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Three-Webs Defined by Symmetric Functions

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Abstract—We study local differential-geometrical properties of curvilinear k-webs defined by symmetric functions (webs $SW(k)$). This class of k-webs contains in particular algebraic rectilinear k-webs defined by algebraic curves of genus 0. On a web $SW(3)$, there are three three-parameter families of closed Thomsen configurations. We find equations of a rectilinear web $SW(k)$ in terms of adapted coordinates and prove that the curvature of a symmetric three-web is a skew-symmetric function with respect to adapted coordinates. In conclusion, we formulate some open problems.

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Introduction. Recall that a k-web $W(k)$ on the plane is a collection of k families of smooth curves in general position. In some recent publications, the k-webs in question are called ordered. We however will follow the terminology introduced by W. Blaschke, the founder of the differential-topological theory of webs. By the domain of a k-web we mean the maximal domain in which the families form transverse foliations, i.e., leaves of a web are pairwise transverse at each point of its domain.

The paper is devoted to the study of local aspects of the theory of symmetric k-webs, i.e., k-webs whose equations for a certain choice of parameters of the families are not changed under any permutation of arguments. Following W. Blaschke [1], we consider webs up to local diffeomorphisms, i.e., up to the widest equivalence relation. Local diffeomorphisms preserve transversality of lines of a k-web, closure or nonclosure of sufficiently small configurations formed by lines of a web.

The most important symmetric webs are rectilinear webs of special form. Recall that a k-web formed by k families of straight lines (not necessary parallel) is called rectilinear and denoted by $LW(k)$. The most difficult problems of the theory of curvilinear webs which have more than centennial history are connected with rectilinear three-webs.

Fist of all this is so-called “anamorphosis problem”, possibility to represent a function in two variables by nomogram. The complete solution to this problem in terms of differential invariants of a web is given in [2] and is contained in [3]. Another problem also came from nomography and is connected with the proof of Gronwall conjecture (1912) [1]. Its positive solution was found in [4]. In [3], it is given in the following formulation: if two rectilinear three-webs are equivalent, then they are projectively equivalent.

An important subclass of rectilinear k-webs is formed by algebraic webs defined by homogeneous algebraic equations of degree k connecting tangential coordinates of current line. In Item 12, we show that if such an equation defines a curve of genus 1 (a curve birationally equivalent to a straight line), then the corresponding rectilinear k-web is symmetric. In the paper, we generalize this property, namely, we investigate symmetric k-webs or webs $SW(k)$ whose functions, for a choice of parameters of families, are symmetric functions. The geometric characteristic of symmetric three-webs is as follows. Such webs carry three three-parameter families of closed Thomsen configurations (Theorems 1 and 3). Equations of a rectilinear three-web are symmetric if and only if all its families belong to one family (Theorem 4). The condition for existence of a symmetric rectilinear three-web (3-web SLW) is given

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in Theorem 5. In Items 7—9, we find equations of a 3-web SLW and those of a semisymmetric 3-web in terms of complete symmetric polynomials. For a k-web SLW, these results are generalized in Theorems 7 and 8. In conclusion, we prove that the curvature of a semisymmetric three-web is a skew-symmetric function of adapted parameters and formulate problems. All functions considered in the paper are assumed to be real analytic.

1. Consider a k-web $W(k)$ formed by k families λ_α, $\alpha = 1, 2, \ldots, k$, of smooth curves. Webs are studied in this paper from local point of view, so it can be assumed that the families λ_α are foliations and are given on the plane with Cartesian coordinates (x, y) by equations

$$\lambda_\alpha(x, y) = u_\alpha, \tag{1}$$

where u_α is the parameter of the family λ_α. Eliminating from Eqs. (1) the coordinates x, y, we obtain $(k-2)$ equations

$$\mathcal{F}_i(u_1, u_2, \ldots, u_k) = 0, \ \ i = 1, 2, \ldots, k-2, \tag{2}$$

connecting the parameters of lines of a k-web $W(k)$ passing through one point. Since leaves of a web are transverse, Eqs. (2) are functionally independent, and all variables in system (2) are essential. A three-web is defined by a single equation of the form (2).

Since webs are considered from local point of view, the domains of functions $\mathcal{F}_i$ are found as follows. Let M_0 be an arbitrary point of the plane at which leaves of foliations λ_α are pairwise transverse. Then transversality of leaves takes place in a neighborhood U of M_0. In this neighborhood, the parameter u_α of the foliation λ_α varies within some interval I_α. Then the domain of $\mathcal{F}_i$ is a domain in $I = I_1 \times I_2 \times \cdots \times I_k$ containing the point $(u_1^0, u_2^0, \ldots, u_k^0)$, where u_α^0 are the parameters of the leaves through the point M_0.

Eliminating from some three Eqs. (1) the variables x, y, we obtain an equation of the form

$$F_{\alpha\beta\gamma}(u_\alpha, u_\beta, u_\gamma) = 0, \tag{3}$$

which is the equation of the three-subweb of the k-web $W(k)$ formed by the families $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$. It is denoted by $W(\alpha, \beta, \gamma)$.

Taking the first and the second families as basis ones, we can write system (2) in the usual form

$$F_3(u_1, u_2, u_3) = 0,\ F_4(u_1, u_2, u_4) = 0, \ldots, F_k(u_1, u_2, u_k) = 0 \tag{4}$$

or

$$u_3 = f_3(u_1, u_2),\ u_4 = f_4(u_1, u_2),\ \ldots, u_k = f_k(u_1, u_2). \tag{5}$$

Systems (4) and (5) are locally equivalent, but, generally speaking, are not equivalent globally.

