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On Analytic Periodic Solutions to Nonlinear Differential Equations With Delay (Advance)

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Abstract—We study a system of the reaction–diffusion type, where diffusion coefficients depend in an arbitrary way on spatial variables and concentrations, while reactions are expressed as homogeneous functions whose coefficients depend in a special way on spatial variables. We prove that the system has a family of exact solutions that are expressed through solutions to a system of ordinary differential equations (ODE) with homogeneous functions in right-hand sides. For a special case of the ODE system we construct a general solution represented by Jacobi higher transcendental functions. We also prove that these periodic solutions are analytic functions that can be expressed near each point on the period by convergent power series.

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INTRODUCTION

Mathematical models of reaction–diffusion processes usually represent systems of nonlinear parabolic differential equations [1]. The exact solution of this class of equations is a very difficult task; one of the most effective techniques for it assumes the reduction of the initial system to a system of ordinary differential equations (ODE), which is possible only under certain conditions [2]. As is indicated in [3], it is actual to study the solvability of systems of reaction–diffusion equations with distributed parameters and to construct their exact periodic solutions. This is important both for applications in the chemical technology and from the theoretical point of view for the qualitative theory of differential equations. The study of these questions on the base of various approaches and techniques is being conducted now (e.g., [4] and references therein). In this paper, we consider a system of reaction–diffusion type, where the diffusion coefficients arbitrarily depend on spatial variables and concentrations, while reaction coefficients are described by homogeneous functions whose parameters depend (in a certain way) on spatial variables. We prove that the system under consideration has a family of exact solutions expressed in terms of solutions to a system of ODE with homogeneous functions in the right-hand sides. For a particular form of this system, we construct a general solution expressed via special Jacobi functions. We prove that solutions represent periodic functions and satisfy nonlinear differential equations with delay (advance), whose value depends on initially stated conditions. We also prove that these periodic solutions to equations with delay (advance) are analytic functions representable in a neighborhood of each point on a period by convergent power series. Note that there is no analog to the general Cauchy theorem on the existence of analytic solutions to ODE [5] for nonlinear equations with deviating argument; there are only some papers [6–9] that deal mainly with linear equations with variable coefficients. Therefore, the analytic periodic solutions constructed in this paper can be of interest for the analytic theory of differential equations with deviating argument.

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1. THE REDUCTION OF A SYSTEM OF REACTION–DIFFUSION EQUATIONS TO A SYSTEM OF ODE

Consider the following system of reaction–diffusion equations:

$$\frac{\partial U_i}{\partial t} = g_i(\mathbf{x}, U)\Delta U_i + a_i(\mathbf{x})F_i(U), \quad i = \overline{1, N}. \tag{1}$$

Here $t \in \mathbb{R}$ is time, $\mathbf{x} \in \mathbb{R}^n$ are independent spatial variables, $U_i(t, \mathbf{x})$ are the desired functions (we treat them as concentrations of interacting substances), and Δ is the Laplace operator. Functions $F_i(U) = F_i(U_1, U_2, \dots, U_N)$ describe interactions. Coefficients $a_i(\mathbf{x})$ can express the dependence of reaction rates on spatial coordinates; it can be connected, e.g., with temperature differences or the presence of physical fields. If there is no reaction, i.e., $F_i(U) \equiv 0$, $i = \overline{1, N}$, system (1) turns into a set of diffusion equations whose coefficients $g_i(\mathbf{x}, U)$ depend on spatial variables and concentrations.

Theorem 1. *Assume that functions $F_i(U)$ are positively homogeneous of order $m_i > 0$, while positive in a domain $\mathbf{x} \in D \subset \mathbb{R}^n$ functions $a_i(\mathbf{x})$ are representable as $a_i(\mathbf{x}) = [b(\mathbf{x})]^{1-m_i}$ for all $i = \overline{1, N}$; here $b(\mathbf{x})$ is some positive harmonic function. Then system* (1) *has the following solution*:

$$U_i(t, \mathbf{x}) = b(\mathbf{x})X_i(t), \quad i = \overline{1, N}, \tag{2}$$

where $X_i(t)$ are arbitrary solutions to the following system of ODE:

$$\dot{X}_i = F_i(X), \quad i = \overline{1, N}. \tag{3}$$

Proof is performed by the direct substitution of solutions (2) in system (1). As a result, we get N identities

