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Bifurcation Formulas and Algorithms of Constructing Central Manifolds of Discrete Dynamical Systems

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Abstract—Ones of the main questions in theory of local bifurcations and its applications are questions about direction of bifurcations (sub- or supercriticality) and on stability of the solutions arising in neighborhood of a nonhyperbolic equilibrium point or cycle dynamic system. We consider problems of local bifurcations in dynamical systems with discrete time. New features are proposed to orientation of bifurcations and properties stability of bifurcation solutions for problems on basic scenarios of bifurcations. We also propose new algorithms for constructing central manifolds of the corresponding problems, allowing to obtain new bifurcation formulas, in particular, formulas to calculate Lyapunov quantities. Proposed algorithms and formulas are based on the common operator method the study of problems on local bifurcations and allow under the new conditions effective qualitative analysis of bifurcations in terms of the initial equations.

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1. INTRODUCTION AND PROBLEM STATEMENT

Consider the following discrete-time dynamic system:

$$x_{n+1} = A(\mu)x_n + a(x_n, \mu), \quad x_n \in \mathbb{R}^N, \quad n = 0, 1, 2, \dots, \tag{1}$$

where $\mu \in \mathbb{R}^K$ is a parameter, the matrix $A(\mu)$ and the function $a(x, \mu)$ are continuously differentiable in x and μ,

$$a(x, \mu) = a_2(x, \mu) + a_3(x, \mu) + \widetilde{a}_4(x, \mu), \tag{2}$$

terms $a_2(x, \mu)$ and $a_3(x, \mu)$ are quadratic and cubic in x, correspondingly, and $\widetilde{a}_4(x, \mu)$ satisfies the correlation $\|\widetilde{a}_4(x, \mu)\| = O(\|x\|^4)$, $x \to 0$, uniformly with respect to μ.

System (1) with any values of the parameter μ has the equilibrium point $x = 0$. Assume that with some $\mu = \mu_0$ this point is *nonhyperbolic*, i.e., the matrix $A(\mu_0)$ has one or more eigenvalues which are modulo equal to unity. In this case, μ_0 is called the *bifurcation point* or the *bifurcation value of the parameter* of system (1). The following nonhyperbolicity cases are the main ones, namely, the matrix $A(\mu_0)$ can have

P1) one simple eigenvalue 1;

P2) one simple eigenvalue -1;

P3) a pair of simple eigenvalues in the form $e^{\pm i2\pi\theta_0}$, where $0 < \theta_0 < 1/2$ and $\theta_0 \neq 1/3, 1/4$.

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In all these cases, the matrix $A(\mu_0)$ is assumed to have no other eigenvalues which are modulo equal to unity.

In all these cases, the codimension of the corresponding bifurcations equals unity. Therefore in what follows we assume that the parameter μ is scalar.

Note that the nonhyperbolicity case, when the matrix $A(\mu_0)$ has a pair of simple eigenvalues in the form $e^{\pm i2\pi\theta_0}$, where θ_0 equals either 1/3 or 1/4, is referred to a *strong resonance*; we do not consider it here. In this connection, note also that the subcase of case P3), when θ_0 is rational, is called a *weak resonance*.

Cases P1)–P3) lead to various scenarios of local bifurcations in system (1). Namely, the occurrence of nonzero equilibrium points near the point $x = 0$ (case P1)), the occurrence of cycles of period 2 (case P2)), and the occurrence of invariant tori (case P3)) (e.g., [1–4]). Strictly speaking, the implementation of the mentioned scenarios requires the fulfillment of certain transversality conditions; we mention some of them below.

There are many papers (e.g., [1–9]) devoted to various aspects of bifurcations in dynamical systems. It is most important to establish the so-called *bifurcation formulas* that allow one to study basic properties of bifurcations such as transversality conditions, the bifurcation direction (sub- or supercriticality), the stability of occurring solutions, etc. Lyapunov quantities give an example of bifurcation formulas.

There are two approaches to establishing bifurcation formulas. The first one implies the construction of such formulas in terms of initial equations. As a rule, the formulas deduced in such a way are rather complicated (e.g., [1], P. 209; [2], P. 99; or [4], P. 110). The main advantage of these formulas is the fact that they allow one to study bifurcations immediately in terms of initial equations.

The second approach is connected with the application of the Central Manifold Theorem [1], [4], [10–12] and the method of normal forms [1, 2, 6, 9, 13]. The bifurcation formulas obtained in such a way appear to be much simpler and sufficiently effective for the bifurcation analysis. However, the use of these formulas for studying concrete equations, as a rule, requires some preliminary transformation of initial equations, which can represent a nontrivial task.

The choice between these two approaches is not evident. Since properties of problems of various classes essentially differ, in certain cases some methods are more preferable than other ones. One should also keep in mind the fact that various applied approaches necessarily lead to the same final formulas (certainly, provided that their comparison is performed correctly).

In this paper, we study the first approach. First, we propose new bifurcation formulas in problems on main scenarios of local bifurcations of system (1) in terms of initial equations. Second, we propose new algorithms for constructing central manifolds of the corresponding nonlinear maps. The proposed formulas and algorithms allow us to perform (in new conditions) an effective qualitative analysis of main bifurcation scenarios and, in particular, to establish new formulas for calculating Lyapunov quantities. The proposed analysis schemes are based on the general operator research method for studying local bifurcations of dynamical systems [15, 16].

The results obtained in this paper are applicable (after some natural modifications) for studying local bifurcations in neighborhoods of nonhyperbolic cycles of discrete systems and for studying analogous problems for systems described by nonautonomous periodic differential equations, in particular, equations in the form

$$\frac{dx}{dt} = A(t,\mu)x + a(x,t,\mu), \quad x \in \mathbb{R}^N,$$

where the matrix $A(t,\mu)$ and the nonlinearity $a(x,t,\mu)$ are T-periodic in t, while $a(x,t,\mu)$ uniformly in t and μ satisfies the correlation $\|a(x,t,\mu)\| = O(\|x\|^2)$ with $x \to 0$.

2. THE STUDY OF MAIN BIFURCATION SCENARIOS

2.1. **Case P1)** (the bifurcation of equilibrium states).

2.1.1. *Bifurcation conditions.* The main scenario here is the bifurcation connected with the occurrence of nonzero equilibrium points near the equilibrium point $x = 0$ of system (1). Let us adduce the relevant notions (e.g., [15, 16]).

A value μ_0 is called a *bifurcation point of an equilibrium state* of system (1) if there exists a sequence $\mu_k \to \mu_0$ such that with each $\mu = \mu_k$ system (1) has a nonzero equilibrium point $x = x_k^*$, while $x_k^* \to 0$. Solutions x_k^* of system (1) and the corresponding values of the parameter μ_k are said to be *bifurcating* solutions of system (1).

Evidently, if μ_0 is a bifurcation point of equilibrium states of system (1), then the matrix $A(\mu_0)$ has the eigenvalue 1. However, a value μ_0, for which the matrix $A(\mu_0)$ has the eigenvalue 1, is not necessarily a bifurcation point of equilibrium states of system (1).

Let us adduce one of variants of a sufficient bifurcation condition (e.g., [15]). With μ close to μ_0 the matrix $A(\mu)$ has a simple real eigenvalue $\lambda(\mu)$ such that the function $\lambda(\mu)$ is continuously differentiable and $\lambda(\mu_0) = 1$. Then if

$$\gamma_0 \equiv \lambda'(\mu_0) \neq 0, \tag{3}$$

then μ_0 is a bifurcation point of equilibrium states of system (1).

2.1.2. *Continuous branches of bifurcating solutions.* Bifurcating solutions of system (1), as a rule, form continuous branches in the following sense. One can find $\varepsilon_0 > 0$ and defined with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ continuous functions $\mu(\varepsilon)$ and $x(\varepsilon)$ such that with $\mu = \mu(\varepsilon)$ system (1) has a nonzero (with $\varepsilon \neq 0$) equilibrium point $x = x(\varepsilon)$ such that $x(0) = 0$ and $\mu(0) = \mu_0$.

