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Special Version of Collocation Method for Integral Equations of the Third Kind With Fixed Singularities in a Kernel

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Abstract—We consider a linear integral equation of the third kind with fixed singularities in its kernel. We offer and prove a special version of collocation method for its approximate solving.

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We study linear integral equation of the third kind with fixed singularities in its kernel (E3KFS)

$$Ax \equiv x(t)\prod_{j=1}^{l}(t-t_j)^{m_j} + \int_{-1}^{1} K(t,s)[(s+1)^{p_1}(1-s)^{p_2}]^{-1}x(s)ds = y(t), \tag{1}$$

where $t \in I \equiv [-1,1]$, $t_j \in (-1,1)$, $m_j \in \mathbb{N}$ $(j=\overline{1,l})$; $p_1, p_2 \in \mathbb{R}^+$, K and y are known continuous functions with certain pointwise "smoothness" properties, $x(t)$ is the desired function, and integral is understood as the Hadamard finite part ([1], pp. 144–150). Equations (1) have extensive applications both in theory and in practice. A number of important problems of elasticity theory, transfer of neutrons, particle scattering (see [2, 3] and references in [3, 4]), and theory of differential equations of mixed type [5] reduces to that equations. As a rule, intrinsic classes of solutions of equations of the third kind are special spaces of distributions (SD) of type D or V. The space of type D (or V) is SD built on the base of the Dirak delta-function (correspondingly, on the base of the Hadamard finite part of integral). The equations under consideration can be solved explicitly only in rare cases. Therefore, the development of effective and theoretically based methods of their approximate solving in SD is an actual subject of mathematical analysis and computational mathematics. A number of results on this subject is obtained in works [6–9], where the direct special methods for solving of E3KFS (1) in a space of type D are proposed and substantiated. The first results on approximate solutions of E3KFS in certain SD X of type V are obtained in works [10, 11], where the authors develop polynomial for solving Eq. (1) in a space X.

In the present paper we use considerations and results of works [8–10], and propose a special version of the collocation method on the base of Hermite interpolation polynomials. This method is well-adapted for approximate solving Eq. (1) in class X. The main attention is paid to the substantiation of the method under consideration in the sense of book [12] (Chap. 1). Namely, we prove theorem on existence and uniqueness of solution to the corresponding equation, find bounds for errors of the approximate solution, and prove the convergence of sequence of approximate solutions to exact one in SD X. We consider also the questions of stability and conditionality of the approximating equations.

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1. Main spaces. Let $C \equiv C(I)$ be a space of continuous on I functions with customary max-norm and $m \in \mathbb{N}$. According to [13], we say that a function $f \in C$ belongs to class $C\{m;0\} \equiv C_0^{\{m\}}(I)$ if it has Taylor's derivative $f^{\{m\}}(0)$ of order m at the point $t=0$ (we put $C\{0;0\} \equiv C$). In the norm

$$\|f\|_{C\{m;0\}} \equiv \|Tf\|_C + \sum_{i=0}^{m-1} |f^{\{i\}}(0)|,$$

where

$$Tf \equiv \left[f(t) - \sum_{i=0}^{m-1} f^{\{i\}}(0) t^i / i! \right] \frac{1}{t^m} \equiv F(t) \in C \ \ (F(0) \equiv \lim_{t\to 0} F(t)),$$

the space $C\{m;0\}$ is complete and normally embedded into C (see, e.g., [14], P. 14).

Furthermore, let $p \in \mathbb{R}^+$ and $g \in C$. According to [13], we denote $g \in C\{p;1\} \equiv C_1^{\{p\}}(I)$ if there exist left Taylor derivatives $g^{\{j\}}(1)$ $(j = \overline{1,[p]})$ at the point $t=1$, and for $p \neq [p]$ ($[\cdot]$ stands for entire part) there exists the limit

$$\lim_{t\to 1-} \left\{ \left[g(t) - \sum_{j=0}^{[p]} g^{\{j\}}(1) \frac{(t-1)^j}{j!} \right] (1-t)^{-p} \right\}.$$

We equip vector space $C\{p;1\}$ by the norm

$$\|g\|_{\{p\}} \equiv \|g\|_{C\{p;1\}} \equiv \|Sg\|_C + \sum_{i=0}^{\lambda} |g^{\{i\}}(1)|, \tag{2}$$

where

$$Sg \equiv \left[g(t) - \sum_{i=0}^{\lambda} g^{\{i\}}(1) \frac{(t-1)^i}{i!} \right] (1-t)^{-p} \equiv G(t) \in C, \tag{3}$$