$$b\dot{X}_i = g_i(\mathbf{x}, U)X_i\Delta b + F_i(bX)\, a_i \quad \text{or} \quad b\dot{X}_i = b^{m_i}F_i(X)\, b^{1-m_i},$$

which are true under the theorem assumptions. As examples of functions $a_i(\mathbf{x})$ satisfying assumptions of Theorem 1 with $N = m_i = 3$, we can choose $a_i(\mathbf{x}) = a(\mathbf{x}) \equiv r^2 = x_1^2 + x_2^2 + x_3^2$ for all $i = \overline{1, N}$. In this case, $b(\mathbf{x}) \equiv \left(x_1^2 + x_2^2 + x_3^2\right)^{-1/2}$. For the indicated function $a(\mathbf{x}) \equiv r^2$, one particular case of system (1) with $N = 3$ and $g_i(\mathbf{x}, U) \equiv 1$ is of interest, namely,

$$\frac{1}{r^2}\frac{\partial U_1}{\partial t} = \Delta U_1 + U_1^2(\lambda U_2 - \mu U_3), \quad \frac{1}{r^2}\frac{\partial U_2}{\partial t} = \Delta U_2 + U_2^2(-\lambda U_1 + \sigma U_3),$$

$$\frac{1}{r^2}\frac{\partial U_3}{\partial t} = \Delta U_3 + U_3^2(\mu U_1 - \sigma U_2). \tag{4}$$

Here $U_i = U_i(t, x_1, x_2, x_3)$, Δ is the Laplace operator in a three-dimensional coordinate space, λ, μ, and σ are nonzero real-valued parameters. By Theorem 1 the solution to system (4) takes the form $U_i = X_i(t)/r$, while functions $X_i = X_i(t)$, $i = \overline{1, 3}$, satisfy the nonlinear system

$$\dot{X}_1 = X_1^2(\lambda X_2 - \mu X_3), \quad \dot{X}_2 = X_2^2(-\lambda X_1 + \sigma X_3), \quad \dot{X}_3 = X_3^2(\mu X_1 - \sigma X_2). \tag{5}$$

The autonomous system of ODE (5) is interesting by the fact that it has the following independent first integrals:

$$I_1 = X_1X_2X_3 = C_1, \quad I_2 = \frac{\sigma}{X_1} + \frac{\mu}{X_2} + \frac{\lambda}{X_3} = C_2; \tag{6}$$

one can use them for the qualitative analysis of the system and for constructing its solutions. Here $C_1 \neq 0$ and $C_2 \neq 0$ are arbitrary real constants.

2. CONSTRUCTION OF PERIODIC SOLUTIONS TO THE SYSTEM OF ODE (5) STATED IN TERMS OF ELLIPTIC JACOBI FUNCTIONS

Let us prove the following theorem.

Theorem 2. *Let parameters of system* (5) *satisfy the condition*

$$\lambda\mu\sigma > 0. \tag{7}$$

Then in the domain $X_1X_2X_3\left(\frac{\sigma}{X_1}+\frac{\mu}{X_2}+\frac{\lambda}{X_3}\right) > 27\lambda\mu\sigma$ *system* (5) *has the following general solution*:

$$X_1(t) = \frac{z_1^*+\sigma}{C_2} - \frac{z_1^*-z_2^*}{C_2}\,\mathrm{sn}^2\,(T,k)\,, \tag{8}$$

$$X_2(t) = \frac{C_1C_2^2}{2\lambda}\,\frac{z_1^*-(z_1^*-z_2^*)\,\mathrm{sn}^2\,(T,k)+\delta\,P\,\mathrm{sn}\,(T,k)\,\mathrm{cn}\,(T,k)\,\mathrm{dn}\,(T,k)}{\left(z_1^*+\sigma-(z_1^*-z_2^*)\,\mathrm{sn}^2\,(T,k)\right)^2}, \tag{9}$$

$$X_3(t) = \frac{C_1C_2^2}{2\mu}\,\frac{z_1^*-(z_1^*-z_2^*)\,\mathrm{sn}^2\,(T,k)-\delta\,P\,\mathrm{sn}\,(T,k)\,\mathrm{cn}\,(T,k)\,\mathrm{dn}\,(T,k)}{\left(z_1^*+\sigma-(z_1^*-z_2^*)\,\mathrm{sn}^2\,(T,k)\right)^2}. \tag{10}$$