Note that the range of the function $\mu(\varepsilon)$ can represent: $(S1)$ an interval $(\mu_0 - \delta_0, \mu_0 + \delta_0)$, $(S2)$ a half-interval $[\mu_0, \mu_0 + \delta_0)$ or $(\mu_0 - \delta_0, \mu_0]$, $(S3)$ a point $\mu = \mu_0$. The latter case is said to be degenerate; it is typical, e.g., for linear or conservative systems. The first two cases take place under certain non-degeneracy conditions with respect to the nonlinear term (2) in system (1). Case $(S1)$ is referred to as a *transcritical bifurcation*, case $(S2)$ is called a *fork-type bifurcation*, and case $(S3)$ is said to be an *explosive bifurcation*.

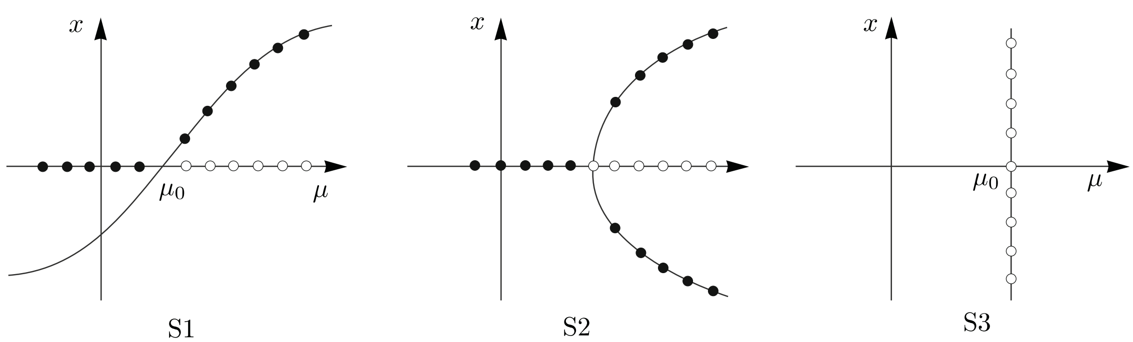

Fig. 1. Continuous branches of bifurcating solutions: $(S1)$ a transcritical bifurcation, $(S2)$ a fork-type bifurcation, $(S3)$ an explosive bifurcation.

In Fig. 1, continuous branches of bifurcating solutions are shown as curves that transversally intersect the axis μ at the point μ_0. Points located on curves correspond to separate bifurcating solutions (equilibrium points) of system (1). Colored points correspond to stable equilibrium points, while non-colored points do to unstable ones.

Therefore, under a transcritical bifurcation, system (1) has a unique continuous branch of bifurcating solutions $x^*(\mu)$ which is defined in some interval $(\mu_0 - \delta_0, \mu_0 + \delta_0)$ so that $x^*(\mu_0) = 0$ and $x^*(\mu) \neq 0$ with $\mu \neq \mu_0$. Under a fork-type bifurcation, system (1) has two continuous branches of bifurcating solutions $x_1^*(\mu)$ and $x_2^*(\mu)$ which are defined in one of half-intervals $[\mu_0, \mu_0 + \delta_0)$ or $(\mu_0 - \delta_0, \mu_0]$ so that $x_j^*(\mu_0) = 0$ and $x_j^*(\mu) \neq 0$ $(j = 1, 2)$ for $\mu \neq \mu_0$, while for other values of μ

system (1) has no bifurcating solutions. Under an explosive bifurcation, with $\mu < \mu_0$ and $\mu > \mu_0$ system (1) has no bifurcating solutions, while with $\mu = \mu_0$ there occurs a continuum of equilibrium points.

2.1.3. *Bifurcation formulas.* Let us establish new formulas for bifurcations of equilibrium states of system (1) which would allow us to decide which of cases ($S1$), ($S2$), or ($S3$) is realized (the bifurcation direction) and to describe stability properties of occurring solutions. To this end, we denote by e and g eigenvectors of the matrix $A_0 = A(\mu_0)$ and the conjugate one $A^*(\mu_0)$ that correspond to the eigenvalue 1. We can choose these vectors in accordance with the equalities

$$\|e\| = 1, \quad (e, g) = 1. \tag{4}$$

Then for calculating value (3) we can use the following formula (see [15], [17]):

$$\gamma_0 = (A'(\mu_0)e, g); \tag{5}$$

here $A'(\mu)$ is the matrix consisting of derivatives of elements of the matrix $A(\mu)$.

In what follows, for simplicity, we use one and the same denotation for the square real matrix and the linear operator generated by it in the standard basis of the space $\mathbb{R}^N$.

Denote by P_0 and P^0 the linear operators that act in $\mathbb{R}^N$ and obey the equalities

$$P_0 x = (x, g)e, \quad P^0 = I - P_0 \tag{6}$$

(if $N = 1$, then $P_0 x = x$ and $P^0 x = 0$). Put $B_0 = I - A_0 + P_0$. By construction, the operator $B_0 : \mathbb{R}^N \to \mathbb{R}^N$ is invertible. Finally, for simplicity, we put

$$a_2 = a_2(e, \mu_0), \quad a_3 = a_3(e, \mu_0), \quad a_2' = a_2'(e, \mu_0); \tag{7}$$

here $a_2'(x, \mu)$ is the Jacobian matrix of the vector function $a_2(x, \mu)$.

Introduce values

$$l_1 = (a_2, g), \quad l_2 = (a_2' B_0^{-1} P^0 a_2, g) + (a_3, g). \tag{8}$$

In assertions given below, we assume that absolute values of eigenvalues of the matrix A_0 different from unity are less than 1.

Theorem 1. *Let $\gamma_0 l_1 \neq 0$. Then μ_0 is a transcritical bifurcation point of system* (1). *If, in addition, $\gamma_0 < 0$ ($\gamma_0 > 0$), then the occurring continuous branch of bifurcating solutions $x^*(\mu)$ with $\mu > \mu_0$ consists of nonstable* (*asymptotically stable*) *equilibrium points, and with $\mu < \mu_0$ it does of asymptotically stable* (*nonstable*) *ones.*

Theorem 2. *Let $l_1 = 0$ and $\gamma_0 l_2 \neq 0$. Then μ_0 is a fork-type bifurcation point of system* (1). *If, in addition, $\gamma_0 l_2 > 0$ ($\gamma_0 l_2 < 0$), then continuous branches of equilibrium points $x_1^*(\mu)$ and $x_2^*(\mu)$ occur with $\mu < \mu_0$ ($\mu > \mu_0$). These equilibrium points are asymptotically stable* (*nonstable*), *if $l_2 < 0$* ($l_2 > 0$).

See proofs of these assertions and other main ones in Section 4.

Remark 1. If the matrix A_0 mentioned in conditions of theorems 1 and 2 has at least one eigenvalue whose absolute value exceeds unity, then the occurring continuous branches of bifurcating solutions of system (1) consist of only nonstable equilibrium points.

Remark 2. Values (8) define the same bifurcation properties as the first and second Lyapunov quantities in the problem on the bifurcation of equilibrium states of system (1) ([2], P. 179). Moreover, in the proof of theorems 1, 2 one ascertains that values (8) coincide with these Lyapunov quantities.

Remark 3. There are only two ways to choose the normalizing rule for vectors e and g in accordance with equalities (4); they differ only in signs. Therefore, in essence, we propose two variants of bifurcation formulas for l_1 and l_2. In these variants, values of l_1 differ only in signs, while values of l_2 coincide.

2.1.4. **Example 1.** Consider the following two-dimensional system (depending on the parameter μ):

$$x_{n+1} = (1+\mu)x_n + 2x_n y_n, \; y_{n+1} = \mu x_n + q y_n + x_n^2 + y_n^2; \tag{9}$$

here q is a fixed value, $|q| < 1$. This system takes form (1) with $N = 2$, $A(\mu) = \left[\begin{smallmatrix} 1+\mu & 0 \\ \mu & q \end{smallmatrix}\right]$, and with the quadratic nonlinearity $a_2(x, y) = \left[\begin{smallmatrix} 2xy \\ x^2+y^2 \end{smallmatrix}\right]$.