$\lambda = \lambda(p) \equiv [p] - (1 + \operatorname{sign}([p]-p))$, $G(1) \equiv \lim_{t\to 1-} G(t)$. Note that the space $C\{p;1\}$ consists of functions representable as

$$g(t) = (1-t)^p G(t) + \sum_{i=0}^{\lambda} b_i (t-1)^i, \tag{4}$$

where $G = Sg \in C$, $b_i = g^{\{i\}}(1)/i!$ $(i = \overline{0,\lambda})$. Obviously, $C\{p;1\}$ with norm (2) is complete, and embedded into C.

The following space is main for our studies:

$$Y \equiv C_{0;1}^{\{m\};\{p\}} \equiv C_{0;1}^{\{m\};\{p\}}(I) \equiv \{ y \in C\{m;0\} | Ty \in C\{p;1\} \}.$$

We equip it by the norm

$$\|y\|_Y \equiv \|Ty\|_{\{p\}} + \sum_{i=0}^{m-1} |y^{\{i\}}(0)|, \ \ y \in Y. \tag{5}$$

Lemma 1 ([6]). (i) *There is valid the relation*

$$\varphi \in Y \Leftrightarrow \varphi(t) = (UV\Phi)(t) + t^m \sum_{j=0}^{\lambda} d_j (t-1)^j + \sum_{i=0}^{m-1} e_i t^i, \tag{6}$$

where $\Phi = ST\varphi \in C$, $\varphi^{\{i\}}(0) = e_i i!$ $(i = \overline{0, m-1})$, $(T\varphi)^{\{j\}}(1) = d_j j!$ $(j = \overline{0,\lambda})$; $Uf = t^m f(t)$, $Vf \equiv (1-t)^p f(t)$;

(ii) *the space Y in norm* (5) *is complete and embedded into $C\{m;0\}$.*

Let $v \in C(I^2)$ and for arbitrarily fixed $s \in I$ the function $v(t,s)$ belongs to $C\{p;1\}$. We say that $v \in C_t^{\{p\}}(I^2)$, if $S_t v \in C$, where S_t is operator (3) applied in argument t. Definition of class $C_s^{\{p\}}(I^2)$ is analogous. Then

$$C_1^{\{p\}}(I^2) \equiv C_t^{\{p\}}(I^2) \cap C_s^{\{p\}}(I^2).$$

On the main space Y we consider the family $X \equiv V^{\{p\}}\{m;0\}$ of distributions $x(t)$ of the form

$$x(t) \equiv z(t) + \sum_{i=0}^{m-1} \gamma_i \,\mathrm{P.\,F.}\, t^{-i-1},$$

where $t \in I$, $z \in C\{p;1\}$, $\gamma_i \in \mathbb{R}$ are arbitrary constants, and P. F. t^{-k} are distributions defined on space Y by the rule

$$(\mathrm{P.\,F.}\, t^{-k}, y) \equiv \mathrm{P.\,F.} \int_{-1}^{1} y(t) t^{-k} dt, \quad y \in Y, \ \ k = \overline{1,m}.$$

The symbol P. F. means here the Hadamard's finite part of integral ([1], pp. 144–150) (in what follows we omit this symbol for brevity). Clearly, the vector space X is the Banach space with regard to the norm

$$\|x\|_X \equiv \|z\|_{\{p\}} + \sum_{i=0}^{m-1} |\gamma_i|.$$

2. An approximating "polynomial" operator. Denote by $\Pi^{ST}_{4n+m+\lambda} \equiv UV(\Pi_{4n-1}) \oplus \Pi_{m+\lambda}$ $(4n+m+\lambda+1)$-dimensional subspace of space Y, where $\Pi_l \equiv \mathrm{span}\{t^i\}_0^l$ is the class of all algebraic polynomials of degree no higher than l. Let $\Gamma_n \equiv \Gamma_{4n+m+\lambda} : Y \to \Pi^{ST}_{4n+m+\lambda}$ be operator mapping any function $g \in Y$ on element of $\Gamma_n g$, uniquely determined by the conditions

$$\begin{gathered}(ST\Gamma_n g - STg)(\nu_j) = 0, \quad (ST\Gamma_n g)^{(k)}(\nu_j) = 0, \quad k = \overline{1,3}, \ \ j = \overline{1,n}, \\ (T\Gamma_n g - Tg)^{\{j\}}(1) = 0, \quad j = \overline{0,\lambda}, \quad (\Gamma_n g - g)^{\{i\}}(0) = 0, \quad i = \overline{0, m-1},\end{gathered} \tag{7}$$

where $\{\nu_j\}_1^n$ is the system of Chebyshev nodes of the first kind.