Equations (2), (4), (5) are called the equations of a k-web, the functions $\mathcal{F}_i$, $F_3, F_4, \ldots, F_k$ and $f_3, f_4, \ldots, f_k$ are called the functions of a k-web. Their form depends on parameterization of the families. It is clear that the functions $\mathcal{F}_i$, and even $F_3, F_4, \ldots, F_k$, are defined not uniquely. We will write the equation of a three-web in the form

$$F(u_1, u_2, u_3) = 0. \tag{6}$$

If two k-webs are equivalent (locally diffeomorphic), then the equations of one of them can be transformed into the equations of the other with the use of coordinate change

$$u_\alpha \to \widetilde{u}_\alpha(u_\alpha), \ \ \alpha = 1, 2, \ldots, k. \tag{7}$$

Such transformations in the theory of quasigroups are called isotopic transformations or an isotopy. On the other hand, change of variables (7) can be viewed as parameter transformation in families of lines of some k-web. In particular, if all functions $\widetilde{u}_\alpha(u_\alpha)$ are the same, an isotopy is called an isomorphism.

In k-dimensional space of parameters u_α, system (2) defines a two-dimensional surface V^2. The coordinate hyperplanes $u_\alpha = \text{const}$ cut on V^2 a k-web $\widehat{W}(k)$ equivalent to the initial k-web $W(k)$.

Equations (5) can be interpreted as equations of a three-web $\widetilde{W}(k)$ on the plane with Cartesian coordinates u_1, u_2 formed by the coordinate net $u_1 = \text{const}$, $u_2 = \text{const}$ and the level lines of the functions $f_3, f_4, \dots, f_k$. This k-web is also equivalent to the initial web $W(k)$ and it is obtained as the projection of the k-web $\widehat{W}(k)$ to the coordinate 2-plane u_1, u_2.

In what follows, instead of a k-web $W(k)$, we will, as a rule, consider equivalent webs $\widehat{W}(k)$ and $\widetilde{W}(k)$ in the space of parameters.

2. We assume that the definitions of a symmetric and a skew-symmetric functions are known. In what follows the following assertions will be useful.

Proposition 1. *Let a function $F(u_1, u_2, \dots, u_k)$ be symmetric with respect to the arguments u_i, u_j and skew-symmetric with respect to the arguments u_i, u_k, all i, j, k are different. Then* $F(u_1, u_2, \dots, u_k) \equiv 0$.

Proof.

$$\begin{aligned} F(\dots, u_i, \dots, u_j, \dots, u_k, \dots) &= F(\dots, u_j, \dots, u_i, \dots, u_k, \dots) \\ &= -F(\dots, u_j, \dots, u_k, \dots, u_i, \dots) = -F(\dots, u_k, \dots, u_j, \dots, u_i, \dots) \\ &= F(\dots, u_k, \dots, u_i, \dots, u_j, \dots) = F(\dots, u_i, \dots, u_k, \dots, u_j, \dots) \\ &= -F(\dots, u_i, \dots, u_j, \dots, u_k, \dots). \quad \square \end{aligned}$$

Proposition 2. *If a function $F(u_1, u_2, \dots, u_k)$ is skew-symmetric with respect to the arguments u_i and u_j, then there exists a real analytic function $\widetilde{F}(u_1, u_2, \dots, u_k)$ symmetric with respect to the arguments u_i, u_j such that*

$$F(u_1, u_2, \dots, u_k) = (u_i - u_j)\widetilde{F}(u_1, u_2, \dots, u_k).$$

Corollary. If a function $F(u_1, u_2, \dots, u_k)$ is skew-symmetric with respect to all arguments, then

$$F(u_1, u_2, \dots, u_k) = \prod_{1 \le i, j \le k,\ i<j} (u_i - u_j)\widetilde{F}(u_1, u_2, \dots, u_k),$$

where $\widetilde{F}$ is a function symmetric with respect to all arguments.

3. Definition 1. A k-web given by Eqs. (2) will be called symmetric and denoted by $SW(k)$ if, for some choice of the parameters of the families, all functions $\mathcal{F}_i$ are symmetric.

The parameters for which the symmetry condition holds will be called *adapted parameters of a symmetric k-web*. Obviously, adapted parameters are defined up to an isomorphism.

Example 1. A three-web is called parallel if it is formed by three families of parallel straight lines. A three-web is called parallelizable or regular if is equivalent to a parallel three-web. Each regular three-web is symmetric since its equation can be reduced to the form $u_1 + u_2 + u_3 = 0$.

Example 2. Consider in the space of parameters the symmetric 4-web $\widetilde{SW}$ given by the equations

$$u_1 + u_2 + u_3 + u_4 = 0, \quad (u_1)^2 + (u_2)^2 + (u_3)^2 + (u_4)^2 = 1.$$

On the plane of the variables u_3, u_4, this web is formed by the coordinate net $u_3 = \text{const}$, $u_4 = \text{const}$ and the level lines of the functions

$$\begin{aligned} u_1 &= 2^{-1}\left(-u_3 - u_4 + \sqrt{-3(u_3)^2 - 3(u_4)^2 - 2u_3u_4 + 2}\right), \\ u_2 &= 2^{-1}\left(-u_3 - u_4 - \sqrt{-3(u_3)^2 - 3(u_4)^2 - 2u_3u_4 + 2}\right). \end{aligned}$$

Eliminating irrationalities, in each case we arrive at the equation

$$(u_3)^2 + (u_4)^2 + u_3u_4 + cu_3 + cu_4 + c^2 = 1/2,$$

which defines a quadratic family of equal ellipses with centers on the straight line $u_3 = u_4$ symmetric with respect to this straight line. Through each point of the domain, there pass two ellipses whose parts are lines $u_1 = \text{const}$ and $u_2 = \text{const}$ of a 4-web $\widetilde{SW}$.