The matrix $A(\mu)$ with $\mu = 0$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = q$. Therefore it should be expected that when the parameter μ runs through $\mu = 0$ in a neighborhood of the equilibrium point $(0, 0)$ of system (9), there occur nonzero equilibrium points. Really, calculations performed by formulas (4)–(8) lead to the equalities $\gamma_0 = 1$, $l_1 = 0$ and $l_2 = 2/(1 - q)$. Therefore Theorem 2 implies that the value $\mu = 0$ is a fork-type bifurcation point for system (9). Since $\gamma_0 > 0$ and $l_2 > 0$, continuous branches of nonzero equilibrium points of system (9) occur with $\mu < 0$, and these equilibrium points are nonstable.

One can confirm the results of the performed research by direct calculations, which demonstrate that in system (9) with $\mu < 0$ there occur two continuous branches of nonzero equilibrium points

$$(x_{1,2}^*(\mu), y_{1,2}^*(\mu)) = \left(\frac{-\mu \pm \sqrt{\mu(q-1)}}{2}, -\frac{\mu}{2} \right)$$

which are nonstable.

2.2. **Case P2)** (the period-doubling bifurcation).

2.2.1. *Bifurcation conditions.* In this case, there takes place a bifurcation connected with the occurrence of cycles of period 2. Let us give the corresponding notion (e.g., [15, 16]).

A value μ_0 is called a *period-doubling bifurcation point* of system (1), if there exists a sequence $\mu_k \to \mu_0$ such that with each $\mu = \mu_k$ system (1) has a nonstationary cycle $x_k^* = \{x_0^k, x_1^k\}$ of period 2, while $\max\limits_{0 \leqslant i \leqslant 1} \|x_i^k\| \to 0$.

Evidently, if μ_0 is a period-doubling bifurcation point of system (1), then the matrix $A(\mu_0)$ has the eigenvalue -1. However, a value μ_0, for which the matrix $A(\mu_0)$ has the eigenvalue -1, is not necessarily a period-doubling bifurcation point of system (1).

There takes place an analog of the sufficient condition (3) for the bifurcation of equilibrium states. Namely, with μ close to μ_0 the matrix $A(\mu)$ has a simple real eigenvalue $\lambda(\mu)$ such that the function $\lambda(\mu)$ is continuously differentiable and $\lambda(\mu_0) = -1$. Then if $\gamma_0 = \lambda'(\mu_0) \neq 0$, then μ_0 is a period-doubling bifurcation point of system (1).

Furthermore, similarly to case P1), here there also occur nonzero branches of bifurcating solutions. Namely, in case P2), as a rule, system (1) has exactly one continuous branch of cycles of period 2 $x^*(\mu) = \{x_1(\mu), x_2(\mu)\}$ defined in one of half-intervals $[\mu_0, \mu_0 + \delta_0)$ or $(\mu_0 - \delta_0, \mu_0]$ so that $x^*(\mu_0) = 0$ and $x^*(\mu) \neq 0$ with $\mu \neq \mu_0$. Moreover, with other values of μ system (1) has no bifurcating solutions.

See Fig. 2 for the scheme of a continuous branch of bifurcating solutions $x^*(\mu) = \{x_1(\mu), x_2(\mu)\}$ of system (1) in the period-doubling bifurcation problem. This branch is formed by two continuous functions $x_1(\mu)$ and $x_2(\mu)$. Arrows connect points that form cycles of period 2 in system (1).

2.2.2. *Bifurcation formulas.* Let us establish new bifurcation formulas for the period-doubling bifurcation problem for system (1) that would allow us to determine the bifurcation direction and to study stability properties of occurring solutions. To this end, denote by e and g eigenvectors of the matrix $A(\mu_0)$ and the conjugate one $A^*(\mu_0)$ that correspond to the eigenvalue -1. We can choose these vectors in accordance with equalities (4). Then we can calculate values $\gamma_0 = \lambda'(\mu_0)$ by formula (5) [15].

Denote by P_0 and P^0 the (acting in $\mathbb{R}^N$) operators defined by equalities (6) as applied to case P2) considered here. By construction, the linear operator $I - A_0^2 + P_0 : \mathbb{R}^N \to \mathbb{R}^N$ is invertible. Put

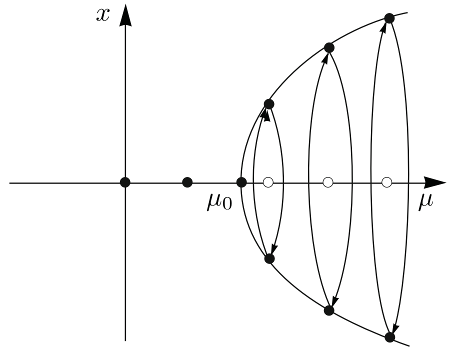

Fig. 2. Continuous branches of bifurcating solutions: a period-doubling bifurcation.

$\Gamma_0 = (I - A_0^2 + P_0)^{-1} : \mathbb{R}^N \to \mathbb{R}^N$. In what follows, for brevity, we use denotations (7) in application to the considered case P2).

Put

$$l_1 = -\frac{(2a_3 + a'_{2x}[I + \Gamma_0(I + A_0)^2]a_2, g)}{2}. \tag{10}$$

In the assertion given below we additionally assume that absolute values of eigenvalues of the matrix A_0 that differ from -1 are less than 1.

Theorem 3. *Let $\gamma_0 l_1 \neq 0$. Then μ_0 is a period-doubling bifurcation point of system* (1). *If, in addition, $\gamma_0 l_1 > 0$ ($\gamma_0 l_1 < 0$), then continuous branches $x^*(\mu) = \{x_1(\mu), x_2(\mu)\}$ of cycles of period 2 occur with $\mu > \mu_0$ ($\mu < \mu_0$). If $l_1 < 0$ ($l_1 > 0$), then cycles $x^*(\mu) = \{x_1(\mu), x_2(\mu)\}$ are asymptotically stable* (*nonstable*).

Here an analog of Remark 1 also takes place, as well as the following analog of Remark 2.

Remark 4. Value (10) defines the same bifurcation properties as the first Lyapunov quantity for the period-doubling bifurcation problem for system (1) ([2], P. 193). Moreover, in the proof of Theorem 3 one makes sure that value (10) coincides with this Lyapunov quantity.

Consider a particular case, when system (1) is scalar, i.e., consider the equation

$$x_{n+1} = \beta_1(\mu)x_n + \beta_2(\mu)x_n^2 + \beta_3(\mu)x_n^3 + O(x_n^4), \quad x_n \in \mathbb{R}^1,$$

where functions $\beta_j(\mu)$ are smooth, while $\beta_1(\mu_0) = -1$. In this case, formula (10) takes a simpler form, namely,

$$l_1 = -(\beta_2^2 + \beta_3), \tag{11}$$

where $\beta_2 = \beta_2(\mu_0)$ and $\beta_3 = \beta_3(\mu_0)$.

2.2.3. **Example 2** (a period-doubling bifurcation in the Hénon map). Consider the Hénon model (e.g., [2]):

$$u_{n+1} = v_n, \quad v_{n+1} = a - \mu u_n - v_n^2, \tag{12}$$

where $0 < a < 3$ and $-1 < \mu < 1$. In what follows, we assume that the value a is fixed, while μ is the bifurcation parameter.

