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Foundations of the Theory of Dual Lie Algebras

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Abstract—In this paper we introduce and study dual Lie algebras, i.e., Lie algebras over the algebra of dual numbers. We establish some fundamental properties of such Lie algebras and compare them with the corresponding properties of real and complex Lie algebras. We discuss the question of classification of dual Lie algebras of small dimension and consider the connection of dual Lie algebras with approximate Lie algebras.

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INTRODUCTION

In this paper, we study Lie algebras defined over the algebra of dual numbers. Some reasons for the consideration of this class of Lie algebras are given in Appendix, where we discuss their applications to the study of approximate Lie algebras of symmetries of various equations (including differential equations). Note that till now dual Lie algebras were not studied in detail though they appeared occasionally in research papers.

Consider two-dimensional algebra $\mathbf{D}$ of dual numbers, that is numbers of the form $a + \varepsilon b$ over $\mathbf{R}$ with generators 1 and ε, where $\varepsilon^2 = 0$. The algebra $\mathbf{D}$ (commutative, associative, with unity) has exact matrix representation with matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$, where $a, b \in \mathbf{R}$. In contrast to the field $\mathbf{R}$, whose automorphism group is trivial, and the field $\mathbf{C}$ of complex numbers, whose automorphism group is isomorphic to $\mathbf{Z}_2$ (the unique nontrivial automorphism is the complex conjugation), the algebra $\mathbf{D}$ has continuous one-dimensional automorphism group consisting of transformations of the form $a + \varepsilon b \to a + \varepsilon\alpha b$ for arbitrary nonzero $\alpha \in \mathbf{R}$.

The main object of our study are arbitrary Lie $\mathbf{D}$-algebras. These Lie algebras (finite-dimensional Lie algebras over $\mathbf{R}$ or $\mathbf{C}$) are in addition $\mathbf{D}$-modules, i.e., admit multiplication by dual numbers compatible with the Lie algebra structure. The $\mathbf{D}$-structure on such an algebra is given by the linear operator of the multiplication by ε, which is denoted by $\mathcal{E}$. It is also natural to study respective Lie subalgebras, ideals, homomorphisms, etc. Note that a Lie algebra can be a trivial $\mathbf{D}$-module, i.e., the corresponding operator $\mathcal{E}$ can be zero. Thus, usual Lie algebras with $\mathcal{E} = 0$ can be viewed as Lie $\mathbf{D}$-algebras. However, in what follows we will be interested mainly in the cases when $\mathcal{E} \neq 0$. What is more, it is useful sometimes to consider only "nondegenerate" $\mathbf{D}$-structures, when the rank of $\mathcal{E}$ is one-half of the dimension of an even-dimensional Lie algebra. This takes place if and only if a Lie algebra L is a free $\mathbf{D}$-module. However, unfortunately, the class of such nondegenerate Lie algebras is not closed with respect to passages to subalgebras, ideals, or quotient algebras. In the general case, after such a passage, the linear operator $\mathcal{E}$ may become degenerate, in particular, zero. Then we have a trivial $\mathbf{D}$-module, and the obtained Lie $\mathbf{D}$-algebra is a Lie algebra without additional $\mathbf{D}$-structure. In Section 1, we study in detail some basic properties of arbitrary Lie $\mathbf{D}$-algebras. Section 2 is devoted to nondegenerate Lie $\mathbf{D}$-algebras. In Section 3, we consider some questions concerning classification of Lie $\mathbf{D}$-algebras.

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Since the algebra $\mathbf{D}$ is not a field (there is no division by numbers of the form εb), we cannot use the usual notion of a basis for its modules. By an essential basis (or a $\mathbf{D}$-basis) for a Lie $\mathbf{D}$-subalgebra we will understand a minimal collection of vectors generating this subalgebra as a $\mathbf{D}$-module. In other words, it is a minimal collection of elements of a Lie algebra which together with the results of their multiplications by ε generate the Lie algebra as a vector space over $\mathbf{R}$. Such bases were introduced in [1] (P. 41) under the name of essential parameters. The number of elements in such a basis (the number of essential parameters) is an analog of the dimension of a Lie algebra L over $\mathbf{D}$. One can use the term "$\mathbf{D}$-dimension" and denote this number by $d(L)$. For example, a Lie $\mathbf{D}$-subalgebra in a Lie algebra of the form $L \otimes \mathbf{D}$ generated by an element of the form εX, where $X \in L$, is one-dimensional over $\mathbf{R}$, and that generated by X is two-dimensional (here $X = X + \varepsilon \cdot 0$).

If a vector space V is a finite-dimensional free $\mathbf{D}$-module, then V is isomorphic to $\mathbf{D}^m$ for some natural number m. In this case, it is natural to consider $\mathbf{D}$-bases in V (systems of free generators of the module), and linear operators can be given by their matrices with elements belonging to $\mathbf{D}$.