Here $C_1 \neq 0$ *and* $C_2 > 0$ *are arbitrary real constants such that* $C_1C_2^3 > 27\lambda\mu\sigma$, *while* $z_1^* > z_2^* > z_3^*$ *are real roots of the cubic equation*

$$z^3 - \frac{C_1C_2^3-12\lambda\mu\sigma}{4\lambda\mu}\,z^2+3\sigma^2 z+\sigma^3 = 0. \tag{11}$$

Functions $\mathrm{sn}(T,k)$, $\mathrm{cn}(T,k)$, *and* $\mathrm{dn}(T,k)$ *are, correspondingly, elliptic sine, cosine, and Jacobi delta amplitude,* $k=\sqrt{\frac{z_1^*-z_2^*}{z_1^*-z_3^*}}$ *is a model of an elliptic function,*

$$P = 2\sqrt{\frac{\lambda\mu}{C_1C_2^3}}\,(z_1^*-z_2^*)\,\sqrt{z_1^*-z_3^*}, \quad T = -\delta\sqrt{\frac{\lambda\mu\,C_1}{C_2}}\sqrt{z_1^*-z_3^*}\,(t-C_3)\,,$$

$\delta = \pm 1$, *and* C_3 *is an arbitrary constant value.*

Proof. Using formulas (6), we find

$$X_2(t) = \frac{C_1C_2\,X_1(t)-\sigma C_1+\delta\sqrt{\Omega}}{2\lambda\,X_1^2(t)}, \quad X_3(t) = \frac{C_1C_2\,X_1(t)-\sigma C_1-\delta\sqrt{\Omega}}{2\mu\,X_1^2(t)}. \tag{12}$$

Here for convenience we denote

$$\Omega = C_1\left(\sigma^2C_1-2\sigma\,C_1C_2\,X_1(t)+C_1C_2^2\,X_1^2(t)-4\lambda\mu\,X_1^3(t)\right), \quad \delta=\pm 1.$$

Taking into account formula (12), we reduce system (5) to the equation $\dot{X}_1 = F\,(X_1)$, where

$$F\,(X_1) = \frac{C_1\,\left[(C_2\,X_1-\sigma)\left(C_1C_2\,X_1-\sigma C_1+\delta\sqrt{\Omega}\right)-4\lambda\mu\,X_1^3\right]}{C_1C_2\,X_1-\sigma C_1+\delta\sqrt{\Omega}}.$$

Put $z = C_2\,X_1(t)-\sigma$ or $X_1(t)=\frac{1}{C_2}(z+\sigma)$. Then

$$\Omega = \frac{4\lambda\mu C_1}{C_2^3}\left(-z^3+\frac{C_1C_2^3-12\lambda\mu\sigma}{4\lambda\mu}\,z^2-3\sigma^2 z-\sigma^3\right). \tag{13}$$

Rewrite the function $F\,(X_1)$ as

$$F\,(X_1) = C_1\,z - \frac{4\lambda\mu C_1}{C_2^3}\,\frac{(z+\sigma)^3}{C_1\,z+\delta\sqrt{\Omega}}.$$

Multiplying the numerator and the denominator of the latter term by the value $C_1\,z-\delta\sqrt{\Omega}$ and taking into account the equality $\delta^2 = 1$, we get the formula

$$F(X_1) = C_1\, z - \frac{4\lambda\mu C_1}{C_2^3}\, \frac{(z+\sigma)^3 (C_1\, z - \delta\sqrt{\Omega})}{C_1^2\, z^2 - \Omega}.$$

Since $C_1^2\, z^2 - \Omega = \frac{4\lambda\mu C_1}{C_2^3}\,(z+\sigma)^3$, we finally get the representation $F(X_1) = \delta\sqrt{\Omega}$, where Ω is defined in formula (13). In this case the ODE $\dot{X}_1 = F(X_1)$ for the new variable z takes the form

$$\dot{z} = C_2 \delta \sqrt{\Omega}.$$

Therefore, we have reduced the evaluation of the function $X_1(t)$ to that of the integral

$$\Phi(z) = \frac{\delta}{2} \int \frac{dz}{\sqrt{\frac{\lambda\mu C_1}{C_2}\left(-z^3 + \frac{C_1 C_2^3 - 12\lambda\mu\sigma}{4\lambda\mu}\, z^2 - 3\sigma^2 z - \sigma^3\right)}}. \tag{14}$$