Proposition 3. *If the functions $\mathcal{F}_i$ are skew-symmetric, then the corresponding k-web is symmetric.*

Proof. By Corollary to Proposition 2, we have

$$\mathcal{F}_i(u_1, u_2, \dots, u_k) = \prod_{1\le i,j\le k,\ i<j} (u_i - u_j)\widetilde{\mathcal{F}}_i(u_1, u_2, \dots, u_k),$$

where the functions $\widetilde{\mathcal{F}}_i$ are symmetric. The domains $u_i - u_j = 0$ are not contained in the domain of the web since in these domains the rank of system (2) becomes lower. Outside these domains, system (2) is equivalent to the system $\widetilde{\mathcal{F}}_i(u_1, u_2, \dots, u_k) = 0$. □

Let Eqs. (2) be symmetric. Then the variables u_α, u_β, u_γ can always be placed onto the first three places and in any order. Therefore all functions $F_{\alpha,\beta,\gamma}$ (see (3)) are solutions to the same system of equations and are symmetric. The same can be said about the functions $f_3, f_4, \dots, f_k$.

Consider first a symmetric three-web $SW(3) \equiv SW$ with Eq. (6). Its symmetry means in particular that if there exists a line from one of the families with adapted parameter u, then the other families also have a line with identical adapted parameter. Therefore, *there is a local bijective correspondence between the families of lines of symmetric three-web under which lines with equal adapted parameters correspond to each other*. In the domain of a web, there arise three special curves $u_i = u_j$ along which lines from different families with equal parameters meet. These lines (not necessarily real) will be called *equilibrium lines*. Obviously, if two equilibrium lines meet at a real point, then through it there passes the third equilibrium line. Such points will be called *umbilical points* of a symmetric three-web.

Theorem 1. *If a configuration T of a symmetric three-web SW contains two corresponding lines from some two families, then the other two pairs of lines of the same families contained in T are also in correspondence, and the figure T is closed. On each symmetric three-web SW, there are at least three three-parameter families of closed (sufficiently small) configurations T containing three pairs of corresponding lines from two families.*

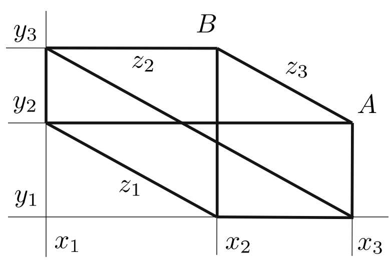

Fig. 1.

Proof. A closed Thomsen configuration (or T) formed by lines of a three-web is shown on Fig. 1. Here lines of the first, second, and third families of a web are depicted by vertical, horizontal, and oblique lines, respectively. The vertical lines are marked by the parameters x_α, the horizontal lines by the parameters y_α, and the oblique lines by the parameters z_α. The figure T is constructed, for example, as follows. We take two arbitrary lines x_1 and y_1 from the first and the second families, then sufficiently close to the point of their intersection we draw oblique lines z_1 and z_2. Through the obtained points of intersection we draw vertical and horizontal lines x_2 and y_2, x_3 and y_3, as is shown on Fig. 1. Then we obtain points A and B. The configuration obtained is called the Thomsen figure or figure T. If the points A and B belong to a line from the third family, one says that the figure T is closed.

If we start the construction with the lines z_1 and y_3, then the closure condition for the same figure T is the existence of a line x_3. Similarly, if we start the construction with the lines from the first and the third families, then the closure condition is the existence of a line from the second family. In this connection, one differs the closure conditions $T(1,2)$, $T(2,3)$, $T(3,1)$.

Generally speaking, Thomsen figures need not to be closed on an arbitrary three-web. If on a three-web all sufficiently small figures T are closed, one says that *closure condition* T holds. Such webs are called hexagonal. Every regular three-web is hexagonal and vice versa: every hexagonal three-web is regular ([3], P. 37).

The condition for three lines with parameters x_1, y_2, z_1 to pass through one point is written, in accordance with the definition of equation of a web, as $F(x_1, y_2, z_1) = 0$. Therefore, the closure condition $T(1,2)$ for the figure T, depicted on Fig. 1, is written in the form of so-called conditional identity: The relations

$$\begin{aligned} &F(x_1, y_2, z_1) = 0, \quad F(x_2, y_1, z_1) = 0, \\ &F(x_1, y_3, z_2) = 0, \quad F(x_3, y_1, z_2) = 0 \end{aligned} \tag{8}$$

must imply existence of z_3 such that

$$F(x_2, y_3, z_3) = 0, \quad F(x_3, y_2, z_3) = 0. \tag{9}$$

Note that, in all, on a three-web there is a four-parameter family of (nonclosed) Thomsen configurations, and the variables x_1, y_1, z_1, and z_2 can be taken as parameters. Locally, system (8), (9) has, generally speaking, a finite number of solutions, i.e., on an arbitrary three-web, there are only finite number of (locally isolated) closed configurations T.

Let, for a symmetric three-web SW, lines x_1 and y_1 be in correspondence, i.e., $x_1 = y_1$. Consider the figure T, depicted on Fig. 1. From equalities (8) by virtue of symmetry of F, we obtain

$$\begin{aligned} &F(x_1, y_2, z_1) = 0, \quad F(x_1, x_2, z_1) = 0, \\ &F(x_1, y_3, z_2) = 0, \quad F(x_1, x_3, z_2) = 0. \end{aligned}$$

Since the lines z_1 and z_2 from the third family are close to the intersection point of the lines x_1 and y_1, from the first two equations, by virtue of the above indicated local uniqueness, we obtain $x_2 = y_2$, and from the second two equations we obtain $x_3 = y_3$, i.e., the two other pairs of lines from the first and the second families in the figure T are in correspondence. Then, by virtue of symmetry of F, equalities (9) hold, i.e., the figure T is closed.

We arrive at a similar result if we assume that $x_2 = y_2$ or $x_3 = y_3$. We have proved the following: *If the Thomsen figure T of a symmetric three-web contains two corresponding lines from the first and the second families, then the other two pairs of lines from the same families in the figure T are also corresponding, and the figure T is closed* (the closure condition $T(1,2)$ holds for these lines). Such a figure is defined by a choice of three pairs of corresponding lines, therefore, they all form a three-parameter family.