The notion of characteristic polynomial is poorly adapted to the use in the $\mathbf{D}$-situation because there is no basis in the usual meaning of this word, therefore, it is difficult to use matrices. As for polynomials and their roots, we have many surprises. A polynomial can have infinitely many roots. So, any dual number of the form $a + \varepsilon b$, where a is a fixed real number and b is an arbitrary real number, is a root of the polynomial $(z - a)^2$.

In this paper, we mainly study real Lie algebras, though similar considerations can be performed for complex Lie $\mathbf{D}$-algebras. We restrict ourselves to the study only of finite-dimensional Lie algebras, though many of the assertions given below hold without this restriction.

1. SOME PROPERTIES OF LIE D-ALGEBRAS

Consider basic properties of finite-dimensional Lie $\mathbf{D}$-algebras, i.e., real Lie algebras which are $\mathbf{D}$-modules. Recall that the algebra $\mathbf{D}$ of dual numbers can be viewed as the result of the Cayley–Dickson doubling process applied to the algebra of real numbers. Therefore, the corresponding Lie algebras can be called dual Lie algebras. We will mainly use this term, thou sometimes we will call them Lie $\mathbf{D}$-algebras or Lie algebras over $\mathbf{D}$.

There are two ways to develop the theory of dual Lie algebras. The first way is a direct one. Since dual Lie algebras till now were not studied as a special object, one can develop their general theory step-by-step following the development of the theory of usual Lie algebras but without essential use of the latter. This way leads to rather large exposition containing many tens of pages with detailed proofs. The second way is much more short. In this way, the theory of dual Lie algebras is constructed on the basis of the well-known theory of "usual" Lie algebras. This way requires far less intermediate results, and the proofs are considerably shorter, which does not depreciate their significance. In this paper, we choose the second way.

A structure of Lie $\mathbf{D}$-algebra on a real Lie algebra L is given by a linear operator $\mathcal{E}$ which will be considered as a multiplication operator by the element $\varepsilon \in \mathbf{D}$ and which has the following two properties:

1. $[\mathcal{E}X, Y] = [X, \mathcal{E}Y] = \mathcal{E}[X, Y]$.

In other words, the operator of $\mathbf{D}$-structure commutes with the operators of the adjoint representation of the Lie algebra L.

2. $\mathcal{E}^2 = 0$.

In terms of a suitable Jordan basis, the matrix of such linear operator $\mathcal{E}$ is the direct sum $\oplus J_2(0) \oplus 0$ of several Jordan blocks $J_2(0)$ of order 2 corresponding to zero eigenvalue and a zero square matrix denoted above as 0, which may be absent in the decomposition. The number of Jordan blocks $J_2(0)$ equals the rank of the matrix of $\mathcal{E}$ and the dimension of the image of $\mathcal{E}$.

Usual Lie algebras can be viewed as trivial examples of Lie $\mathbf{D}$-algebras whose operator $\mathcal{E}$ is zero.

The simplest nontrivial examples of Lie $\mathbf{D}$-algebras are Lie algebras of the form $L \otimes \mathbf{D}$, the tensor products of real Lie algebras and the algebra $\mathbf{D}$. The commutator is defined in a natural way: $[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes (\alpha\beta)$. Lie $\mathbf{D}$-algebras obtained by means of this construction are even-dimensional viewed as real Lie algebras. However, there are odd-dimensional Lie $\mathbf{D}$-algebras, for example, any one-dimensional Lie subalgebra in a Lie $\mathbf{D}$-algebra spanned by an element of the form εX. The above indicated Lie algebras $L \otimes \mathbf{D}$ are obviously free $\mathbf{D}$-modules. The corresponding operator $\mathcal{E}$ has maximal possible rank which equals the dimension of the initial Lie algebra L.

Clearly, odd-dimensional Lie algebras cannot be obtained with the use of the operation of tensor product, which can be called the operation of dualization of the initial Lie algebra. However, there are even-dimensional Lie $\mathbf{D}$-algebras which cannot be obtained as a result of dualization of a Lie algebra. As an example, one can take the Lie algebra $\mathbf{R}^2$ with trivial $\mathbf{D}$-structure. There are also examples of such algebras with nontrivial $\mathbf{D}$-structure. The simplest example of minimal possible dimension is the four-dimensional Lie algebra L_4 with basis $X, Y, \varepsilon X, \varepsilon Y$ and defining relation $[X, Y] = \varepsilon X$. The other nontrivial brackets are zero. One can easily see that, viewed as a real Lie algebra, the algebra L_4 is isomorphic to the Lie algebra $N(3, \mathbf{R}) \oplus \mathbf{R}$, where $N(3, \mathbf{R})$ is a three-dimensional nilpotent Lie algebra which, in terms of an appropriate basis U, V, W, is defined by the relation $[U, V] = W$. Note that the $\mathbf{D}$-structure on the Lie algebra in question is degenerate (the rank of $\mathcal{E}$ equals 1). Since Lie algebras of the form $L \otimes \mathbf{D}$ always have nondegenerate $\mathbf{D}$-structures, the Lie algebra L_4 in question cannot be obtained by dualization. In the case of Lie $\mathbf{D}$-algebras with nondegenerate $\mathbf{D}$-structure, examples of such kind can also be constructed, but they are more complicated.