Under the theorem assumptions the discriminant of Eq. (11) takes the form

$$\mathrm{D} = \frac{\sigma^3 C_1^2 C_2^6 \left(C_1 C_2^3 - 27\lambda\mu\sigma\right)}{16\lambda^3\mu^3};$$

it is positive, because the cubic equation (11) has three real roots $z_1^* > z_2^* > z_3^*$, and integral (14) is reducible to the quadrature form

$$\sqrt{\frac{C_2}{\lambda\mu\, C_1}}\, \frac{\delta\, F(\varphi, k)}{\sqrt{z_1^* - z_3^*}} = -\,(t - C_3)\,, \quad \varphi = \arcsin\sqrt{\frac{z_1^* - z}{z_1^* - z_2^*}}, \quad k = \sqrt{\frac{z_1^* - z_2^*}{z_1^* - z_3^*}}.$$

Here $F(\varphi, k) = \int\limits_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$ is the elliptic integral of the first kind. Inverting this elliptic integral, we find the function $X_1(t)$ which takes the form (8). Using correlation (8), by formulas (12) we evaluate functions $X_2(t)$ and $X_3(t)$ that obey formulas (9) and (10). $\square$

Evidently, each particular solution that is obtained from the (earlier calculated) general solution (8)–(10) to system (5) is periodic; its period τ depends on parameters λ, μ, and σ and on constant values C_1 and C_2. This property follows from the periodic behavior of Jacobi functions $\mathrm{sn}(T,k)$, $\mathrm{cn}(T,k)$, and $\mathrm{dn}(T,k)$ and from the linear dependence of T on time t. As is well-known [10], for real T, functions $\mathrm{sn}(T,k)$ and $\mathrm{cn}(T,k)$ have real period of $4K$, while the function $\mathrm{dn}(T,k)$ has the real period of $2K$; here K is a complete elliptic integral of the first kind, namely,

$$K = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}, \quad k^2 = \frac{z_1^* - z_2^*}{z_1^* - z_3^*},$$

and periodicity correlations take the form

$$\mathrm{sn}(T + 4K, k) = \mathrm{sn}(T,k), \quad \mathrm{cn}(T + 4K, k) = \mathrm{cn}(T,k), \quad \mathrm{dn}(T + 2K, k) = \mathrm{dn}(T,k).$$

Here real values z_1^*, z_2^*, and z_3^* that define the model of the elliptic function k and satisfy the chain of inequalities $z_1^* > z_2^* > z_3^*$ are roots of the cubic equation (11) whose coefficients depend on parameters λ, μ, and σ and constant values C_1 and C_2. Consequently, the value of the complete elliptic integral K and real periods $4K$ and $2K$ of the evaluated elliptic functions depend on the indicated parameters and constants. Moreover, the value of the period τ in the scale of the initial time t takes the form $\tau = \frac{4K}{\theta}$, where $\theta = \sqrt{\frac{\lambda\mu\, C_1}{C_2}}\sqrt{z_1^* - z_3^*}$.

Remark. Functions (9) and (10) with the constant $C_3 = 0$ are such that

$$X_2(-t) = \frac{\mu}{\lambda}\, X_3(t), \quad X_3(-t) = \frac{\lambda}{\mu}\, X_2(t). \tag{15}$$

If $C_3 = 0$, then the change of the variable $t \to -t$ is equivalent to the transform $T \to -T$. Since the function $\mathrm{sn}(T, k)$ is odd, while functions $\mathrm{cn}(T, k)$ and $\mathrm{dn}(T, k)$ are even, equalities (15) immediately follow from formulas (9) and (10).

Along with the solution that is representable in terms of elliptic Jacobi functions by formulas (8)–(10), system (5) has a particular solution that can be expressed in terms of trigonometric functions. The following assertion is valid.

Proposition 1. *With $\lambda\mu\sigma > 0$ system* (5) *has the following particular solution*:

$$X_1(t) = \frac{\sigma}{C_2}\,\frac{3\left(4\cos^2 T - 3\right)}{4\cos^2 T},$$

$$X_2(t) = \frac{\mu}{C_2}\,\frac{6\cos T\left[-9\cos T + 8\cos^3 T - 3\sqrt{3}\,\delta\,\sin T\right]}{9 - 24\cos^2 T + 16\cos^4 T}, \tag{16}$$

$$X_3(t) = \frac{\lambda}{C_2}\,\frac{6\cos T\left[-9\cos T + 8\cos^3 T + 3\sqrt{3}\,\delta\,\sin T\right]}{9 - 24\cos^2 T + 16\cos^4 T},$$

where $T = \delta\frac{9\sqrt{3}\lambda\mu\sigma}{2\,C_2^2}\,(t - C_3)$ and C_3 is an arbitrary constant.