In a similar way we can consider a figure T which has two corresponding lines from the second and the third families, for example, $z_1 = y_3$. Then we arrive at the closure condition $T(2,3)$, and all three pairs of lines from the second and the third families in this figure T will be corresponding. Such figures also form a three-parameter family. Finally, there are as many closed figures T which contain three pairs corresponding lines from the first and the third families. □

Remark 1. As we have already mention, all configurations T are closed on a regular three-web. Thus, a web SW in question is close in this sense to a regular web, but, generally speaking, is not a regular web (since Thomsen figures built on non-corresponding lines, generally speaking, are not closed).

Remark 2. Theorem 1 shows that the symmetry condition for the function of a web has invariant geometric sense: A great number of Thomsen configurations are closed. This property of a web remains valid under local diffeomorphisms, i.e., it holds for all equivalent webs. Hence it follows that the definition of a symmetric three-web is correct though it uses some distinguished parameters. As has already been shown, symmetric Eq. (6) remains symmetric under isotopic transformations (7) for which the functions $\widetilde{u}_\alpha(u_\alpha)$, $\alpha = 1, 2, 3$, coincide, i.e., an isotopy is an isomorphism.

The obtained result can be formulated as follows.

Theorem 2. *Using change of variables* (7)*, one cannot, generally speaking, make an arbitrary function* $F(u_1, u_2, u_3)$ *into a symmetric one.*

Remark 3. As one can see from Fig. 1, the corresponding lines with parameters $x_2 = y_2$ in a closed figure T meet at a point which, by definition, belongs to the corresponding equilibrium line. Consequently, in a neighborhood of this point, the equilibrium line is real. Since the figure T is small, the other two lines of intersection of corresponding lines contained in this figure T, $x_1 = y_1$ and $x_3 = y_3$, also belong to the same equilibrium line. Thus, all closed Thomsen figures on a symmetric three-web "are attached" to equilibrium lines.

Remark 4. For a symmetric k-web, a similar result takes place. In fact, in case of symmetry, each of Eqs. (3) is an equation of some symmetric three-subweb $W(\alpha, \beta, \gamma)$ formed by lines from families with corresponding numbers. By Theorem 1, on each of these three-subwebs, there are $3 \cdot \infty^3$ closed configurations T formed by the pairs of corresponding lines from some two families.

4. It turns out that the converse of Theorem 1 is true. Preliminarily, we give the following definition.

Definition 2. Let between two families λ_α and λ_β of lines of a three-web W a local bijective correspondence $\varphi_\gamma : \lambda_\alpha \to \lambda_\beta$ be established, where α, β, γ are all different and form an even permutation of 1, 2, 3. We say that a figure T belongs to φ_γ if it contains three lines from the family λ_α and the three corresponding lines from the family λ_β.

For example, the figure on Fig. 1 belongs to the mapping $\varphi_3 : \lambda_1 \to \lambda_2$ if $y_\alpha = \varphi_3(x_\alpha)$.

Theorem 3 (the converse of Theorem 1)**.** *Let, on a three-web W formed by families* λ_α*, there exist three local bijections*

$$\varphi_3 : \lambda_1 \to \lambda_2, \quad \varphi_1 : \lambda_2 \to \lambda_3, \quad \varphi_2 : \lambda_3 \to \lambda_1$$

such that figures T belonging to them are closed. Then the three-web W is symmetric.

Proof. Let ℓ_0 be an arbitrary line from the family λ_1 of a three-web W. We ascribe to the lines ℓ_0, $\varphi_3(\ell_0)$, and $\varphi_1(\varphi_3(\ell_0))$ equal parameter u_0. Then the families can be parameterized in a neighborhood of the point u_0 in such a way that the corresponding lines $\ell \in \lambda_1$, $\varphi_3(\ell)$, and $\varphi_1(\varphi_3(\ell))$ have equal parameter u.

Consider on W an arbitrary figure T depicted on Fig. 1, and let the lines of the first and the second families of this figure be corresponding, i.e., $x_1 = y_1$, $x_2 = y_2$, and $x_3 = y_3$. By the assumptions of the theorem, this figure must be closed, and for it conventional identity (5), (6) must hold. In the case in question, the corresponding equalities take the form

$$\begin{aligned} F(x_1, x_2, z_1) = 0, &\quad F(x_2, x_1, z_1) = 0, \\ F(x_1, x_3, z_2) = 0, &\quad F(x_3, x_1, z_2) = 0, \\ F(x_2, x_3, z_3) = 0, &\quad F(x_3, x_2, z_3) = 0. \end{aligned}$$

Each pair of these equalities means that the function $u_3 = f(u_1, u_2)$ such that

$$F(u_1, u_2, f(u_1, u_2)) \equiv 0$$

is symmetric. Hence it follows that the function $F(u_1, u_2, u_3)$ is symmetric with respect to the first two arguments.

Considering, for the figure T, the closure conditions $T(2, 3)$ and $T(3, 1)$, we arrive at the conclusion that the function $F(u_1, u_2, u_3)$ is symmetric with respect to the other pairs of arguments. □

5. Definition 3. A three-web whose Eq. (6), for some choice of family parameters, is symmetric with respect to some two variables will be called semisymmetric and denoted by $SSW(3)$ or briefly SSW.

Let Eq. (6) be symmetric with respect to the first two variables. Repeating the arguments from Theorem 1, we arrive at the conclusion that on the corresponding three-web SSW there is a three-parameter family of figures T which contain three pairs of corresponding lines from one and two families.

As is known ([3], P. 45), the closure condition for figures T has an algebraic analog. Solving Eq. (6) of a three-web SSW with respect to u_3 in the form $u_3 = f(u_1, u_2)$, we obtain the equation of the coordinate quasigroup q_{12} of this web. By virtue of symmetry, this quasigroup is commutative. In particular, if a three-web SSW is symmetric, then all six its coordinate quasigroups q_{12}, q_{21}, q_{23}, q_{32}, q_{31}, q_{13} are commutative.