For an arbitrary Lie $\mathbf{D}$-algebra L, we denote by $L_{\mathbf{R}}$ the realification of L, i.e., the Lie algebra L viewed over $\mathbf{R}$ when its $\mathbf{D}$-structure is ignored. As will be shown below, it is convenient to prove properties of Lie $\mathbf{D}$-algebras applying results known for real Lie algebras to Lie algebras $L_{\mathbf{R}}$. However, there are results which are specific to Lie $\mathbf{D}$-algebras.

Let L be a finite-dimensional Lie $\mathbf{D}$-algebra. Consider the linear subspace $P = \mathrm{Im}(\mathcal{E})$, the image of the operator $\mathcal{E}$, and denote by Q its kernel $\mathrm{Ker}(\mathcal{E})$. The condition $\mathcal{E}^2 = 0$ is equivalent to the relation $P \subseteq Q$.

Obviously, the subspace P is zero if and only if $\mathcal{E}$ is the zero operator. Note that the number of essential parameters $d(L)$ for a Lie $\mathbf{D}$-algebra (mentioned above) equals the real dimension of the vector space L/P. Now we pass to the detailed study of properties of Lie $\mathbf{D}$-algebras.

Proposition 1. *Let L be a Lie $\mathbf{D}$-algebra, and let U be an arbitrary linear subspace in L. Then $\mathcal{E}(U)$ is an abelian subalgebra in L.*

Proof. We have $[\mathcal{E}(U), \mathcal{E}(U)] = \mathcal{E}^2([U, U]) = \{0\}$. □

For an arbitrary linear subspace $U \subseteq L$, we let $\widehat{U} = U + \mathcal{E}U$. As a matter of fact, $\widehat{U}$ is $\mathbf{D} \cdot U$, i.e., the minimal $\mathbf{D}$-subspace containing U or the $\mathbf{D}$-saturation of U.

Proposition 2. *Let U be an ideal in a Lie $\mathbf{D}$-algebra. Then*

i) *$\mathcal{E}(U)$ is an ideal in L,*

ii) *$\widehat{U}$ is an ideal in L (and $[\widehat{U}, L] \subseteq U$).*

Proof. i) We have $[\mathcal{E}(U), L] = \mathcal{E}([U, L])$. But U is an ideal in L, therefore $[U, L] \subset U$, whence we have $[\mathcal{E}(U), L] \subset \mathcal{E}(U)$, i.e., $\mathcal{E}(U)$ is an ideal in L.

ii) We have $[\widehat{U}, L] = [U + \mathcal{E}U, L] \subseteq [U, L] + [\mathcal{E}U, L]$. But $[U, L] \subseteq U$ because U is an ideal, and $[\mathcal{E}U, L] = [U, \mathcal{E}L] \subseteq U$. We obtain $[\widehat{U}, L] \subseteq \widehat{U}$, i.e., $\widehat{U}$ is an ideal in L. In addition, $[\widehat{U}, L] \subseteq U$. □

In particular, from Propositions 1 and 2 it follows that the above introduced subspace P is an abelian ideal in L. This statement, as well as Corollary 1 stated below, was first mentioned in [2]. In [2], the authors use a notation for the decomposition $L = L_0 \oplus L_1$ into direct sum which is nonstandard for the theory of Lie algebras, though in fact they implicitly mean a disjoint union of two subsets one of which, L_0, is not a subspace.

Corollary 1. If L is a Lie $\mathbf{D}$-algebra and $\mathcal{E} \neq 0$, then L considered as a Lie algebra over $\mathbf{R}$ cannot be semisimple.

Proof. If $\mathcal{E} \neq 0$, the subspace P is a nontrivial abelian ideal, which implies that the Lie algebra L cannot be semisimple. □

In addition to statement ii) of Proposition 2, the following proposition holds.

