$,

$$f_1(e^{p_k})\leq f_1(e)=\frac{e^{\frac{e}{5}}-1}{e}<1.$$

Therefore, no iteration of Theorem 12 allows one to reveal the oscillation of Eq. (18).

Remark. In papers [12] and [13], one studies Eq. (6) under narrower assumptions than those considered in this paper, namely, functions p_i and h_i are assumed to be continuous. However, it is evident that one can make functions h_1 and h_2 in Eq. (18) continuous without violating the estimates obtained in example 4.

3. A DISTRIBUTED DELAY EQUATION

Let us find out whether the assumption of the delay concentration in Eq. (1) (made in theorems 1–5) is important. Another question is whether theorems 4 and 5 can be generalized for integro-differential equations and equations with distributed delay.

3.1. *The main result.* Consider the functional differential equation

$$\dot{x}(t) + \int_{h(t)}^{t} x(s)\, d_s r(t,s) = 0, \quad t \geq 0, \tag{19}$$

under the following assumptions: the function $r(t,\cdot)$ has a bounded variation, $r(t,0) = 0$, the function $\rho(t) = \operatorname{var}_{h(t)\leq s\leq t} r(t,s)$ is locally summable, the function $h(t)$ is measurable, $h(t) \leq t$, and the integral is understood in the Riemann–Stieltjes sense.

In what follows, we assume that the function $r(t,\cdot)$ is monotonically increasing and $\lim\limits_{t\to+\infty} h(t) = +\infty$.

Eq. (19) includes Eqs. (1) and (6) as particular cases. The definition of a solution to Eq. (19) verbatim repeats the definitions for Eqs. (1) and (6). Under commonly accepted assumptions, for any bounded Borel initial function, Eq. (19) has a unique solution.

Analogously to formula (9), we put

$$P_0(t,s) = 1, \quad P_{k+1}(t,s) = \exp\left\{\int_s^t \left(\int_{h(\zeta)}^{\zeta} P_k(\zeta,\eta)\, d_\eta r(\zeta,\eta)\right) d\zeta\right\}, \quad k \in \mathbb{N}_0.$$

Similarly to Lemma 1, we deduce the following assertion.

Lemma 2. *Let x be a solution to Eq.* (19) *and $x(t) > 0$ for $t \geq t_0$. Then there exists $t_1 \geq t_0$ such that for $t \geq s \geq t_1$ it holds that*

$$x(t)P_k(t,s) \leq x(s). \tag{20}$$

Proof. Applying the induction principle with respect to k, we rewrite Eq. (19) in the form

$$\dot{x}(t) + \rho(t)x(t) = \int_{h(t)}^{t} (x(t) - x(\eta))\, d_\eta r(t,\eta).$$

Since with $t \geq t_0$ the solution $x(t)$ is monotonically decreasing, it holds that $x(t) \leq x(\eta)$, consequently, $\dot{x}(t) + \rho(t)x(t) \leq 0$. By the formula

$$x(t) = \exp\left(-\int_{t_0}^{t} \rho(\zeta)\, d\zeta\right) x(t_0) + \int_{t_0}^{t} \exp\left(-\int_s^t \rho(\zeta)\, d\zeta\right) \int_{h(s)}^{s} (x(s) - x(\eta))\, d_\eta r(s,\eta)\, ds,$$

we get the following representation of the solution:

$$x(t) \leq x(s)\exp\left(-\int_s^t \rho(\zeta)\, d\zeta\right) = x(s)P_1(t,s),$$

i. e., for $k = 1$ the assertion of the lemma is valid.

Assume that inequality (20) is valid for some $k \geq 1$. Then $x(t)P_k(t,\eta) \leq x(\eta)$, consequently,

$$\dot{x}(t) + x(t)\int_{h(t)}^{t} P_k(t,\eta)\, d_\eta r(t,\eta) \leq 0.$$

Applying the Grownwall–Bellman inequality, we conclude that

$$x(t) \le x(s) \exp\left\{ -\int_s^t \left(\int_{h(\zeta)}^{\zeta} P_k(\zeta, \eta)\, d_\eta r(\zeta, \eta) \right) d\zeta \right\}.$$

Taking into account the definition of P_{k+1}, we conclude that $x(s) \ge x(t)P_{k+1}(t, s)$. □

As in Section 1, we denote $H(t) = \{s \mid h(s) \le t \le s\}$.

Theorem 13. *If the inequality*

$$\overline{\lim_{t\to\infty}} \int_{H(t)} \int_{h(s)}^{t} P_k(t, \eta)\, d_\eta r(s, \eta)\, ds > 1$$

is valid for some $k \in \mathbb{N}_0$*, then Eq.* (19) *is oscillating.*

Proof. Assume that the desired assertion is false, i. e., the solution to Eq. (19) remains positive, beginning with some point. Since $\lim\limits_{t\to+\infty} h(t) = +\infty$, this solution is monotonically decreasing, beginning with some point T. Without loss of generality, we assume that

$$\int_{H(T)} \int_{h(s)}^{T} P_k(T, \eta) d_\eta(s, \eta)\, ds > 1.$$

Formula (19) implies that

$$x(t) = x(T) - \int_T^t \int_{h(s)}^{s} x(\eta)\, d_\eta r(s, \eta) ds > 0, \quad t \ge T.$$