6. It is convenient to consider rectilinear three-webs on the projective plane. We write the equations of the families of straight lines forming a rectilinear three-web LW in the form

$$\lambda_\alpha : a_\alpha(u_\alpha)x_1 + b_\alpha(u_\alpha)x_2 + c_\alpha(u_\alpha)x_3 = 0, \ \ \alpha = 1, 2, 3, \tag{10}$$

where x_1, x_2, x_3 are projective coordinates. Eliminating them from Eqs. (10), we obtain the following equation of a three-web LW

$$\begin{vmatrix} a_1(u_1) & b_1(u_1) & c_1(u_1) \\ a_2(u_2) & b_2(u_2) & c_2(u_2) \\ a_3(u_3) & b_3(u_3) & c_3(u_3) \end{vmatrix} = 0. \tag{11}$$

Note that all minors of second order in this determinant differ from zero. In fact, let, for example, $a_1b_2 - a_2b_1 = 0$, then Eq. (11) falls into two equations, one of which contains only the variables u_1 and u_2, and the other one only the variables u_2 amd u_3. Therefore, none of them can be an equation of a web.

Assume that all a_α in Eqs. (10) are different from zero, i.e., consider a domain in which there are no horizontal straight lines belonging to the three-web in question. Let us introduce new parameters $v_\alpha = b_\alpha/a_\alpha$, then Eq. (11) takes the form

$$\begin{vmatrix} 1 & v_1 & p_1(v_1) \\ 1 & v_2 & p_2(v_2) \\ 1 & v_3 & p_3(v_3) \end{vmatrix} = 0$$

or

$$p_1(v_1)(v_3 - v_2) + p_2(v_2)(v_1 - v_3) + p_3(v_3)(v_2 - v_1) = 0. \tag{12}$$

The parameters v_α will be called *canonical parameters of a rectilinear three-web LW*, and Eq. (12) the *canonical equation* of this web.

Consider the excluded case: Let, for example, $a_3 = 0$, then the third family is the pencil with vertex $(0, 0, 1)$. (In this case, the functions a_1 and a_2 must be nonzero, otherwise two families or all three families are the same pencil, i.e., there is no three-web.) Denote a three-web in question by LW_0 and introduce on it canonical parameters $v_1 = -b_1/a_1$, $v_2 = -b_2/a_2$, $v_3 = c_3/b_3$. Then Eqs. (10) of the families of a three-web take the form

$$x_1 + v_1x_2 + p_1(v_1)x_3 = 0, \ \ x_1 + v_2x_2 + p_2(v_2)x_3 = 0, \ \ x_2 = -v_3x_3.$$

Eliminating the coordinates, we obtain the canonical equation of a three-web LW_0:

$$v_3 = \frac{p_1(v_1) - p_2(v_2)}{v_1 - v_2}. \tag{13}$$

7. Theorem 4. *Equation* (11) *is symmetric with respect to the variables u_1 and u_2 if and only if $(a_1, b_1, c_1) = (a_2, b_2, c_2)$, i.e., lines from the first two families of a three-web LW belong to the same family.*

Proof. Sufficiency is obvious: If $(a_1, b_1, c_1) = (a_2, b_2, c_2)$, then Eq. (11) is not changed under the change $u_1 \leftrightarrow u_2$. Conversely, let Eq. (11) be symmetric with respect to u_1 and u_2, i.e., together with

Eq. (11) the following equation holds

$$\begin{vmatrix} a_1(u_2) & b_1(u_2) & c_1(u_2) \\ a_2(u_1) & b_2(u_1) & c_2(u_1) \\ a_3(u_3) & b_3(u_3) & c_3(u_3) \end{vmatrix} = 0.$$

Comparing in these equations the coefficients of the variables (a_3, b_3, c_3), we arrive at the proportionality of the two vector products: $(a_1(u_1), b_1(u_1), c_1(u_1)) \times (a_2(u_2), b_2(u_2), c_2(u_2))$ and $(a_1(u_2), b_1(u_2), c_1(u_2)) \times (a_2(u_1), b_2(u_1), c_2(u_1))$. This is possible only on condition that all four vectors belong to a plane, which in particular implies that $(a_1(u_2), b_1(u_2), c_1(u_2)) = p(a_1(u_1), b_1(u_1), c_1(u_1)) + q(a_2(u_2), b_2(u_2), c_2(u_2))$. Since the latter equality must hold for any u_1 and u_2, we have $p = 0$, $(a_1(u_2), b_1(u_2), c_1(u_2)) = q(a_2(u_2), b_2(u_2), c_2(u_2))$. Consequently, $(a_1, b_1, c_1) = q(a_2, b_2, c_2)$. One can let $q = 1$ because the coefficients in an equation of a straight line are defined up to a factor. □

Corollary 1. A three-web LW is symmetric if and only if, for any $\alpha = 1, 2, 3$, we have $a_\alpha = a$, $b_\alpha = b$, $c_\alpha = c$. Denote such three-webs by SLW. All three families of straight lines of a web SLW belong to one family

$$a(u)x + b(u)y + c(u)z = 0, \tag{14}$$

and Eq. (11) for them takes the form

$$\begin{vmatrix} a(u_1) & b(u_1) & c(u_1) \\ a(u_2) & b(u_2) & c(u_2) \\ a(u_3) & b(u_3) & c(u_3) \end{vmatrix} = 0. \tag{15}$$

In terms of the canonical parameters, Eq. (14) takes the form

$$x + vy + p(v)z = 0, \tag{16}$$

and the equation of a web SLW takes the form

$$F(v_1, v_2, v_3) \equiv p(v_1)(v_2 - v_3) + p(v_2)(v_3 - v_1) + p(v_3)(v_1 - v_2) = 0. \tag{17}$$