Proposition 3. *Let U be a linear subspace in a Lie $\mathbf{D}$-algebra L. Then $\widehat{U}$ is a Lie $\mathbf{D}$-subalgebra in L.*

Proof. We have $[\widehat{U}, \widehat{U}] = [U + \mathcal{E}U, U + \mathcal{E}U] \subseteq [U, U] + [\mathcal{E}U, U] + [\mathcal{E}U, \mathcal{E}U]$. But $[\mathcal{E}U, U] = \mathcal{E}[U, U] \subseteq \widehat{U}$ and $[\mathcal{E}U, \mathcal{E}U] = \mathcal{E}^2[U, U] = 0$. Therefore, $[\widehat{U}, \widehat{U}] \subseteq \widehat{U}$, i.e., $\widehat{U}$ is a Lie subalgebra. □

Consider briefly the above introduced subspace $Q = \operatorname{Ker} \mathcal{E}$.

Proposition 4. *For an arbitrary Lie $\mathbf{D}$-algebra L, the subspace $Q = \operatorname{Ker} \mathcal{E}$ is an ideal in L, generally speaking, a nonabelian one. The center $Z(Q)$ contains the ideal P.*

Proof. Let $X \in L$, $Y \in Q$. By conditions, $\mathcal{E}Y = 0$. Then $\mathcal{E}[X, Y] = [X, \mathcal{E}Y] = 0$, i.e., $[X, Y] \in Q$. Thus, Q is an ideal. If $\mathcal{E} = 0$, then the ideal Q coincides with the initial Lie algebra L, and it is nonabelian if L is nonabelian.

Now, let us show that $P \subseteq Z(Q)$. Let $X \in P$, $Y \in Q$. Then $X = \mathcal{E}Z$ for some $Z \in L$. Therefore, $[X, Y] = [\mathcal{E}Z, Y] = [Z, \mathcal{E}Y] = 0$, which means that any element of P is central in Q. □

The notions of solvable and nilpotent Lie algebras are defined for the class of Lie $\mathbf{D}$-algebras in a natural way by means the of sequences of the corresponding commutants. On can easily see that all members of central series (upper and lower) and of the series of commutants for a Lie $\mathbf{D}$-algebra are Lie $\mathbf{D}$-subalgebras. We show this for the center $Z(L)$: let $X \in Z(L)$, then, for any element $Y \in L$, we have $[\mathcal{E}X, Y] = \mathcal{E}[X, Y] = 0$. Therefore, $\mathcal{E}X \in Z(L)$.

The following, more general, proposition is proved in a similar way.

Proposition 5. *Let L be a Lie $\mathbf{D}$-algebra, and let U be a Lie $\mathbf{D}$-subalgebra in L. Then the centralizer $Z_L(U)$ and the normalizer $N_L(U)$ of U are Lie $\mathbf{D}$-subalgebras in L.*

Further, one can introduce the notion of solvable radical as the greatest solvable $\mathbf{D}$-ideal. It turns out that the $\mathbf{D}$-radical in a Lie $\mathbf{D}$-algebra coincides with the usual radical in $L_{\mathbf{R}}$. A similar situation takes place in the case of the nilradical, the greatest nilpotent ideal of $L_{\mathbf{R}}$. The following two lemmas demonstrate some basic $\mathbf{D}$-properties of these ideals.

Lemma 1. *Let L be a Lie $\mathbf{D}$-algebra, and let R be its radical, the greatest solvable ideal in $L_{\mathbf{R}}$. Then R is the $\mathbf{D}$-radical, the greatest solvable $\mathbf{D}$-ideal in L. If N is the nilradical in $L_{\mathbf{R}}$, then it is also the $\mathbf{D}$-nilradical.*

Proof. Consider the Lie subalgebra $\widehat{R} = R + \mathcal{E}R$. The ideal R is a $\mathbf{D}$-ideal if and only if $\widehat{R} = R$. But, by Proposition 1, $\mathcal{E}(R)$ is contained in the abelian ideal P of L. We have $P \subseteq R$ and $\widehat{R} \subseteq R$. Therefore, $\widehat{R} = R$.

In the case of the nilradical, the proof is simpler. The nilradical consists of elements of the radical for which the operator of adjoint representation restricted to the radical has only zero eigenvalues. On multiplication by elements of $\mathbf{D}$, nonzero eigenvalues cannot appear. Therefore, $\mathbf{D} \cdot N = N$. □

Lemma 2. *Let L be a Lie $\mathbf{D}$-algebra, and let N be the nilradical of L. Then N contains the abelian ideal $P = \operatorname{Im} \mathcal{E}$. In particular, $\dim N \geq \dim P$.*

Proof. By Proposition 1, the subspace P is an abelian ideal in L. Therefore, P is contained in the radical of L. Let $X \in P$. Then $\operatorname{ad}_X(L) \subseteq P$, therefore $\operatorname{ad}_X^2(L) = \{0\}$, i.e., the operator ad_x is nilpotent. Therefore, X is contained in N. □

The statement of Lemma 2 can also be proved as follows: Any nilpotent ideal of a Lie algebra is always contained in its radical. Therefore, it is contained in its nilradical. This follows from the well-known description of the nilradical in a solvable Lie algebra: It is maximal among all nilpotent ideals of the radical.