Lemma 2. *The operator Γ_n acts by the rule*

$$\Gamma_n g \equiv \Gamma_{4n+m+\lambda}(g;t) = (UV\Phi_n STg)(t) + \sum_{j=0}^{\lambda} (Tg)^{\{j\}}(1) \frac{t^m (t-1)^j}{j!} + \sum_{i=0}^{m-1} g^{\{i\}}(0) \frac{t^i}{i!}, \tag{8}$$

where $\Phi_n \equiv \Phi_{4n-1} : C \to \Pi_{4n-1}$ is interpolation operator, which maps any function $f \in C$ on polynomial $\Phi_n f \in \Pi_{4n-1}$, uniquely determined by $4n$ equalities

$$(\Phi_n f)(\nu_j) = f(\nu_j), \quad (\Phi_n f)^{(k)}(\nu_j) = 0, \quad k = \overline{1,3}, \ \ j = \overline{1,n}. \tag{9}$$

Proof. The desired element $\Gamma_n g \in \Pi^{ST}_{4n+m+\lambda}$ is representable as

$$\begin{aligned}\Gamma_n g \equiv t^m (1-t)^p \sum_{i=0}^{4n-1} \alpha_i t^i + \sum_{i=0}^{m+\lambda} \beta_i t^i &= \sum_{i=0}^{m-1} \beta_i t^i + t^m \left[(1-t)^p \sum_{i=0}^{4n-1} \alpha_i t^i + \sum_{j=0}^{\lambda} \beta_{m+j} t^j \right] \\ &= \sum_{i=0}^{m-1} \beta_i t^i + t^m \left[(1-t)^p \sum_{i=0}^{4n-1} \alpha_i t^i + \sum_{j=0}^{\lambda} \rho_j (t-1)^j \right] \equiv \sum_{i=0}^{m-1} \beta_i t^i + t^m P(t).\end{aligned} \tag{10}$$

According to (4), we have

$$(1-t)^p \sum_{i=0}^{4n-1} \alpha_i t^i = (VSP)(t), \quad \rho_j = \frac{1}{j!} P^{\{j\}}(1), \quad j = \overline{0, \lambda}. \tag{11}$$

By means of representation (10) and known structure of elements of space $C\{m;0\}$ (see, e.g., [14], P. 12) we obtain

$$P(t) = (T\Gamma_n g)(t), \quad \beta_i = \frac{1}{i!} (\Gamma_n g)^{\{i\}}(0), \quad i = \overline{0, m-1}, \tag{12}$$

and by virtue of equalities (9) and (7)

$$ST\Gamma_n g = \Phi_n STg. \tag{13}$$

Then representation (10) by means of equalities (11)–(13) and (7) implies relation (8). □

The following lemma describes an approximative properties of operator Γ_n in the space Y.

Lemma 3. *Any function $y \in Y$ satisfies the estimate*

$$\|y - \Gamma_n y\|_Y \leqslant d_1 \omega(STy; \Delta_n); \tag{14}$$

here and below d_i, $i = \overline{1, 2}$, are certain constants independent on natural parameter n, $\omega(f; \Delta_n)$ is continuity module of the function $f \in C$ with step $\Delta_n \equiv (\ln n)/n$, $n = 2, 3, \ldots$

Lemma 3 follows immediately from relations (6), (8), (5), (2) and bound [15]

$$\|f - \Phi_n f\|_C \leqslant d_1 \omega(f; \Delta_n), \quad f \in C. \tag{15}$$

Let us introduce the set

$$R_{4n+m+\lambda} \equiv \left\{ y_n \in \Pi^{ST}_{4n+m+\lambda} \mid (STy_n)^{(k)}(\nu_j) = 0, \quad k = \overline{1, 3}, \ j = \overline{1, n} \right\}.$$