System (12) has an equilibrium point $(u^*(\mu), v^*(\mu))$, where

$$u^*(\mu) = v^*(\mu) = \frac{-(1+\mu) + \sqrt{(1+\mu)^2 + 4a}}{2}.$$

Changing variables $u = x + u^*(\mu)$ and $v = y + v^*(\mu)$ in formula (12), we get the system

$$x_{n+1} = y_n, \ y_{n+1} = -\mu x_n - 2u^*(\mu)y_n - y_n^2,$$

i.e., a system in form (1) with $N = 2$ and

$$A(\mu) = \begin{bmatrix} 0 & 1 \\ -\mu & -2u^*(\mu) \end{bmatrix}, \quad a(w,\mu) = a_2(w) = \begin{bmatrix} 0 \\ -y^2 \end{bmatrix}; \tag{13}$$

here $w = (x, y)$. The matrix $A(\mu)$ with $\mu = \mu_0 = 2\sqrt{a/3} - 1$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -\mu_0$. Therefore it is natural to expect that when the parameter μ runs through $\mu = \mu_0$ in a neighborhood of the equilibrium point $(u^*(\mu), v^*(\mu))$ of system (12) there occur cycles of period 2.

Really, calculations by formulas (4)–(7) and (10) lead to equalities $\gamma_0 = \frac{3}{2(1-\mu_0)}$ and $l_1 = \frac{1}{2(\mu_0^2-1)}$. Then $l_1\gamma_0 < 0$ and $l_1 < 0$. Hence and from Theorem 3 it follows that cycles of period 2 in a neighborhood of the equilibrium point $(u^*(\mu), v^*(\mu))$ of system (12) occur with $\mu < \mu_0$ and they are asymptotically stable.

2.3. **Case P3)** (Andronov–Hopf bifurcation).

2.3.1. *Bifurcation conditions.* For simplicity, we restrict ourselves to considering the cases when system (1) is two-dimensional, namely,

$$x_{n+1} = A(\mu)x_n + a(x_n, \mu), \quad x_n \in \mathbb{R}^2, \quad n = 0, 1, 2, \dots; \tag{14}$$

here we assume that

$$A(\mu) = \rho(\mu) \begin{bmatrix} \cos 2\pi(\theta_0 + \omega(\mu)) & -\sin 2\pi(\theta_0 + \omega(\mu)) \\ \sin 2\pi(\theta_0 + \omega(\mu)) & \cos 2\pi(\theta_0 + \omega(\mu)) \end{bmatrix},$$

where the value θ_0 satisfies conditions of case P3), and smooth functions $\rho(\mu)$ and $\omega(\mu)$ do conditions $\rho(\mu_0) = 1$ and $\omega(\mu_0) = 0$.

The main bifurcation scenario in case P3) is the Andronov–Hopf bifurcation (see [1–3]). Namely, when the value of the parameter μ runs through μ_0 in a neighborhood of the equilibrium point $x = 0$ of system (14), there usually occurs a closed invariant curve $\Gamma(\mu)$ that confines the attraction (repulsion) domain of the fixed point $x = 0$ of the system (Fig. 3).

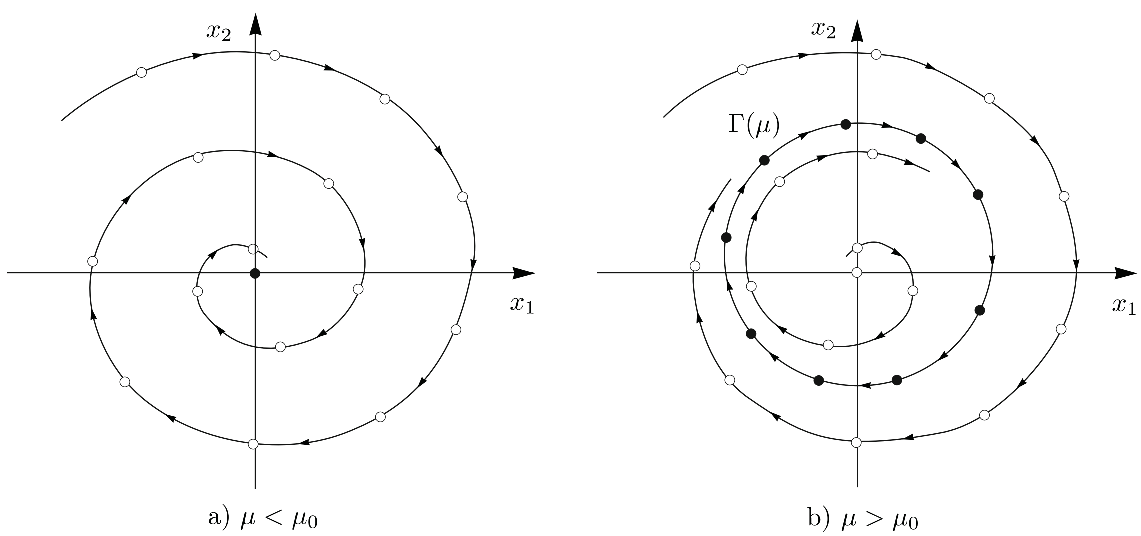

Fig. 3. The Andronov–Hopf bifurcation.

The curve $\Gamma(\mu)$ smoothly depends on μ and contracts to the point $x = 0$ as $\mu \to \mu_0$. The dynamic behavior of system (14) on the curve $\Gamma(\mu)$ can be rather complex, including various families of quasiperiodic and periodic trajectories. In a general case, when μ tends to μ_0 on the curve $\Gamma(\mu)$, there occur and disappear long-period cycles of system (14). The mentioned effect (the subfurcation of periodic oscillations) was first described by V. S. Kozyakin [18].

The invariant curve $\Gamma(\mu)$ usually occurs in one of following three cases: (S1) $\mu \in (\mu_0, \mu_0 + \delta)$, (S2) $\mu \in (\mu_0 - \delta, \mu_0)$, or (S3) $\mu = \mu_0$, where $\delta > 0$. Case (S3) is called degenerate, it is typical for linear and conservative systems. Cases (S1) and (S2) require the fulfillment of a certain nondegeneracy condition for the nonlinear part of system (14). In addition, each value of μ in the corresponding interval corresponds to one closed invariant curve $\Gamma(\mu)$. Note also that an invariant curve $\Gamma(\mu)$ can be an attractor of system (14) (i.e., it can attract all trajectories that start in some neighborhood of this curve); in this case we say that the curve $\Gamma(\mu)$ is *asymptotically stable.* We also define the notion of a *nonstable* curve $\Gamma(\mu)$ in a natural way.

2.3.2. *Bifurcation formulas.* Let us establish new bifurcation formulas for the Andronov–Hopf bifurcation problem for system (14), which would allow us to determine the bifurcation direction and to establish stability properties of the invariant curve $\Gamma(\mu)$. For simplicity, we restrict ourselves to the case, when the nonlinearity $a(x, \mu)$ in this equation starts with the cubic term. i.e.,

$$a(x, \mu) = a_3(x, \mu) + \widetilde{a}_4(x, \mu). \tag{15}$$

Put $\chi(\varphi) = (a_3(e(\varphi), \mu_0), h(\varphi))$, where $e(\varphi) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, $h(\varphi) = \begin{bmatrix} \cos(\varphi + 2\pi\theta_0) \\ \sin(\varphi + 2\pi\theta_0) \end{bmatrix}$,

while $\gamma_0 = \rho'(\mu_0)$ and

$$L_1 = \frac{1}{2\pi} \int_0^{2\pi} \chi(\varphi)\, d\varphi. \tag{16}$$

Theorem 4. *Let* $\gamma_0 L_1 \neq 0$. *Then* μ_0 *is an Andronov–Hopf bifurcation point of system* (14). *If, in addition,* $\gamma_0 L_1 < 0$ $(\gamma_0 L_1 > 0)$, *then the closed invariant curve* $\Gamma(\mu)$ *occurs with* $\mu > \mu_0$ $(\mu < \mu_0)$. *This curve is asymptotically stable, if* $L_1 < 0$; *it is nonstable, if* $L_1 > 0$.

Remark 5. Value (16) defines the same bifurcation properties as the first Lyapunov quantity of system (14) in the Andropov–Hopf bifurcation problem ([2], P. 222). Moreover, in the proof of Theorem 4 one makes sure that (16) coincides with this Lyapunov quantity.