With the notions of semisimple and simple Lie $\mathbf{D}$-algebras which are understood in the usual sense, there are connected some difficulties. For such Lie algebras, we necessarily have $\mathcal{E} = 0$ (see Corollary 1). But then, in the theory of Lie $\mathbf{D}$-algebras, many facts have usual sense. For example, the Levi decomposition $L = S + R$ takes place. In this case, the radical R is a Lie $\mathbf{D}$-subalgebra, but the semisimple part S cannot be a Lie $\mathbf{D}$-subalgebra since, if the Lie subalgebra $\mathcal{E}(S)$ is nonzero, it cannot be contained in S. As an example, consider the Lie algebra $S \otimes \mathbf{D}$, where S is a semisimple real Lie algebra. Then from the structure of the tensor product it follows that $\mathcal{E}(S) \neq \{0\}$, but $\mathcal{E}S \cap S = \{0\}$. This situation is a typical one, see Proposition 6 below.

Let us study the problem concerning semisimple subalgebras in a Lie $\mathbf{D}$-algebra L in detail.

Proposition 6. *Let S be a semisimple Lie subalgebra in a finite-dimensional Lie $\mathbf{D}$-algebra L, more precisely, in $L_{\mathbf{R}}$. Then*

1. $\mathcal{E}(S) \cap S = \{0\}$,

2. *the Lie algebra $\widehat{S}$ has Levi decomposition $S + A$, it is decomposed into a semidirect sum of the Lie subalgebra S and some abelian ideal $A \subseteq \widehat{S}$.*

Proof. 1. We have $[\mathcal{E}(S), S] = \mathcal{E}([S, S]) \subseteq \mathcal{E}(S)$. Hence it follows that the subspace $\mathcal{E}(S)$ is invariant with respect to the action of S induced by the adjoin representation of L. On the other hand, $\mathcal{E}(S)$ is contained in the abelian ideal $P = \mathcal{E}(L)$ of L (see Corollary 1 above). Since the semisimple Lie algebra S has no nontrivial abelian ideals, we have $\mathcal{E}S \cap S = \{0\}$, which proves the first statement.

2. Consider the Lie subalgebra $\widehat{S} \subseteq L$. By definition, it is the sum of the two subspaces S and $A = \mathcal{E}(S)$, which have, as it has been shown above, trivial intersection. In addition, the subspace A is an abelian Lie algebra invariant under the action of S. All these facts mean that the Lie subalgebra $\widehat{S}$ is a semidirect sum of the Lie subalgebra S and an abelian ideal A which is its radical. □

If the $\mathbf{D}$-structure on $\widehat{S}$ is of maximal rank, then $\dim A \geq \dim S$. This follows from the fact that the image ot the restriction of the operator $\mathcal{E}$ to S is an abelian ideal which is contained in A and has, by virtue of the nondegeneracy of the $\mathbf{D}$-structure, the dimension no less than the dimension of S. The equality takes place if $L = S \otimes \mathbf{D}$, where S is an arbitrary semisimple Lie algebra.

Consider the Levi decomposition $L_{\mathbf{R}} = S + R$ for a realified Lie $\mathbf{D}$-algebra L. By Proposition 6, the intersection of $\mathcal{E}(S)$ and S is always trivial, i.e., the Lie subalgebra S is maximally far from algebras closed with respect to the multiplication by elements of $\mathbf{D}$. In addition, S is isomorphic to the quotient algebra of a Lie $\mathbf{D}$-algebra L (including the case when L is nondegenerate) by an $\mathbf{D}$-ideal (radical), which can be degenerate.

Further, for solvable Lie $\mathbf{D}$-algebras, there is an analog of the Lie theorem (which is usually formulated for complex Lie algebras, but there is an analog for real Lie algebras). The classical Lie theorem for a complex linear Lie algebra L or for a solvable Lie algebra L with given linear representation states that there exists one-dimensional L-invariant subspace. Then we deduce by induction that all matrices of this linear Lie algebra L are reduced to triangular form in terms of some basis. This is the Lie theorem. A real solvable Lie algebra can have no one-dimensional invariant subspace, but there always exists an invariant subspace of dimension no more than two. Then, by induction, it is proved that, in terms of an appropriate basis, such a Lie algebra is reduced to quasitriangular (over $\mathbf{C}$, to triangular) form: Blocks of order 1 or 2 stand along the diagonal and the elements below are zeros. A similar statement holds for Lie $\mathbf{D}$-algebras since it is possible to prove that the indicated invariant space can be chosen to be $\mathbf{D}$-saturated. For simplicity of exposition, we prove this assertion only in the case of adjoint representation.