Lemma 4. *Operator $\Gamma_n : Y \to R_{4n+m+\lambda}$ is a projection.*

Proof. Let $y_n \in R_{4n+m+\lambda}$ be an arbitrary element, and $H_{ji}(t)$, $i = \overline{0, 3}$, $j = \overline{1, n}$, be fundamental Hermite polynomials of degree $4n-1$ on nodes $\{\nu_j\}_1^n$ (see, e.g., [16], P. 63). Then, obviously,

$$(STy_n)(t) = \sum_{j=1}^{n} \sum_{i=0}^{3} (STy_n)^{(i)}(\nu_j) H_{ji}(t) = \sum_{j=1}^{n} (STy_n)(\nu_j) H_{j0}(t) \equiv (\Phi_n STy_n)(t). \tag{16}$$

From relations (8), (16) and (6) we obtain $\Gamma_n y_n = y_n$. □

3. Collocation method using Hermite interpolation polynomials. Let E3KFS (1) be given. For simplification of calculations and formulations we put $l = 1$, $t_1 = 0$, $p_1 = 0$, i.e., we consider the equation

$$Ax \equiv (Ux)(t) + (Kx)(t) = y(t), \ \ t \in I, \tag{17}$$

$$Kx \equiv \int_{-1}^{1} K(t,s)(1-s)^{-p} x(s) ds,$$

where $m \in \mathbb{N}$, $p \in \mathbb{R}^+$; $y \in Y$, K is a given function satisfying the conditions

$$K \in C_s^{\{p\}}(I^2), \psi_i(t) \equiv K_s^{\{i\}}(t, 1) \in Y, \ \ i = \overline{0, \lambda},$$

$$u \equiv S_s K \in C_t^{\{m\}}(I^2), \ \ \theta_i(s) \equiv u_t^{\{i\}}(0, s) \in C\{m; 0\}, \ \ i = \overline{0, m-1}; \tag{18}$$

$$v \equiv T_t u \in C_t^{\{p\}}(I^2), \ \ \varphi_i(s) \equiv v_t^{\{i\}}(1, s) \in C\{m; 0\}, \ \ i = \overline{0, \lambda},$$

$$h \equiv S_t v \in C_s^{\{m\}}(I^2),$$

and $x \in X$ is the desired distribution. The Fredholm properties and sufficient conditions for continuous reversibility of the operator $A : X \to Y$ are proved in [10]; a method for evaluation of the exact solution of E3KFS (1) in the class X is described in the same paper.

An approximate solution of Eq. (17) is the element

$$x_n \equiv x_n(t; \{c_i\}) \equiv g_n(t) + \sum_{i=0}^{m-1} c_{i+\lambda+4n+1} t^{-i-1}, \tag{19}$$

$$g_n(t) \equiv (Vz_n)(t) + \sum_{i=0}^{\lambda} c_{i+4n}(t-1)^i, \quad z_n(t) \equiv \sum_{i=0}^{4n-1} c_i t^i, \quad n = 2, 3, \dots\ . \tag{20}$$

We find the unknown parameters $c_i = c_i^{(n)}$, $i = \overline{0, m+\lambda+4n}$, from the system of linear algebraic equations (SLAE)

$$(STAx_n - STy)(\nu_j) = 0, \quad (STUx_n)^{(k)}(\nu_j) = 0 \quad (k = \overline{1,3},\ \ j = \overline{1,n}),$$

$$(TAx_n - Ty)^{\{j\}}(1) = 0, \quad j = \overline{0, \lambda}, \tag{21}$$

$$(Ax_n - y)^{\{i\}}(0) = 0, \quad i = \overline{0, m-1},$$

where $\{\nu_j\}_1^n$ is the used above system of nodes.

The following theorem describes properties of the computational algorithm (17)–(21).