We get this particular solution by choosing initial data $X_1(t_0)$, $X_2(t_0)$, and $X_3(t_0)$ at some time moment $t_0 \geq 0$ so as to fulfill the equality $C_1C_2^3 - 27\lambda\mu\sigma = 0$ with constant values C_1 and C_2 of first integrals (6). In this case, the cubic equation (11) takes the form

$$z^3 - \frac{15}{4}\sigma z^2 + 3\sigma^2 z + \sigma^3 = 0$$

and has roots $z^*_{1,2} = 2\sigma$ and $z^*_3 = -\sigma/4$. Note that in this case the corresponding Cauchy problem for system (5) is not globally solvable in the interval $[t_0, +\infty)$, solutions explode.

3. NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAY (ADVANCE)

Let us prove that (evaluated in the previous Section) functions (8)–(10) that solve system (5) satisfy the following differential equations with delay (advance):

$$\dot{X}_1(t) = \frac{\lambda\mu}{\sigma}\,X_1^2(t)\big[X_1\,(t - T^*) - X_1\,(t + T^*)\big], \tag{17}$$

$$\dot{X}_2(t) = -\frac{\lambda\sigma}{\mu}\,X_2^2(t)\big[X_2\,(t + T^*) - X_2\,(t + 2T^*)\big], \tag{18}$$

$$\dot{X}_3(t) = \frac{\mu\sigma}{\lambda}\,X_3^2(t)\big[X_3\,(t - T^*) - X_3\,(t - 2T^*)\big], \tag{19}$$

where $\lambda \neq 0$, $\mu \neq 0$, $\sigma \neq 0$, and $T^* \neq 0$ are real parameters.

Theorem 3. *Differential equations with delay* (*advance*) (17)–(19)*, where*

$$T^* = C_3 - \frac{\delta}{\sqrt{z_1^* - z_3^*}}\sqrt{\frac{C_2}{\lambda\mu C_1}}\;\mathrm{sn}^{-1}(\eta), \quad \eta = \sqrt{\frac{z_1^* - z_3^*}{z_1^* + \sigma}}, \quad \delta = \pm 1, \tag{20}$$

while $\mathrm{sn}^{-1}(\eta)$ is the inverse function for the elliptic Jacobi sine function, have exact solutions that obey formulas (8)–(10)*, correspondingly.*

Proof. Let us calculate values of function (8) at time moments $\widehat{t} = t \pm T^*$, where T^* obeys correlation (20). Using addition formulas for elliptic Jacobi functions [10]

$$\operatorname{sn}(\widehat{t}, k) = \frac{\operatorname{sn}(t,k)\operatorname{cn}(T^*,k)\operatorname{dn}(T^*,k) \pm \operatorname{sn}(T^*,k)\operatorname{cn}(t,k)\operatorname{dn}(t,k)}{1 - k^2 \operatorname{sn}^2(t,k)\operatorname{sn}^2(T^*,k)},$$

$$\operatorname{cn}(\widehat{t}, k) = \frac{\operatorname{cn}(t,k)\operatorname{cn}(T^*,k) \mp \operatorname{sn}(t,k)\operatorname{sn}(T^*,k)\operatorname{dn}(t,k)\operatorname{dn}(T^*,k)}{1 - k^2 \operatorname{sn}^2(t,k)\operatorname{sn}^2(T^*,k)},$$

$$\operatorname{dn}(\widehat{t}, k) = \frac{\operatorname{dn}(t,k)\operatorname{dn}(T^*,k) \mp k^2\operatorname{sn}(t,k)\operatorname{sn}(T^*,k)\operatorname{cn}(t,k)\operatorname{cn}(T^*,k)}{1 - k^2 \operatorname{sn}^2(t,k)\operatorname{sn}^2(T^*,k)},$$

and correlations for roots of the cubic equation (11)