The function $F(v_1, v_2, v_3)$ in the left-hand side of Eq. (17) is skew-symmetric. Consequently, in accordance with Proposition 3, it can be written in the form $F = (v_1 - v_2)(v_2 - v_3)(v_3 - v_1)\widetilde{F}$, where $\widetilde{F}$ is a symmetric function. Substituting the Taylor series

$$p(v) = a_0 + a_1 v + a_2 v^2 + \cdots, \ \ a_k \in \mathbb{R}, \tag{18}$$

into (17), after not difficult transformations, we obtain

$$\widetilde{F} \equiv a_2 + a_3(v_1 + v_2 + v_3) + a_4 h_2(v_1, v_2, v_3) + a_5 h_3(v_1, v_2, v_3) + \ldots = 0, \tag{19}$$

where

$$h_m(v_1, v_2, v_3) \equiv \sum_{0 \le i,j,k \le 2m-1}^{i+j+k=2m-1} v_1^i v_2^j v_3^k \tag{20}$$

is the complete symmetric polynomial of degree m in variables v_1, v_2, v_3. In particular, if $p(v)$ is a polynomial of degree three, then, by the Graf—Sauer theorem, a rectilinear three-web SLW is regular. In fact, $p(v) = a_0 + a_1 v + a_2 v^2 + a_3 v^3$, and Eq. (20) becomes linear.

Note that, for a web SLW, its canonical coordinates are also adapted coordinates.

Corollary 2. A three-web LW_0 defined by Eq. (13) is semisymmetric if and only if in this equation $p_1 = p_2$ and, therefore, it is of the form

$$v_3 = \frac{p(v_1) - p(v_2)}{v_1 - v_2} \equiv f(v_1, v_2). \tag{21}$$

We will denote such three-webs by SLW_0.

If the function $p(v)$ is not linear, then, on the plane of parameters v_1 and v_2, Eq. (21) defines a rectifiable three-web $\widetilde{SLW}_0$ equivalent to a three-web SLW_0 and formed by the Cartesian net $v_1 =$ const, $v_2 =$ const and the level lines of the function f, which are symmetric with respect to straight lines $v_1 + v_2 = 0$ and $v_1 - v_2 = 0$.

8. Rectilinear symmetric three-webs SLW can be classified by the form of the function $p(v)$.

Theorem 5. *A three-web SLW whose lines belong to family* (16) *exists if and only if the function* $u = p(v)$ *has an inflection point.*

Proof. A symmetric three-web SLW exists in a domain D if through each point of D there pass three straight lines belonging to family (16). The parameters of the lines through a point $M(x, y, z)$ are found from Eq. (16). But the roots of Eq. (16) are points of intersection of the graph of $u = p(v)$ and the straight line $x_1 + vx_2 + ux_3 = 0$ in the plane of variables u, v. If $u = p(v)$ has a point of inflection, then, in a neighborhood of this point, there exists a straight line m which intersects the graph at three points and conversely. Then, by continuity, all straight lines sufficiently close to m (for different values of the parameters x_1, x_2, x_3) also intersect the graph at three points. Consequently, a symmetric three-web SLW exists. □

For example, if, in Eq. (16), we have $p(v) = v^{2m}$, $m \in \mathbb{N}$, then there is no three-web SLW because the function $p(v)$ has no inflection points in a neighborhood of $(0, 0)$. In this case, Eq. (17) takes the form

$$v_1^{2m+1}(v_3 - v_2) + v_2^{2m+1}(v_1 - v_3) + v_3^{2m+1}(v_2 - v_1) = 0.$$

Cancelling out the factors $v_1 - v_2$, $v_2 - v_3$, $v_3 - v_1$, we obtain the equation $h_{2m-1}(v_1, v_2, v_3) = 0$.

9. Theorem 6. *A rectifiable three-web* $\widetilde{SLW}_0$ *given by Eq.* (21) *is regular if and only if* $p(v)$ *is a polynomial of the second degree.*

Proof. If the function $p(v)$ is a polynomial of the second degree, then the right-hand side is linear, consequently, the three-web $\widetilde{SLW}_0$ if regular. Conversely, let a web $\widetilde{SLW}_0$ defined by Eq. (21) be regular. Then, by the Graf–Sauer theorem, all three families of the web must belong to one cubic family, and the third family is linear, a pencil. Therefore, the other two families, the first and the second, must belong to a quadratic family. This means (16) that the function $p(v)$ must be a polynomial of the second degree. □

In particular, if, in Eq. (21), $p(v) = v^{m+1}$, $m \in \mathbb{N}$, then

$$f = (v_1)^m + (v_1)^{m-1}v_2 + (v_1)^{m-2}(v_2)^2 + \cdots + (v_2)^m \equiv h_m(v_1, v_2)$$

is the complete symmetric polynomial. In the general case, if $p(v)$ is a real analytic function of the form (19), then

$$f = a_1 + \sum_{k=2,3,\dots} a_k h_{k-1}(v_1, v_2).$$

10. The notions introduced are generalized for rectilinear k-webs $LW(k)$ formed by k families of straight lines.

Consider the general situation when in the domain of a web $LW(k)$ there are no horizontal lines. Then the equations of the families can be written in the form (16)

$$x + v_\alpha y + p_\alpha(v_\alpha) = 0, \tag{22}$$

and here and in what follows $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, k$.

Eliminating from each triple of equations the variables x and y, we obtain the equations of a web $LW(k)$ in the form (17)

$$p_\alpha(v_\alpha)(v_\gamma - v_\beta) + p_\beta(v_\beta)(v_\alpha - v_\gamma) + p_\gamma(v_\gamma)(v_\beta - v_\alpha) = 0. \tag{23}$$

Among these equations there are $(k-2)$ independent, for example,

$$p_1(v_1)(v_3 - v_2) + p_2(v_2)(v_1 - v_3) + p_3(v_3)(v_1 - v_2) = 0,$$
$$p_1(v_1)(v_4 - v_2) + p_2(v_2)(v_1 - v_4) + p_4(v_4)(v_1 - v_2) = 0,$$
$$\dots$$
$$p_1(v_1)(v_k - v_2) + p_2(v_2)(v_1 - v_k) + p_k(v_k)(v_1 - v_2) = 0.$$

Then, as above (Theorem 4), we require that the functions in the left-hand sides of the latter equations be symmetric. As a result, we find that all functions p_α are the same, $p_\alpha = p$, i.e., all families (23) belong to one family (16). Such rectilinear k-webs will be called symmetric and denoted by $SLW(k)$.