Proposition 7. *Let L be a real solvable finite-dimensional Lie $\mathbf{D}$-algebra. Then, in L, there exists an abelian $\mathbf{D}$-ideal of dimension 1 or 2. If L is a complex Lie $\mathbf{D}$-algebra, then there exists a $\mathbf{D}$-ideal of dimension 1.*

Proof. Consider the realification $L_{\mathbf{R}}$ of L. First, we assume that the linear operator $\mathcal{E}$ is nontrivial. Then $P = \operatorname{Im}\mathcal{E}$ is an abelian ideal of positive dimension. By virtue of the above mentioned real analog of the Lie theorem, there exists an L-invariant subspace W in P of real dimension 1 or 2. Since P is an abelian ideal, the subspace W is also an abelian ideal. In addition, $\mathcal{E}(P) = \{0\}$ since $\mathcal{E}^2 = 0$. Therefore, $\mathcal{E}(W) = \{0\}$, and W is a $\mathbf{D}$-ideal or real dimension 1 or 2.

In case when the operator $\mathcal{E}$ is trivial, the desired assertion is exactly the above mentioned real analog of the Lie theorem.

In the case of a complex Lie algebra, the arguments are similar, the initial subspace W in this case is one-dimensional. □

Let us show that a real Lie $\mathbf{D}$-algebra can have no one-dimensional ideal. For this, consider the Lie algebra $E(2) \otimes \mathbf{D}$, where $E(2)$ is the Lie algebra of motions of the Euclidean plane. It is isomorphic to the Lie algebra $so(2) + \mathbf{R}^2$. Obviously, $E(2)$ contains no one-dimensional ideals. Then one can easily check that the Lie $\mathbf{D}$-algebra $E(2) \otimes \mathbf{D}$ also has no one-dimensional ideals.

Consider for Lie $\mathbf{D}$-algebras an analog of the Ado theorem on existence of an exact finite-dimensional linear representation for an arbitrary real finite-dimensional Lie algebra.

For the Lie algebra $gl_n(\mathbf{R})$ of linear transformations of the vector space $\mathbf{R}$, consider the corresponding Lie $\mathbf{D}$-algebra $gl_n(\mathbf{D})$, which is the Lie algebra of linear transformations of the $\mathbf{D}$-vector space $\mathbf{D}^n$, the free $\mathbf{D}$-module. Note that it is a nondegenerate Lie $\mathbf{D}$-algebra. It can be represented by matrices consisting of blocks of order 2 each of which is of the form $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. Further, one can similarly introduce analogs of other classical matrix Lie algebras. Denote by $N(n, \mathbf{D})$ the nilpotent Lie algebra consisting of nilpotent upper-triangular matrices with elements from $\mathbf{D}$. It is isomorphic to $N(n, \mathbf{R}) \otimes \mathbf{D}$ and can be represented as the set of block-nilpotent matrices whose nonzero blocks are matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. Similarly, denote by $T(n, \mathbf{D})$ the set of upper-triangular matrices with elements from $\mathbf{D}$. It is a solvable (and even a triangular) Lie algebra whose matrix representation also consists from block matrices of order 2. Recall that a real Lie algebra is called triangular (sometimes, completely solvable) if all characteristic roots of the operators of its adjoint representation are real.

Proposition 8. *Let L be a finite-dimensional Lie algebra over $\mathbf{D}$. Then it is isomorphic to a Lie $\mathbf{D}$-subalgebra in $gl_n(\mathbf{D})$. In other words, L has exact finite-dimensional $\mathbf{D}$-linear representation.*

If L is nilpotent, then it is isomorphic to a Lie $\mathbf{D}$-subalgebra in $N(n, \mathbf{D})$. If L is triangular, considered as a real Lie algebra, then it is isomorphic to a Lie $\mathbf{D}$-subalgebra in $T(n, \mathbf{D})$.

Proof. Consider L as a Lie algebra $L_{\mathbf{R}}$ over $\mathbf{R}$. By the classical Ado theorem, there exists an embedding of $L_{\mathbf{R}}$ into the Lie algebra $gl(n, \mathbf{R})$ for some $n \in \mathbf{N}$. It is clear that there exists a natural embedding $L \subseteq L_{\mathbf{R}} \otimes \mathbf{D} \subseteq gl(n, \mathbf{D})$. Then we obtain a $\mathbf{D}$-embedding of L into $gl(n, \mathbf{D})$.