2.3.3 **Example 3** (the Zaslavsky model)**.** Consider the Zaslavsky model that depends on a scalar parameter μ ([8], P. 74)

$$u_{n+1} = u_n + \mu v_n - \mu \sin u_n, \;\; v_{n+1} = \mu v_n - 3\mu \sin u_n. \tag{17}$$

System (17) has the zero equilibrium point $u = v = 0$ with any μ. The Jacobian matrix of the right-hand side of the system at this point takes the form $A(\mu) = \begin{bmatrix} 1-\mu & \mu \\ -3\mu & \mu \end{bmatrix}$. With $\mu = \mu_0 = 1/2$ the matrix $A(\mu_0)$ has eigenvalues $e^{\pm i\pi/3}$, i.e., we get case P3) with $\theta_0 = 1/6$. Therefore one should expect that as μ runs through μ_0 in a neighborhood of the zero equilibrium point of system (17) the Andronov–Hopf bifurcation scenario is implemented. Let us study this issue.

System (17) with $\mu = \mu_0$ in the Jordan basis of the matrix $A(\mu_0)$ takes the form

$$w_{n+1} = B_0 w_n + a_3(w_n) + O(|w_n|^4),$$

where $w = \begin{bmatrix} x \\ y \end{bmatrix}$, $B_0 = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, $a_3(w) = \frac{1}{48} \begin{bmatrix} x^3 \\ -\sqrt{3}x^3 \end{bmatrix}$. Calculations performed by formula (16) give $L_1 = -1/128$. Furthermore, the value $\gamma_0 = \rho'(\mu_0)$ here equals 3/2. Hence and from Theorem 4 we deduce that the value $\mu_0 = 1/2$ is an Andronov–Hopf bifurcation point of system (17). In addition, a closed invariant curve $\Gamma(\mu)$ in a neighborhood of the zero equilibrium point of this system occurs with $\mu > \mu_0$, and this curve is asymptotically stable.

3. THE CONSTRUCTION OF CENTRAL MANIFOLDS

To prove the above assertions, we proceed from initial problems to reduced equations on the corresponding central manifolds. To this end, it is necessary to approximate these manifolds. In this section, we propose new approaches to such constructions. Though this section is an auxiliary one, the proposed schemes are of independent interest.

Consider the map $F:\mathbb{R}^N\to\mathbb{R}^N$ defined by the formula

$$F(x) = A_0x + a_2(x) + a_3(x) + \widetilde{a}_4(x), \tag{18}$$

where A_0 is a square matrix, functions $a_2(x)$ and $a_3(x)$ are, correspondingly, quadratic and cubic terms with respect to x, while $\widetilde{a}_4(x)$ is smooth and satisfies the correlation $\|\widetilde{a}_4(x)\| = O(\|x\|^4)$, $x\to 0$.

Assume that the spectrum σ of the operator $A_0:\mathbb{R}^N\to\mathbb{R}^N$ consists of two nonempty parts, i.e., $\sigma=\sigma_0\cup\sigma^0$, where σ_0 contains eigenvalues that are modulo equals 1, while σ^0 does the rest eigenvalues. Denote by E_0 and E^0 root subspaces of the matrix A_0 that correspond, respectively, to parts σ_0 and σ^0 of its spectrum. Let k_0 and k^0 be dimensions of subspaces E_0 and E^0. The space $\mathbb{R}^N$ is representable as the direct sum $\mathbb{R}^N = E_0\bigoplus E^0$ of invariant for the operator $A_0:\mathbb{R}^N\to\mathbb{R}^N$ subspaces E_0 and E^0. Denote by $P_0:\mathbb{R}^N\to E_0$ and $P^0:\mathbb{R}^N\to E^0$ the corresponding projection operators.

In accordance with the central manifold theorem (e.g., [1], [4, 10, 12]) one can find a δ_0-neighborhood $T(0,\delta_0)$ of the point $x=0$ such that map (18) has in the ball $T(0,\delta_0)$ a smooth invariant k_0-dimensional manifold W_c containing the point $x=0$ and tangential at this point to the subspace E_0; the invariance of the manifold W_c for map (18) means that if $x\in W_c$ and $F(x)\in T(0,\delta_0)$, then $F(x)\in W_c$. The manifold W_c is said to be *central* for map (18) in a neighborhood of the fixed point $x=0$; it can be defined by the equation $v=\psi(u)$, where $u\in E_0$, $v\in E^0$, while the function $\psi(u)$ is smooth and satisfies equalities $\psi(0)=0$ and $\psi'(0)=0$.

Here we consider the same main cases as those considered above when studying bifurcations of system 1), namely, the cases when the matrix A_0 has

P1) one simple eigenvalue 1;

P2) one simple eigenvalue -1;

P3′) a pair of simple eigenvalues $e^{\pm\varphi_0 i}$, where $0<\varphi_0<\pi$.

We assume that absolute values of the rest eigenvalues of the matrix A_0 differ from 1. Let us construct an approximation of the central manifold W_c.

3.1. Case P1).

There exist eigenvectors e and g of matrices A_0 and A_0^*, respectively, that correspond to the simple eigenvalue 1 and satisfy equalities (4). The subspace E_0 is one-dimensional and contains the vector e. Finally, projection operators P_0 and P^0 obey equalities (6).

Since the subspace E_0 is one-dimensional, one can define vectors $u\in E_0$ by the equality $u=\varepsilon e$, where $\varepsilon\in(-\infty,+\infty)$. Correspondingly, any vector $x\in\mathbb{R}^N$ is uniquely representable as the sum $x=\varepsilon e+v$ so that $\varepsilon=(x,g)$ and $v=P^0x$. Finally, one can define the central manifold by the equality

$$W_c = \{x:\, x=\varepsilon e+\psi(\varepsilon)\}, \tag{19}$$

where

S_1) the function $\psi(\varepsilon)$ takes on values in the subspace E^0;

S_2) the function $\psi(\varepsilon)$ is smooth, while $\psi(0)=0$ and $\psi'(0)=0$;

S_3) there exists $\delta_0>0$ such that if $x_1\in T(0,\delta_0)$, $x_1=\varepsilon_1 e+\psi(\varepsilon_1)$ for some ε_1 and, in addition, $x_2=F(x_1)\in T(0,\delta_0)$, then $x_2=\varepsilon_2 e+\psi(\varepsilon_2)$ with some ε_2.

Taking into account the above requirements, it is natural to construct the desired function in the form

$$\psi(\varepsilon) = \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \widehat{\psi}_4(\varepsilon), \tag{20}$$

where coefficients $\psi_2, \psi_3 \in E^0$ have to be defined, while the function $\widehat{\psi}_4(\varepsilon)$ takes on values in the subspace E^0, it is smooth and satisfies the correlation $\|\widehat{\psi}_4(\varepsilon)\| = O(\varepsilon^4)$, $\varepsilon \to 0$.

Put $B_0 = I - A_0 + P_0$. By construction, the operator $B_0 : \mathbb{R}^N \to \mathbb{R}^N$ is invertible, while subspaces E_0 and E^0 are invariant for it. In what follows, for brevity, we write

$$a_2 = a_2(e), \quad a_3 = a_3(e), \quad a_2' = a_2'(e). \tag{21}$$

Theorem 5. *Assume that matrix A_0 have a simple eigenvalue 1, while absolute values of its rest eigenvalues differ from 1. Then the central manifold W_c obeys equality (19), where $\psi(\varepsilon)$ is function (20), while*

$$\psi_2 = B_0^{-1}P^0a_2, \quad \psi_3 = B_0^{-1}P^0[-2(a_2, g)(A_0\psi_2 + a_2) + a_2'\psi_2 + a_3]. \tag{22}$$

3.2. **Case P2).** There exist eigenvectors e and g of matrices A_0 and A_0^*, respectively, that correspond to the simple eigenvalue -1 and satisfy equalities (4). Here the subspace E_0 (as well as in case P1)) is one-dimensional and contains the vector e. Finally, projection operators P_0 and P^0 obey the same equalities (6).