Theorem 1. *If homogeneous E3KFS $Ax = 0$ has only null solution in X (for instance, under assumptions of theorem 2 from [10]), then for any $n \in \mathbb{N}$, $n \geq n_0$, SLAE (21) has a unique solution $\{c_j^*\}$, and the sequence of approximate solutions $x_n^* \equiv x_n(t, \{c_j^*\})$ converges to the exact solution $x^* = A^{-1}y$ in norm of space X with the rate*

$$\|x_n^* - x^*\| = O\Big\{\omega_t(h; \Delta_n) + \sum_{j=0}^{\lambda} \omega(\alpha_j; \Delta_n) + \sum_{i=0}^{m-1} \omega(\beta_i; \Delta_n) + \omega(STy; \Delta_n)\Big\}, \tag{22}$$

where $\omega_t(h; \Delta)$ is partial module of continuity of the function $h(t,s)$ in variable t, $h \equiv S_t v$, $\alpha_j \equiv ST\psi_j$, $j = \overline{0, \lambda}$, $\beta_i \equiv ST\Phi_i$, $i = \overline{0, m-1}$, and

$$\Phi_i(t) \equiv \int_{-1}^{1} K(t,s)(1-s)^{-p} s^{-i-1} ds \in Y, \ \ i = \overline{0, m-1}.$$

Proof. We consider E3KFS (17) as linear operator equation of the form

$$Ax \equiv Ux + Kx = y, \quad x \in X \equiv V^{\{p\}}\{m; 0\}, \ \ y \in Y \equiv C_{0;1}^{\{m\};\{p\}}, \tag{23}$$

where operator $A : X \to Y$ is continuously reversible.

Let $X_n \subset X$ stand for $(4n + m + \lambda + 1)$-dimensional subspace of elements (19) such that $(STUx_n)^{(k)}(\nu_j) = 0$, $k = \overline{1,3}$, $j = \overline{1,n}$, and $Y_n \subset Y$ stand for the class $R_{4n+m+\lambda}$.

We show first that system (19)–(21) is equivalent to linear functional equation

$$A_n x_n \equiv \Gamma_n A x_n = \Gamma_n y, \quad x_n \in X_n, \ \ \Gamma_n y \in Y_n. \tag{24}$$

Let $x_n^* \equiv x_n(t; \{c_i^*\}) \in X$ be solution to Eq. (24), i.e., $\Gamma_n(Ax_n^* - y) = 0$, what by virtue of (8) and (6) means validity of relations

$$(\Phi_n(STAx_n^* - STy))(t) \equiv 0,$$

$$(TAx_n^* - Ty)^{\{j\}}(1) = 0, \quad j = \overline{0, \lambda}, \tag{25}$$

$$(Ax_n^* - y)^{\{i\}}(0) = 0, \quad i = \overline{0, m-1}.$$

Equality (9) implies that the first identity is equivalent to the system

$$(STAx_n^* - STy)(\nu_j) = 0,\ (\Phi_n STAx_n^* - \Phi_n STy)^{(k)}(\nu_j) = 0, \quad k = \overline{1,3},\ \ j = \overline{1,n}. \qquad (26)$$

Then by means of (17), (19) and (20) we obtain the relations

$$STAx_n^* = z_n^* + STKx_n^*,\ \ z_n^* = STUx_n^* \in R_{4n-1},\ \ \Phi_n z_n^* = z_n^*.$$

Consequently, the conformity of meanings of derivatives at the nodes in system (26) means that $(z_n^*)^{(k)}(\nu_j) = (STUx_n^*)^{(k)}(\nu_j) = 0$, $k = \overline{1,3}$, $j = \overline{1,n}$. It follows from these relations and from (25) and (26) that SLAE (21) has the solution $c_i = c_i^*$, $i = \overline{0, m+\lambda+4n}$, i.e., the solution to Eq. (24) is solution to system (19)–(21). The converse is obvious.

It remains to prove existence, uniqueness, and convergence of solutions to Eqs. (24).

Let us research now the question on closeness of the operators A and A_n on subspace X_n. We apply formulas (23), (19), (24), projection property of the operator $\Gamma_n : Y \to Y_n$, (6), relations (8), (5) and (2), and sequentially find for arbitrary $x_n \in X_n$

$$\|Ax_n - A_n x_n\|_Y = \|Kx_n - \Gamma_n Kx_n\|_Y = \|STKx_n - \Phi_n STKx_n\|_C. \qquad (27)$$

As is known [10],

$$STKx_n = \int_{-1}^{1} h(t,s) g_n(s)ds + \sum_{j=0}^{\lambda} \lambda_j(g_n)\alpha_j(t) + \sum_{i=0}^{m-1} c_{i+\lambda+4n+1}\beta_i(t), \qquad (28)$$

where

$$\lambda_j(g) \equiv \int_{-1}^{1} (Sg)(s)(s-1)^j \frac{1}{j!}ds + \sum_{k=0}^{\lambda} g^{\{k\}}(1)\beta_{jk},\ \beta_{jk} \equiv \int_{-1}^{1} (s-1)^{j+k}\frac{1}{j!k!}(1-s)^{-p}ds,\ \ j,k = \overline{0,\lambda}.$$