In the cases when L is nilpotent or triangular, the corresponding statements of Proposition 8 are proved in a similar way on the basis of the corresponding well-known statements of the theory of Lie algebras that specify the Ado theorem in these cases. □

For an arbitrary dual Lie algebra L, denote by $\operatorname{Der}(L, \mathbf{D})$ the set of its derivations compatible with the structure of $\mathbf{D}$-algebra. Elements of $\operatorname{Der}(L, \mathbf{D})$ are $\mathbf{D}$-linear mappings $d : L \to L$ for which $d([X, Y]) = [d(X), Y] + [X, d(Y)]$. It is clear that $\operatorname{Der}(L, \mathbf{D}) \subseteq \operatorname{Der}(L)$. In addition, it is obvious that $\operatorname{Der}(L, \mathbf{D})$ is a Lie algebra admitting multiplication by elements of $\mathbf{D}$. In other words, $\operatorname{Der}(L, \mathbf{D})$ is a dual Lie algebra. It is the centralizer in $\operatorname{Der}(L)$ of the linear operator $\mathcal{E}$.

$\operatorname{Der}(L, \mathbf{D})$ contains the ideal $I(L)$ consisting of inner derivations. If a Lie algebra L is nonabelian, then this ideal is nonzero. The quotient algebra $\operatorname{Der}(L, \mathbf{D})/I(L)$ is called the Lie algebra of outer derivations. It is isomorphic to the space $H^1(L, L, \mathrm{ad})$ of cohomologies of L with coefficients in L viewed as an ad-module. It should be also noted that the algebra of derivations of the dual Lie algebra L is a Lie algebra of the Lie group of automorphisms of L. In addition, a dual structure can be introduces in a natural way on the Lie group of automorphisms.

2. ON NONDEGENERATE DUAL LIE ALGEBRAS

Recall that a Lie $\mathbf{D}$-algebra (dual Lie algebra) L is called nondegenerate if the linear operator $\mathcal{E}$ defining the $\mathbf{D}$-structure on L has rank n equal the one-half of the dimension of L. If a Lie $\mathbf{D}$-algebra L is even-dimensional, then, as one can easily see, L is nondegenerate if ind only if the operator $\mathcal{E}$ has maximal rank.

Further, for a nondegenerate Lie $\mathbf{D}$-algebra, the subspaces (in fact, ideals) $P = \operatorname{Im}\mathcal{E}$, and $Q = \operatorname{Ker}\mathcal{E}$ coincide, and their coincidence can be regarded as an equivalent definition of their nondegeneracy.

Obviously, the linear operator $\mathcal{E}$ is nondegenerate if and only if its matrix can be represented in the form of direct sum $\oplus J_2(0)$ of Jordan blocks of order 2 corresponding to zero eigenvalue. Another useful form of the matrix of this linear operator is $\left(\begin{smallmatrix} 0 & 0 \\ E & 0 \end{smallmatrix}\right)$, where E is the identity matrix of order n and 0 is the zero matrix of the same order.

Consider examples of nondegenerate Lie $\mathbf{D}$-algebras. For an arbitrary real Lie algebra L, the Lie algebra $L \otimes \mathbf{D}$ is a nondegenerate Lie $\mathbf{D}$-algebra. Such an algebra can be regarded as the Lie algebra of a Lie group of the form $T(G)$, a tangent Lie group. The matter is that on the total space $T(G)$ of the tangent bundle of an arbitrary connected Lie group G, as is known, one can introduce a structure of Lie group. Algebraically, $T(G)$ has the form of a semidirect product $G \cdot g$ of the initial Lie group G and an abelian ideal which is identified with the Lie algebra g of G regarded as a vector space. The action of G on g defining this semidirect product is the adjoint action of G on its Lie algebra. From the viewpoint of the approach used in this paper, the Lie algebra of a Lie group $T(G)$ is exactly the Lie algebra $g \otimes \mathbf{D}$. Therefore, there are reasons to call nondegenerate dual Lie algebras of the form $L \otimes \mathbf{D}$ tangent Lie algebras.

We give another series of examples which are not obtained by means of the above described construction. Consider an arbitrary vector space V and a skew-symmetric operation on it, i.e., a linear mapping $\mu : V \wedge V \to V$. We let $L = V \oplus V$ and introduce on V the following commutator operation: If X and Y are elements of the first summand, then $[X, Y]$ is defined to be the element $\mu(X, Y)$ lying in the second summand. The commutator of two elements belonging to different direct summands is defined to be zero, as well as the commutator of two elements belonging to the second summand. Thus, we have introduced on L a structure of nilpotent Lie algebra of the nilpotency class 2. The obtained Lie algebra L can be considered as a dual Lie algebra. For this, we define the action of the linear operator $\mathcal{E}$ on L as follows: It is trivial on the second direct summand and sends elements X of the first direct summand into the same elements viewed as elements of the second summand. One can easily check that, as a result, we obtain a Lie $\mathbf{D}$-algebra.