In a similar way to Theorem 5, we prove the following theorem.

Theorem 7. *A symmetric k-web $SLW(k)$ all of whose families of straight lines belong to family* (16) *exists if and only if there exists a straight line intersecting the graph of the function* $p(v)$ *at least at k points.*

Equations (24) for a symmetric k-web take the form

$$p(v_\alpha)(v_\gamma - v_\beta) + p(v_\beta)(v_\alpha - v_\gamma) + p(v_\gamma)(v_\beta - v_\alpha) = 0. \tag{24}$$

Cancelling out the factors indicated in brackets, we obtain the equation

$$f_2(v_\alpha, v_\beta, v_\gamma) \equiv a_2 + a_3(v_\alpha + v_\beta + v_\gamma) + a_4 h_2(v_\alpha, v_\beta, v_\gamma) + \dots = 0, \tag{25}$$

where h_m, as above (20), are the complete symmetric polynomials. Independent Eqs. in (25) and (24) are obtained, for example, when $\alpha = 1$, $\beta = 2$, $\gamma = 3, 4, \dots, k$,

$$f_2(v_1, v_2, v_3) = 0,\ f_2(v_1, v_2, v_4) = 0, \dots,\ f_2(v_1, v_2, v_k) = 0. \tag{26}$$

Equations (26) will be called the first system of equations of a symmetric k-web $SLW(k)$. We denote this system by Σ_1.

11. The indicated collection of independent equations of a symmetric three-web $LW(k)$ is not unique. Denote by $h_m(v_1, v_2, \dots, v_n)$ the complete symmetric polynomial of degree m in the variables $v_1, v_2, \dots, v_n$. The following properties of the polynomials h_m can be proved either by means of immediate calculations or by induction:

1. $h_m(v_1, v_2, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_n) = h_m(v_1, v_2, \dots v_{i-1}, v_{i+1}, \dots, v_n)$,
2. $h_m(v_1, v_2, \dots, v_n, u_1) = h_m(v_1, \dots, v_n) + u_1 h_{m-1}(v_1, \dots, v_n)$
$+(u_1)^2 h_{m-2}(v_1, \dots, v_n) + \dots + (u_1)^{m-1} h_1(v_1, \dots, v_n) + (u_1)^m$,
3. $h_m(v_1, v_2, \dots, v_n, u_1, u_2) = h_m(v_1, v_2, \dots, v_n)$
$+h_1(u_1, u_2) h_{m-1}(v_1, \dots, v_n) + h_2(u_1, u_2) h_{m-2}(v_1, \dots, v_n) + \dots$
$+h_{m-1}(u_1, u_2) h_1(v_1, \dots, v_n) + h_m(u_1, u_2)$.

Consider the fraction

$$\frac{f_2(v_1, v_2, v_3) - f_2(v_1, v_2, v_4)}{v_3 - v_4}.$$

Let us substitute into it expansions (25) and make use of property 2 for each h_k appearing in these expansions. Cancelling out by $v_3 - v_4$, we obtain the equality

$$\begin{aligned} a_3 + a_4(h_1(v_1, v_2) + h_1(v_3, v_4)) \\ + a_5(h_2(v_1, v_2) + h_1(v_1, v_2) h_1(v_3, v_4) + h_2(v_3, v_4)) + a_6(h_3(v_1, v_2) \\ + h_2(v_1, v_2) h_1(v_3, v_4) + h_1(v_1, v_2) h_2(v_3, v_4) + h_3(v_3, v_4)) + \dots = 0, \end{aligned}$$

which, by property 3, is equivalent to the relation

$$f_3(v_1, v_2, v_3, v_4) \equiv a_3 + a_4 h_1(v_1, v_2, v_3, v_4) + a_5 h_2(v_1, v_2, v_3, v_4) + \cdots = 0. \qquad (27)$$

Consider then the expression

$$\frac{f_3(v_1, v_2, v_3, v_4) - f_3(v_1, v_2, v_3, v_5)}{v_4 - v_5}.$$

Repeating reasoning, we obtain

$$f_4(v_1, v_2, v_3, v_4, v_5) \equiv a_4 + a_5 h_1(v_1, v_2, v_3, v_4, v_5) + a_6 h_2(v_1, v_2, v_3, v_4, v_5) + a_7 h_3(v_1, v_2, v_3, v_4, v_5) + \cdots = 0. \qquad (28)$$

Continuing calculations, we obtain

$$f_{k-1}(v_1, v_2, \ldots, v_k) \equiv a_{k-1} + a_k h_1(v_1, v_2, \ldots, v_k) + a_{k+1} h_2(v_1, v_2, \ldots, v_k) + \cdots = 0. \qquad (29)$$

Denote by Σ_2 the system of $(k-2)$ equations which includes the first Eq. of (26) and Eqs. (27)–(29).

The system Σ_2 is equivalent to the system Σ_1. In fact, from the equations Σ_2 one can get the equations Σ_1. Consider one of the equalities leading to Eq. (29),

$$f_{k-1}(v_1, v_2, \ldots, v_k) = \frac{f_{k-2}(v_1, v_2, \ldots, v_{k-1}) - f_{k-2}(v_1, v_2, \ldots, v_{\xi-1}, v_{\xi+1}, \ldots, v_{k-1}, v_k)}{v_\xi - v_k}.$$

If the functions of the form Σ_2 are given, then from the latter equality we find the functions $f_{k-2}(v_1, v_2, \ldots, v_{\xi-1}, v_{\xi+1}, \ldots, v_{k-1}, v_k)$. Then in terms of these functions we, in a similar way, express all the functions f_{k-3} depending on $(k-2)$ arguments, and so on until we find all the functions f_2.

Thus, the following theorem holds.