Using the inclusion $H(T) \subset [T, \infty)$ and the definition of the set $H(T)$, we conclude that

$$x(T) \ge \int_T^\infty \int_{h(s)}^{s} x(\eta)\, d_\eta r(s, \eta)\, ds \ge \int_{H(T)} \int_{h(s)}^{s} x(\eta)\, d_\eta r(s, \eta)\, ds \ge \int_{H(T)} \int_{h(s)}^{T} x(\eta)\, d_\eta r(s, \eta)\, ds.$$

By Lemma 2 it holds that $x(\eta) \ge x(T)P_k(T, \eta)$, consequently, taking into account conditions of the theorem, we conclude that

$$x(T) \ge \int_{H(T)} \int_{h(s)}^{T} x(\eta)\, d_\eta r(s, \eta)\, ds \ge x(T) \int_{H(T)} \int_{h(s)}^{T} P_k(T, \eta)\, d_\eta r(s, \eta)\, ds > x(T),$$

which is impossible. □

Corollary 2. If

$$\overline{\lim_{t\to\infty}} \int_{H(t)} \int_{h(s)}^{t} d_\eta r(s, \eta)\, ds > 1,$$

then Eq. (19) is oscillating.

Proof follows from the evident inequality $P_k(t, \eta) \ge 1$ and the monotonicity of the function $r(t, \cdot)$.

For Eq. (1) Corollary 2 coincides with Theorem 5.

3.2. *The integro-differential equation.* Consider the equation

$$\dot{x}(t) + \int_{h(t)}^{t} k(t, s)x(s)\, ds = 0, \quad t \ge 0, \tag{21}$$

which is a particular case of Eq. (19).

For Eq. (21) functions $P_k(t,s)$ take the form

$$P_0(t,s)=1, \quad P_{k+1}(t,s)=\exp\left\{\int_s^t\left(\int_{h(\zeta)}^{\zeta}P_k(\zeta,\eta)k(\zeta,\eta)\,d\eta\right)d\zeta\right\}, \quad k\in\mathbb{N}_0.$$

Applying Theorem 13 to Eq. (21), we deduce the following oscillation conditions.

Theorem 14. *Let $k(t,s)\ge 0$. If the inequality*

$$\varlimsup_{t\to\infty}\int_{H(t)}\int_{h(s)}^{t}P_k(t,\eta)k(s,\eta)\,d\eta\,ds>1$$

is valid for some $k\in\mathbb{N}_0$, then Eq. (21) *is oscillating.*

Theorem 14 implies the well-known oscillation conditions for Eq. (21).

Corollary 3 ([14])**.** Let $k(t,s)\ge 0$. If

$$\varlimsup_{t\to\infty}\int_{H(t)}\int_{h(s)}^{t}k(s,\eta)\,d\eta\,ds>1,$$

then Eq. (21) is oscillating.

Proof follows from the evident inequality $P_k(t,\eta)\ge 1$ and the nonnegativity of the function $k(t,s)$.

In the paper [15], one considers an equation with a distributed delay stated differently from Eq. (21), namely,

$$\dot{x}(t)+\int_0^{b(t)}R(t,s)x(t-s)\,ds=0, \quad t\ge 0. \tag{22}$$

Here the function b is continuous and positive on the semiaxis $[0,\infty)$, while the function R is piecewise continuous on the segment $[0,\infty)\times[0,b(t)]$.

Theorem 15 ([15])**.** *Let $R(t,s)\ge 0$, $\lim\limits_{t\to\infty}(t-b(t))=\infty$ and let the function b be monotonically increasing on $[0,\infty)$. If*

$$\varlimsup_{t\to\infty}\int_0^{b(t)}\int_t^{t+s}R(\eta,s)\,d\eta\,ds>1,$$

then Eq. (22) *is oscillating.*

Let us prove that Theorem 15 follows from Corollary 3. Note that one can reduce Eq. (22) to form (21) by changing variables $t-s=\tau$ in the integral and using denotations $t-b(t)=h(t)$ and $R(t,t-\tau)=k(t,\tau)$.

Since the function b is monotonically increasing, it holds that $b(\eta)\ge b(t)$ for $\eta\ge t$, and if $t\le\eta\le t+b(t)$, then $\eta-b(\eta)\le t\le\eta$, consequently, $\eta\in H(t)$. Since the function R is nonnegative, we get the following chain of inequalities:

$$\int_0^{b(t)}\int_t^{t+s}R(\eta,s)\,d\eta\,ds=\int_t^{t+b(t)}\left(\int_{\eta-t}^{b(t)}R(\eta,s)\,ds\right)d\eta=$$
$$=\int_t^{t+b(t)}\int_{\eta-b(t)}^{t}R(\eta,\eta-\tau)\,d\tau\,d\eta\le\int_t^{t+b(t)}\int_{h(\eta)}^{t}k(\eta,\tau)\,d\tau\,d\eta\le\int_{H(t)}\int_{h(\eta)}^{t}k(\eta,\tau)\,d\tau\,d\eta,$$

whence it follows that assumptions of Theorem 15 ensure the validity of conditions in Corollary 3. The converse assertion is false, because the monotonicity of the function $b(t)=t-h(t)$ in Corollary 3 is not required.

CONCLUSION

Oscillation conditions for Eqs. (1), (6), and (19) usually represent estimates of integrals of equation coefficients over segments that depend on delays. In this paper, we prove that by using sets H and H_i in place of integration intervals, we get much wider classes of oscillating equations, and the well-known results appear to be corollaries of new ones.

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