Similarly to case P1), here we will seek for the equation for the central manifold W_c in the form (20). Put

$$B_1 = I - A_0, \quad B_2 = I + A_0 + P_0. \tag{23}$$

By construction, operators $B_1 : \mathbb{R}^N \to \mathbb{R}^N$ and $B_2 : \mathbb{R}^N \to \mathbb{R}^N$ are invertible, while subspaces E_0 and E^0 are invariant for them.

Theorem 6. *Assume that matrix A_0 has the simple eigenvalue -1, while absolute values of its rest eigenvalues differ from 1. Then the central manifold W_c obeys equality (19), where $\psi(\varepsilon)$ is function (20), while*

$$\psi_2 = B_1^{-1}P^0a_2, \quad \psi_3 = B_2^{-1}P^0[-2(a_2, g)(A_0\psi_2 + a_2) - a_2'\psi_2 - a_3]. \tag{24}$$

3.2.1. **Example 4** (the central manifold in the Hénon model). Consider the Hénon model (12) under assumptions of Example 2. The corresponding matrix $A_0 = A(\mu_0)$ (see (13)) has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -\mu_0$. Let us construct the central manifold W_c for system (12) with $\mu = \mu_0$.

Calculations by formulas (4), (6), (23), and (24) lead to the following equalities:

$$e = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad g = \frac{\sqrt{2}}{\mu_0 - 1}\begin{bmatrix} \mu_0 \\ 1 \end{bmatrix}, \quad \psi_2 = \frac{1}{2(\mu_0^2 - 1)}\begin{bmatrix} 1 \\ -\mu_0 \end{bmatrix}. \tag{25}$$

Therefore the desired central manifold $W_c = \{x : x = \varepsilon e + \varepsilon^2\psi_2 + O(\varepsilon^3)\}$, where e and ψ_2 are vectors indicated in (25).

3.3. **Case P3′)** should be considered only with $N \geqslant 3$.

Since the matrix A_0 has a pair of simple eigenvalues $e^{\pm i\varphi_0}$, there exist nonzero vectors $e, g, e^*, g^* \in \mathbb{R}^N$ such that

$$A_0(e + ig) = e^{i\varphi_0}(e + ig), \quad A_0^*(e^* + ig^*) = e^{-i\varphi_0}(e^* + ig^*); \tag{26}$$

here A_0^* is the transposed matrix. We can treat vectors e, g, e^*, and g^* as normalized ones in accordance with equalities

$$\|e\| = \|g\| = 1, \quad (e, e^*) = (g, g^*) = 1, \quad (e, g^*) = (g, e^*) = 0. \tag{27}$$

Assume that E_0 is the (two-dimensional) root subspace of the operator A_0 that corresponds to simple eigenvalues $e^{\pm i\varphi_0}$; for its basis we can use vectors e and g. The space $\mathbb{R}^N$ is representable in the form $\mathbb{R}^N = E_0 \oplus E^0$, where E^0 is the additional invariant for A_0 subspace of dimension $N-2$.

The equality $\mathbb{R}^N = E_0 \oplus E^0$ defines projection operators $P_0 : \mathbb{R}^N \to E_0$ and $P^0 : \mathbb{R}^N \to E^0$ so that $P^0 = I - P_0$, while the operator P_0 is representable as $P_0 x = (x, e^*)e + (x, g^*)g$; the latter formula follows from the fact that by assumption vectors e, g, e^*, and g^* are chosen in accordance with equalities (27).

The central manifold in the considered case takes the form

$$W_c = \{x : x = u + \psi(u), \quad u \in E_0\}, \tag{28}$$

where $\|u\| < \delta$ with some $\delta > 0$, while

S_1') the function $\psi(u)$ is defined in the circle $\|u\| < \delta$ in the subspace E_0 and takes on values in the subspace E^0;

S_2') the function $\psi(u)$ is smooth, while $\psi(0) = 0$ and $\psi'(0) = 0$;

S_3') there exists $\delta_0 > 0$ such that if $x_1 \in T(0, \delta_0)$, $x_1 = u_1 + \psi(u_1)$ for some $u_1 \in E_0$ and, in addition, $x_2 = F(x_1) \in T(0, \delta_0)$, then $x_2 = u_2 + \psi(u_2)$ with some $u_2 \in E_0$.

Taking into account the stated requirements, we construct the desired function in the form

$$\psi(u) = \psi_2(u) + \psi_3(u) + \widehat{\psi}_4(u), \tag{29}$$

where functions $\psi_2(u)$ and $\psi_3(u)$ (they are quadratic and cubic, correspondingly) have to be defined, while the function $\widehat{\psi}_4(u)$ takes on values in the subspace E^0, it is smooth and satisfies the correlation $\|\widehat{\psi}_4(u)\| = O(\|u\|^4)$, $u \to 0$.

In order to define functions $\psi_2(u)$ and $\psi_3(u)$, denote by Q_0 the contraction of the operator $A_0 : \mathbb{R}^N \to \mathbb{R}^N$ on the subspace E_0, i.e., $Q_0 : E_0 \to E_0$ and $Q_0 u = A_0 u$ for $u \in E_0$. The operator Q_0 has a pair of simple eigenvalues $e^{\pm i\varphi_0}$, consequently, it is invertible.

Furthermore, denote by F_p the set of homogeneous of order p (p is a positive integer number) functions which are defined in the subspace E_0 and take on values in the subspace E^0, i.e.,

$$F_p = \{\varphi(u) \mid \varphi :\ E_0 \to E^0,\ \varphi(\alpha u) \equiv \alpha^p \varphi(u)\}.$$

For each p the set F_p forms a linear space with usual operations of addition of elements and multiplication by real values. Denote by L a linear operator that acts in the space F_p and maps each function $\psi(u) \in F_p$ to a function $L_p\psi(u) \in F_p$:

$$L\psi(u) = \psi(u) - A_0\psi(Q_0^{-1}u). \tag{30}$$

Lemma. *The linear operator $L : F_p \to F_p$ defined by equality* (30) *is invertible.*

Denote by L^{-1} the inverse operator for (30). In what follows, for simplicity, we denote by L and L^{-1} the operators that act in spaces F_p of the corresponding operators independently of the value p.

Put

$$b_2(u) = P^0 a_2(Q_0^{-1}u); \tag{31}$$

the inclusion $b_2(u) \in F_2$ is valid by construction. In what follows, assume that

$$\psi_2(u) = L^{-1} b_2(u), \tag{32}$$

where L^{-1} is the inverse operator for (30) with $p = 2$.

Put

$$b_3(u) = P^0 a_3(Q_0^{-1}u) + A_0\psi_2'(Q_0^{-1}u) f_2(u) + P^0 a_2'(Q_0^{-1}u)[f_2(u) + \psi_2(Q_0^{-1}u)],$$

where $f_2(u) = -Q_0^{-1}P_0a_2(Q_0^{-1}u)$. Then $b_3(u) \in F_3$ and, consequently, the following function is defined:

$$\psi_3(u) = L^{-1}b_3(u), \tag{33}$$

where L^{-1} is the inverse operator for that (30) with $p = 3$.

Theorem 7. *Assume that matrix A_0 has a pair of simple eigenvalues $e^{\pm i\varphi_0}$, where $0 < \varphi_0 < \pi$, while absolute values of its rest eigenvalues differ from 1. Then the central manifold W_c obeys equality* (28)*, where $\psi(u)$ is function* (29)*, while functions $\psi_2(u)$ and $\psi_3(u)$ obey equalities* (32) *and* (33)*, correspondingly.*

3.3.1. *Construction of functions $\psi_2(u)$ and $\psi_3(u)$.* The proof of Theorem 7 given in Section 4, in essence, proposes a way for constructing functions $\psi_2(u)$ and $\psi_3(u)$ that obey equalities (32) and (33). Let us adduce the corresponding scheme, restricting ourselves to constricting the function $\psi_2(u)$.