$$z_1^* + z_2^* + z_3^* = \frac{C_1 C_2^3 - 12\lambda\mu\sigma}{4\lambda\mu}, \quad z_1^* z_2^* z_3^* = -\sigma^3, \quad z_1^* z_2^* + z_1^* z_3^* + z_2^* z_3^* = 3\sigma^2,$$

by certain transforms we get the equality $X_1(t + \delta T^*) = f(T)$, $\delta = \pm 1$, where

$$f(T) = \frac{C_1 C_2^2 \sigma}{2\lambda\mu}\, \frac{z_1^* - (z_1^* - z_2^*)\operatorname{sn}^2(T,k) - \delta\, P\, \operatorname{sn}(T,k)\operatorname{cn}(T,k)\operatorname{dn}(T,k)}{\left(z_1^* + \sigma - (z_1^* - z_2^*)\operatorname{sn}^2(T,k)\right)^2}.$$

Comparing $f(T)$ with functions (9) and (10), we get correlations

$$X_2(t) = \frac{\mu}{\sigma}\, f(T)\Big|_{\delta=-1}, \quad X_3(t) = \frac{\sigma}{\lambda}\, f(T)\Big|_{\delta=1}. \tag{21}$$

Formula (21) implies that

$$X_2(t) = \frac{\mu}{\sigma}\, X_1(t - T^*), \quad X_3(t) = \frac{\lambda}{\sigma}\, X_1(t + T^*). \tag{22}$$

Hence,

$$X_2(t + T^*) = \frac{\mu}{\sigma}\, X_1(t), \quad X_3(t - T^*) = \frac{\lambda}{\sigma}\, X_1(t), \tag{23}$$

$$X_2(t + 2T^*) = \frac{\mu}{\sigma}\, X_1(t + T^*), \quad X_2(t) = \frac{\mu}{\lambda}\, X_3(t - 2T^*). \tag{24}$$

Taking into account formulas (22)–(24), we conclude that solutions (8)–(10) to system (5) also solve differential equations with delay (advance) (17)–(19). □

4. ANALYTIC PERIODIC SOLUTIONS TO EQUATIONS WITH DELAY (ADVANCE)

For systems of ODE with analytic right-hand sides, including (5), the Cauchy theorem [5] is valid. It guarantees the existence and uniqueness of analytic solutions. Note that there is still no analog of the general theorem on analytic solutions to nonlinear equations with delay (advance). These questions were studied by several mathematicians (e.g., [6–9] and references therein). They mainly considered linear equations of various types with variable coefficients.

Using the connection between solutions to system (5) and equations with delay (advance) (17)–(19) that was established in the previous Section, we conclude that constructed periodic solutions (8)–(10) are analytic functions of time. Really, since they are solutions to system (5), they allow the application of the Cauchy theorem mentioned above. Due to the periodicity of solutions they are located in a bounded domain; therefore by using the Cauchy theorem we consecutively express solutions in terms of power series that converge in some neighborhood of each point $t_0 \in [0, \tau]$, where $\tau > 0$ is a real period of solutions to equations with delay (advance) (8)–(10); it is defined by the choice of initial values $X_1(0)$, $X_2(0)$, and $X_3(0)$.

Example. For values of parameters λ=1, μ=2, and σ=1 and initial conditions $X_1(0) = 1$, $X_2(0) = 1$, and $X_3(0) = 1$, the first nine terms of the power series are the following ones:

$$X_1(t) = 1 - t + \frac{4}{3}\,t^3 - t^4 - \frac{8}{15}\,t^5 + \frac{4}{3}\,t^6 - \frac{148}{315}\,t^7 - \frac{4}{5}\,t^8 + \frac{2668}{2835}\,t^9,$$

$$X_2(t) = 1 + t^2 + \frac{1}{3}\,t^4 - \frac{7}{45}\,t^6 - \frac{26}{315}\,t^8,$$

$$X_3(t) = 1 + t - \frac{4}{3}\,t^3 - t^4 + \frac{8}{15}\,t^5 + \frac{4}{3}\,t^6 + \frac{148}{315}\,t^7 - \frac{4}{5}\,t^8 - \frac{2668}{2835}\,t^9.$$

Note that one can treat these polynomials with respect to nonlinear equations with delay (advance) (17)–(19) as polynomial quasisolutions that were studied in [8] for linear functional differential equations with variable coefficients. However, as distinct from [8], here the convergence of series is guaranteed, i.e., solutions to equations with delay (advance) are analytic.

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