By virtue of relations (28) and (15) we sequentially obtain

$$\begin{aligned}
&\|STKx_n - \Phi_n STKx_n\|_C \\
&= \max_{t\in I}\left| \int_{-1}^{1} (h - \Phi_n^t h)(t,s) g_n(s)ds + \sum_j \lambda_j(g_n)(\alpha_j - \Phi_n\alpha_j)(t) + \sum_i c_{i+\lambda+4n+1}(\beta_i - \Phi_n\beta_i)(t)\right| \\
&\leqslant 2d_1\|g_n\|_C\omega_t(h;\Delta_n) + d_1\sum_j |\lambda_j(g_n)|\omega(\alpha_j;\Delta_n) + d_1\sum_i |c_{i+\lambda+4n+1}|\omega(\beta_i;\Delta_n) \\
&\leqslant 2^{p+1}d_1\|g_n\|_{\{p\}}\omega_t(h;\Delta_n) + d_1\|g_n\|_{\{p\}}(2^{\lambda+1}+\beta)\sum_j \omega(\alpha_j;\Delta_n) + d_1\|x_n\|_X\sum_i \omega(\beta_i;\Delta_n) \\
&\leqslant d_1\|x_n\|\left\{2^{p+1}\omega_t(h;\Delta_n) + (2^{p+1}+\beta)\sum_j \omega(\alpha_j;\Delta_n) + \sum_i \omega(\beta_i;\Delta_n)\right\} \\
&\leqslant d_2\left\{\omega_t(h;\Delta_n) + \sum_j \omega(\alpha_j;\Delta_n) + \sum_i \omega(\beta_i;\Delta_n)\right\}\|x_n\|. \qquad (29)
\end{aligned}$$

We use here notation $\beta \equiv \max_{0\leqslant j,k\leqslant\lambda} |\beta_{jk}|$, $d_2 \equiv d_1(2^{p+1}+\beta)$. Consequently, relations (27) and (29) yield the bounds

$$\varepsilon_n \equiv \|A - A_n\|_{X_n\to Y} \leqslant d_2\left\{\omega_t(h;\Delta_n) + \sum_j \omega(\alpha_j;\Delta_n) + \sum_i \omega(\beta_i;\Delta_n)\right\}. \qquad (30)$$

Then inequalities (14) and (30) allow us to conclude by means of theorem 7 from [12] (P. 19) that the theorem under consideration is valid with estimate (22).

Corollary 1. If h (invariable t), α_j, β_i, $STy \in C^{(1)}$, then under assumptions of Theorem 1 we have $\|x_n^* - x^*\| = O(\Delta_n)$.

Under assumptions of Theorem 1 the corresponding approximating operators A_n $(n \geq n_1)$ are continuously reversible, and the inverse operators are bounded in norm in the aggregate: $\|A_n^{-1}\| = O(1)$ $(A_n^{-1} : Y_n \to X_n)$. Consequently, by virtue of the results of [12] (pp. 23–24) there is valid

Theorem 2. *Under assumptions of Theorem* 1 *there are valid assertions*:

i) *the direct method* (19)–(21) *for E3KFS* (17) *is stable with respect to small perturbations of elements of SLAE* (21);

ii) *if E3KFS* (17) *is well-conditioned, then SLAE* (21) *is well-conditioned, too.*

Remark 1. By definition of the norm in $X \equiv V^{\{p\}}\{m; 0\}$ the convergence of sequence (x_n^*) to $x^* = A^{-1}y$ in metrics of X implies customary convergence in the space of of generalized functions, i.e., weak convergence.

Remark 2. Approximation of solution to operator equations $Ax = y$ implies intrinsic question on the rate of convergence of discrepancy $\rho_n^A(t) \equiv (Ax_n^* - y)(t)$ of the method under consideration. One of results of this kind follows easily from the main Theorem 1; namely, it yields the following simple

Corollary 2. If initial data $(h, \alpha_j, \beta_i, STy)$ of Eq. (17) belong to class $C^{(1)}$, then under assumptions of Theorem 1 there is valid the bound $\|\rho_n^A\|_Y = O(\Delta_n)$.

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