Denote by E_c the complexification of the real linear space E.

1. Choose a basis in the subspace E_0 consisting of vectors $e, g \in \mathbb{R}^N$ (see (26)). Then for $u \in E_0$ it holds that $u = u_1e + u_2g$ with some real u_1 and u_2. Let us write the vector defined by equality (31) in the form $b_2(u) = b_2(u_1, u_2)$.

2. By putting $z = u_1 + iu_2$ we define the function $\widetilde{b}_2(z) = b_2((z+\overline{z})/2, (z-\overline{z})/2i)$ and represent it as follows: $\widetilde{b}_2(z) = c_1z^2 + c_2z\overline{z} + \overline{c}_1\overline{z}^2$, where $c_1, c_2 \in E_c^0$ (here, in fact, $c_2 \in E^0$).

3. Equations
$$\varphi_1 = e^{-2i\varphi_0}A_0\varphi_1 + c_1,\ \ \varphi_2 = A_0\varphi_2 + c_2, \tag{34}$$
allow us to define vectors $\varphi_1, \varphi_2 \in E_c^0$. These equations are uniquely solvable, because the operator $A_0 : E_c^0 \to E_c^0$ has no eigenvalues that are modulo equal to 1.

4. Define the function $\widetilde{\psi}(z) = \varphi_1z^2 + \varphi_2z\overline{z} + \overline{\varphi}_1\overline{z}^2$.

5. Put $\psi_2(u) = \widetilde{\psi}(u_1 + iu_2)$.

3.3.2. **Example 4** (construction of the central manifold). Consider operator (18) in the form

$$F(x) = A_0x + a_2(x), \quad x \in \mathbb{R}^3, \tag{35}$$

where

$$A_0 = \begin{bmatrix} \cos\varphi_0 & -\sin\varphi_0 & 0 \\ \sin\varphi_0 & \cos\varphi_0 & 0 \\ 0 & 0 & k_0 \end{bmatrix}, \quad a_2(x) = \begin{bmatrix} 0 \\ 0 \\ x_1^2 + x_2^2 + x_3^2 \end{bmatrix};$$

here $0 < \varphi_0 < \pi$ and $k_0 \neq \pm 1$.

Then (see formula (26)) $e = e^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $g = g^* = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. Then the vector $u \in E_0$ is representable as $u = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$, while the operator Q_0^{-1} takes the form $Q_0^{-1}u = \begin{bmatrix} cu_1\cos\varphi_0 + u_2\sin\varphi_0 \\ -u_1\sin\varphi_0 + u_2\cos\varphi_0 \\ 0 \end{bmatrix}$.

Therefore

$$b_2(u) = b_2(u_1, u_2) = \begin{bmatrix} 0 \\ 0 \\ u_1^2 + u_2^2 \end{bmatrix}, \quad \widetilde{b}_2(z) = b_2((z+\overline{z})/2, (z-\overline{z})/2i) = \begin{bmatrix} 0 \\ 0 \\ z\overline{z} \end{bmatrix}.$$

Then $c_1 = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right]$, $c_2 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$. Solving system (34), we get $\varphi_1 = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right]$, $\varphi_2 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1/(1-k_0) \end{smallmatrix}\right]$. Therefore, according to formula (28), the central manifold of map (35) obeys the equality

$$W_c = \left\{ x : x = \begin{bmatrix} u_1 \\ u_2 \\ (u_1^2 + u_2^2)/(1-k_0) \end{bmatrix} + O(\|u\|^3) \right\}.$$

4. PROOFS OF MAIN ASSERTIONS

Proofs of Theorems 1–3. Analogous assertions proved in [15] imply that in assumptions of Theorems 1–3 the value μ_0 is the point of the corresponding bifurcation of system (1) and there occur continuous branches of bifurcating solutions. Therefore it remains only to verify assertions of theorems 1–3 with respect to the stability property of occurring solutions. For proving these assertions we use a scheme based on the transition from initial problems to reduced equations on the corresponding one-dimensional central manifolds (theorems 5 and 6) and on the analysis of obtained Lyapunov quantities. Let us restrict ourselves to the description of this scheme for proving Theorem 3.

For simplicity, consider the case when system (1) is two-dimensional, i.e., $N = 2$. Moreover, for simplicity, we also assume that with $\mu = \mu_0$ the matrix $A(\mu)$ takes the form $A_0 = A(\mu_0) = \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix}$, where $b \neq \pm 1$. Finally, assume that in nonlinearity (2) the quadratic and cubic nonlinearities with $\mu = \mu_0$ are represented, correspondingly, as follows:

$$a_2(x) = \begin{bmatrix} a_{20}x_1^2 + 2a_{11}x_1x_2 + a_{02}x_2^2 \\ b_{20}x_1^2 + 2b_{11}x_1x_2 + b_{02}x_2^2 \end{bmatrix},$$

$$a_3(x) = \begin{bmatrix} a_{30}x_1^3 + 3a_{21}x_1^2x_2 + 3a_{12}x_1x_2^2 + a_{03}x_2^3 \\ b_{30}x_1^3 + 3b_{21}x_1^2x_2 + 3b_{12}x_1x_2^2 + b_{03}x_2^3 \end{bmatrix}. \tag{36}$$

We get

$$e = g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P^0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$a_2 = \begin{bmatrix} a_{20} \\ b_{20} \end{bmatrix}, \quad a_2' = 2\begin{bmatrix} a_{20} & a_{11} \\ b_{20} & b_{11} \end{bmatrix}, \quad a_3 = \begin{bmatrix} a_{30} \\ b_{30} \end{bmatrix}.$$

Then Eq. (1) reduced with $\mu = \mu_0$ on the corresponding one-dimensional central manifold (Theorem 6) is scalar and representable in the form $\varepsilon_{n+1} = G(\varepsilon_n)$, where $G(\varepsilon) = -\varepsilon + (a(\varepsilon e +$

$\psi(\varepsilon), \mu_0), g)$; here $\psi(\varepsilon)$ is function (20) with coefficients (24). Since with small $|\varepsilon|$ we get the correlation

$$G(\varepsilon) = -\varepsilon + \varepsilon^2(a_2, g) + \varepsilon^3[(a_2'\psi_2, g) + (a_3, g)] + O(\varepsilon^4),$$

the indicated reduced equation is representable as follows:

$$\varepsilon_{n+1} = -\varepsilon_n + \gamma_2\varepsilon_n^2 + \gamma_3\varepsilon_n^3 + O(\varepsilon_n^4), \qquad (37)$$

where $\gamma_2 = (a_2, g)$ and $\gamma_3 = (a_2'\psi_2, g) + (a_3, g)$.

In [2] (P. 114) one proves that the first Lyapunov quantity of the one-dimensional equation (37) obeys the formula

$$l_1 = -(\gamma_2^2 + \gamma_3), \qquad (38)$$

which coincides with (11).

It remains to make sure that values (10) and (38) coincide. Taking into account formulas (24), by immediate calculations we conclude that expressions (10) and (38) equal one and the same value, namely,

$$l_1 = -\Big(a_{20}^2 + a_{30} + \frac{2}{1-b}\, a_{11}b_{20}\Big).$$

The analysis of the calculated Lyapunov quantity with the help of the corresponding assertions of the theory of local bifurcations (e.g., [2], P. 193) completes the proof of Theorem 3.

Proof of Theorem 4. For simplicity, assume that in nonlinearity (15) the function $a_3(x, \mu)$ with $\mu = \mu_0$ obeys equality (36). For this case, in [1] (P. 209) one gives the following formula for the Lyapunov quantity L_1:

$$L_1 = \frac{3}{8}[(a_{30} + a_{12} + b_{21} + b_{03})\cos 2\pi\varphi_0 + (b_{30} + b_{12} - a_{21} - a_{03})\sin 2\pi\varphi_0]. \qquad (39)$$

Therefore we can prove formula (16) by substituting (36) in (16) and calculating the obtained integral. As a result, we get the value that coincides with (39).