Theorem 8. *The equations Σ_2 are equations of a symmetric rectilinear k-web $SLW(k)$.*

Finally, let us present one more collection of equations of a symmetric k-web $SLW(k)$ (denote it by Σ_3): take the sum of all equations of the form $f_2(v_\alpha, v_\beta, v_\gamma) = 0$, the sum of all equations of the form $f_3(v_\alpha, v_\beta, v_\gamma, v_\delta) = 0$ and so on. One can show that

1) the left-hand sides of the equations obtained are expressed only in terms of the basis symmetric polynomials in all variables $v_1, v_2, \ldots, v_k$;

2) these equations are functionally independent.

12. A rectilinear k-web LW is called algebraic if it is formed by straight lines belonging to a curve of class k, a so-called set of straight lines $ax + by + cz = 0$ of the projective plane whose tangential coordinates a, b, c are connected by a homogeneous algebraic equation $S(a, b, c) = 0$ of degree k. Assume that the curve defined by the equation $S(a, b, c) = 0$ is unicursal (of genus 0), i.e., admits a rational parameterization: $a = a(u)$, $b = b(u)$, $c = c(u)$. Then without loss of generality we can assume that, in equations of family of lines (14), the coefficients $a(u), b(u), c(u)$ are polynomials of the same degree, say m.

Show that such a web LW is symmetric. Fix in the domain of LW a point (x, y, z), then the parameters of straight lines from family (14) passing trough it are the roots $u_1, u_2, \ldots, u_m$ of Eq. (14). By the Vieta theorem, the basic symmetric polynomials $\sigma_1, \sigma_2, \ldots, \sigma_m$ in the variables $u_1, u_2, \ldots, u_m$ are expressed in terms of the coefficients of Eq. (14), which are linear functions of (x, y, z). To get equations of a web LW, we need to eliminate from these equations the variables (x, y, z). Obviously, the equations obtained are symmetric with respect to the variables $u_1, u_2, \ldots, u_m$, i.e., the web LW in question is symmetric. In addition, one can show that, in this case, the algebraic equation $S(a, b, c) = 0$ defining LW also has degree m: eliminating from the equations $a = a(u)$, $b = b(u)$, $c = c(u)$ the parameter u, we arrive at a homogeneous equation of degree m.

We have proved the following theorem.

Theorem 9. *An algebraic k-web LW defined by an algebraic curve of genus zero is a symmetric web.*

In particular, let an algebraic k-web be given by Eq. (16), where $p(v)$ is a polynomial of degree k. Then the left-hand side of Eq. (16) is also a polynomial of degree k, and its coefficients of powers $v^2, \dots, v^k$ are constant. Therefore, the symmetric polynomials $\sigma_1, \sigma_2, \dots, \sigma_{k-2}$ in the roots $v_1, v_2, \dots, v_k$ of Eq. (16) are also constant, $\sigma_i = c_i, i = 1, 2, \dots, k-2$. These are equations of the corresponding k-web $SLW(k)$.

13. Consider some differential-geometric aspects of the theory of symmetric webs. Recall that all functions under consideration are real analytic.

One can easily prove the following proposition.

Proposition 4. *A real analytic function $f(x, y)$ is symmetric if and only if its first partial derivatives are connected by the relation*

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(y, x).$$

In addition, for a symmetric real analytic function, the following equalities hold:

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{\partial^2 f}{\partial y^2}(y, x),\ \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(y, x),\ \frac{\partial^3 f}{\partial x^2 \partial y}(x, y) = \frac{\partial^3 f}{\partial x \partial y^2}(y, x), \dots .$$

Corollary. In the expansion of $f(x, y)$ in Taylor series the coefficients of monomials $x^n y^m$ and $x^m y^n$ are equal. Thus, each of the polynomials of degree n of this expansion is a symmetric polynomial and, therefore, is expressed in terms of the basic symmetric polynomials $\sigma_1 = x + y$ and $\sigma_2 = xy$.

The basic relative differential invariants of a three-web are its curvature and covariant derivatives of the curvature with respect to the canonical torsion-free connection Γ (the Chern connection). Relative differential invariants of a k-web are relative differential invariants of its three-subwebs and anharmonic relations of quadruples of directions of the lines forming the web.

Let a three-web be given, as above, by the equation

$$u_3 = f(u_1, u_2).$$

We let, as in [3] (P. 64),

$$\omega_1 = f_1 du_1,\ \ \omega_2 = f_2 du_2,\ \ \omega = -\frac{f_{12}}{f_1 f_2}(\omega_1 + \omega_2).$$

Then the exterior differentials of the forms ω_1, ω_2, and ω have the form $d\omega_1 = \omega_1 \wedge \omega$, $d\omega_2 = \omega_2 \wedge \omega$, and $d\omega = b\omega_1 \wedge \omega_2$, where

$$b = -\frac{1}{f_1 f_2} \frac{\partial^2}{\partial u_1 \partial u_2} \ln \frac{f_1}{f_2}. \tag{30}$$

The quantity b is called the curvature of a three-web and is the basic relative invariant of a three-web.

As a consequence of Proposition 4 and formula (30), we conclude that for a semisymmetric three-web SSW defined by equation (14) the following relation holds: $b(u_1, u_2) = -b(u_2, u_1)$, i.e., the following theorem holds.

Theorem 10. *The curvature of a semisymmetric three-web is a skew-symmetric function of adapted parameters.*

From (30) it follows that

$$b(u_1, u_1) = 0. \tag{31}$$

Recall that, in Item 3, the lines $u_i = u_j$ were called equilibrium lines. Therefore, equality (31) means that *along an equilibrium line, the curvature of a semisymmetric three-web equals zero.*

In conclusion, we formulate some problems.

1. To classify symmetric three-webs by their equilibrium lines and by types of umbilic points (see Item 3).

2. To characterize the class of symmetric three-webs in terms of differential invariants. In particular, to clear up the question whether the statement converse to Theorem 10 is true.

3. To generalize the theory developed to the case of $(n+1)$-webs in the sense of V. V. Goldberg formed by $n+1$ hypersubmanifolds of an n-dimensional manifold.

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