For completing the proof of Theorem 4, it remains to study the obtained Lyapunov quantity with the help of the corresponding assertions of the theory of local bifurcations (e.g., [2], P. 222).

Proof of Theorems 5 and 6 follow one and the same scheme, so we restrict ourselves to proving the first of them.

Assume that the function $\psi(\varepsilon)$ in (19) obeys equality (20), where coefficients ψ_2 and ψ_3 are vectors (22). For proving the theorem, it suffices to verify item S_3) in Definition (19) for the central manifold W_c.

Let $x = \varepsilon e + \psi(\varepsilon)$ with some small ε. Put $y = F(x)$. Then it suffices to prove that $y = \delta e + \psi(\delta)$ with some small δ.

Taking into account equality (20), we get the formula

$$F(x) = F(\varepsilon e + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \widehat{\psi}_4(\varepsilon)) = A_0(\varepsilon e + \varepsilon^2\psi_2 + \cdots) \\ + a_2(\varepsilon e + \varepsilon^2\psi_2 + \cdots) + a_3(\varepsilon e + \varepsilon^2\psi_2 + \cdots) + \widetilde{a}_4(\varepsilon e + \varepsilon^2\psi_2 + \cdots),$$

where the symbol "$\cdots$" stands for terms $\varepsilon^3\psi_3 + \widehat{\psi}_4(\varepsilon)$. Hence, taking into account equalities (which use denotations (21))

$$A_0 e = e, \quad a_2(\varepsilon e + \varepsilon^2\psi_2 + \cdots) = \varepsilon^2(a_2 + \varepsilon a_2'\psi_2) + O(\varepsilon^4), \quad a_3(\varepsilon e + \varepsilon^2\psi_2 + \cdots) = \varepsilon^3 a_3 + O(\varepsilon^4),$$

we get the formula

$$F(x) = \underbrace{\varepsilon e + \varepsilon^2 P_0 a_2 + \varepsilon^3 P_0(a_2'\psi_2 + a_3) + P_0 b_4(\varepsilon)} \\ + \underbrace{\varepsilon^2 P^0 a_2 + \varepsilon^3 P^0(a_2'\psi_2 + a_3) + \varepsilon^2 A_0\psi_2 + \varepsilon^3 A_0\psi_3 + P^0 b_4(\varepsilon)},$$

where the function $b_4(\varepsilon)$ is smooth and satisfies the correlation $\|b_4(\varepsilon)\| = O(\varepsilon^4)$, $\varepsilon \to 0$. The first group of terms in the obtained equality belongs to the subspace E_0, while the second does to E^0. For completing the proof of the theorem, it remains to make sure that the mentioned groups of terms are representable in the form $\delta e + \psi(\delta)$ with some δ; in other words, we have to prove that there exists δ (it is small, if so is ε) such that

$$\begin{gathered}\delta = \varepsilon + \varepsilon^2(a_2, g) + \varepsilon^3(a_2'\psi_2 + a_3, g) + (b_4(\varepsilon), g),\\ \psi(\delta) = \varepsilon^2 P^0 a_2 + \varepsilon^3 P^0(a_2'\psi_2 + a_3) + \varepsilon^2 A_0\psi_2 + \varepsilon^3 A_0\psi_3 + P^0 b_4(\varepsilon).\end{gathered} \tag{40}$$

With small δ the first equation in this system has a unique solution with respect to ε:

$$\varepsilon = \delta + \delta^2 k_2 + \delta^3 k_3 + O(\delta^4), \tag{41}$$

where $k_2 = -(a_2, g)$, $k_3 = 2(a_2, g)^2 - (a_2'\psi_2 + a_3, g)$ (one can verify this correlation by substituting (41) in the first equation in system (40) and deducing the equality $\delta = \delta$ accurate to $O(\delta^4)$). It remains to make sure that by substituting (41) in the second equation in system (40) we get the following equality (analogous to (20)):

$$\psi(\delta) = \delta^2\psi_2 + \delta^3\psi_3 + \widehat{\psi}_4(\delta). \tag{42}$$

Really,

$$\psi(\delta) = \delta^2(A_0\psi_2 + P^0 a_2) + \delta^3[A_0\psi_3 + 2k_2(A_0\psi_2 + P^0 a_2) + P^0(a_2'\psi_2 + a_3)] + O(\delta^4).$$

In view of formula (22) the latter equality coincides with (42). □

Proof of the lemma. Consider the equation

$$\psi(u) = A_0\psi(Q_0^{-1}u) + b(u) \tag{43}$$

with respect to the unknown function $\psi(u) \in F_p$ and with a given one $b(u) \in F_p$. For proving the lemma, it suffices to make sure that Eq. (43) is uniquely solvable.

Let vectors $e, g \in \mathbb{R}^N$ (see (26)) form a basis in the subspace E_0. Then any vector $u \in E_0$ is uniquely representable as $u = u_1 e + u_2 g$ with some real u_1 and u_2. Proceeding in formula (43) from u to the complex variable $z = u_1 + iu_2$, we get the equivalent (in a natural sense) equation

$$\psi(z) = A_0\psi(e^{-i\varphi_0}z) + b(z), \tag{44}$$

where, for simplicity, we use the same denotations for the corresponding functions.

For definiteness, we put $p = 2$, i.e., functions $\psi(z)$ and $b(z)$ in Eq. (44) are quadratic with respect to the variable z, they take on values in the complexification E_c^0 of the subspace E^0. These functions are representable in the form

$$\psi(z) = \varphi_1 z^2 + \varphi_2 z\overline{z} + \overline{\varphi}_1\overline{z}^2, \quad b(z) = c_1 z^2 + c_2 z\overline{z} + \overline{c}_1\overline{z}^2,$$

with some $\varphi_1, \varphi_2, c_1, c_2 \in E_c^0$ (here, in fact, $\varphi_2, c_2 \in E^0$). Since

$$\psi(e^{-i\varphi_0}z) = \varphi_1 e^{-2i\varphi_0}z^2 + \varphi_2 z\overline{z} + \overline{\varphi}_1 e^{2i\varphi_0}\overline{z}^2,$$

Eq. (44) is equivalent to the system

$$\varphi_1 = e^{-2i\varphi_0}A_0\varphi_1 + c_1,\ \varphi_2 = A_0\varphi_2 + c_2,\ \overline{\varphi}_1 = e^{2i\varphi_0}A_0\overline{\varphi}_1 + \overline{c}_1.$$

This system is uniquely solvable, because the operator $A_0 : E_c^0 \to E_c^0$ has no eigenvalues which are modulo equals 1. □

Proof of Theorem 7. Assume that the function $\psi(u)$ in (28) obeys equality (29), while functions $\psi_2(u)$ and $\psi_3(u)$ do equalities (32) and (33). For proving the theorem, it suffices to verify item S_3) in Definition (28) of the central manifold W_c. In turn, for this it suffices to prove that if $x = u + \psi(u)$ with some small $\|u\|$, $u \in E_0$, then the vector $y = F(x)$ is representable in the form $y = v + \psi(v)$ with some small $\|v\|$, $v \in E_0$. One can prove this fact by the same scheme as that used for proving an analogous assertion in Theorem 5.

CONCLUSION

In this paper, we propose new formulas for calculating Lyapunov quantities in problems on the main scenarios of local bifurcations of system (1) in terms of initial equations. Namely, we propose formula (8) for the first and second Lyapunov quantities in the problem on bifurcation of equilibrium states, formula (10) for the first Lyapunov quantity in the problem on the period-doubling bifurcation, and formula (16) for the first Lyapunov value in the problem on the Andronov–Hopf bifurcation. We prove assertions (Theorems 1–4) that allow one to perform (in new conditions) an effective qualitative analysis of the main bifurcation scenarios. We propose new algorithms for constructing central manifolds of the corresponding nonlinear maps in the main cases of the degeneration of the linearized operators. These algorithms are described in Theorems 5–7.

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