The following proposition gives a criterion of existence of a nondegenerate dual structure on a Lie algebra.

Proposition 9. *Let L be an even-dimensional Lie $\mathbf{D}$-algebra. Then the following statements are equivalent*:

i) *L is a nondegenerate Lie $\mathbf{D}$-algebra,*

ii) $\operatorname{Im}\mathcal{E} = \operatorname{Ker}\mathcal{E}$,

iii) *L is a free $\mathbf{D}$-module of rank n equal to one-half of the dimension of L,*

iv) *in terms of an appropriate basis of L, the matrices of operators of the adjoint representation have block form $\left(\begin{smallmatrix} A & 0 \\ C & A \end{smallmatrix}\right)$, where A, C are square matrices of order n.*

Proof. Statements i) and ii) are obviously equivalent. In statement iii), the fact that $\mathbf{D}$-module L is free means that L can be represented as the direct sum of a number of copies of the algebra $\mathbf{D}$. But then statements i) and iii) are obviously equivalent. To complete the proof of Proposition 9, let us prove that statements i) and iv) are equivalent.

By definition, a Lie algebra is dual if and only if the operator $\mathcal{E}$ commutes with operators $\operatorname{ad}_X (X \in L)$ of adjoint representation. We choose a basis in a Lie algebra L over $\mathbf{R}$ such that the matrix of $\mathcal{E}$ is of the form $\left(\begin{smallmatrix} 0 & 0 \\ E & 0 \end{smallmatrix}\right)$ and find the centralizer of this matrix in the algebra $M_{2n}(\mathbf{R})$ of all square matrices of order $2n$ equal to the dimension of L. It can be done with reference to general results on centralizers of matrices (see, for example, [3]) or to general results on centralizers in arbitrary reductive Lie algebras. It is useful to perform direct matrix calculations, which is rather simple.

Consider an arbitrary matrix U from $M_{2n}(\mathbf{R})$ written in block form: $U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

The commutativity condition $U\mathcal{E} = \mathcal{E}U$, where $\mathcal{E}$ is the above indicated matrix of special block form, gives matrix equalities $A = D$ and $B = 0$ with no restrictions on C, which exactly means that U is of the form $\left(\begin{smallmatrix} A & 0 \\ C & A \end{smallmatrix}\right)$. □

One can prove a criterion of existence of a dual structure on an arbitrary Lie algebra L which is similar to statement iv) of Proposition 9. This criterion states that matrices of adjoint representation of L have a special block form more complicated than in the nondegeneral case.

From statement iv) of Proposition 9, we obtain the following the next corollary.

Corollary 2. For a nondegenerate Lie $\mathbf{D}$-algebra L, the modules P and L/P are isomorphic.

Consider the following construction. Let L be a nondegenerate Lie $\mathbf{D}$-algebra. Then $P = \operatorname{Im}\mathcal{E}$ is its abelian ideal of dimension which equals one-half of the dimension of L. The dimension of the Lie algebra $L^\sharp = L/P$ is also one-half of the dimension of L. The operator $\mathcal{E}$ induces on $L^\sharp$ the zero linear operator, i.e., $L^\sharp$ is a usual "completely degenerate" Lie algebra. The initial Lie algebra L viewed as a Lie algebra over $\mathbf{R}$ is an abelian extension of $L^\sharp$. Since the kernel of this extension is abelian, the Lie algebra $L^\sharp$ acts in a natural way on the vector space P by linear transformations. Thus, we can consider P as an $L^\sharp$-module.

From statement iv) of Proposition 9, we obtain the following corollary.

Corollary 3. For a nondegenerate Lie $\mathbf{D}$-algebra L, the adjoint representation of $L^\sharp = L/Im\mathcal{E}$ on itself and its representation on P are equivalent.

If a nondegenerate Lie $\mathbf{D}$-algebra L is of the form $L = M \otimes \mathbf{D}$ for a real Lie algebra M, then $L^\sharp = M$, and L is a semidirect sum of M and an abelian ideal which coincides with the Lie algebra M regarded as a vector space. In addition, the characteristic class $c(L)$ is trivial. In this case matrices C from statement iv) of Proposition 9 are zero. For arbitrary nondegererate dual Lie algebras, occurence of nonzero matrices C can be considered as a result of deformations of Lie algebras of the form $M \otimes \mathbf{D}$.

Conversely, if $c(L) = 0$, then the above indicated extension splits, and L decomposes into a semidirect sum of a Lie subalgebra isomorphic to $L^\sharp$ and an abelian ideal. Therefore, such an algebra L is isomorphic to a dual Lie algebra of the form $L^\sharp \otimes \mathbf{D}$.

An arbitrary nondegenerate Lie $\mathbf{D}$-algebra L can have no representation in the form of a semidirect sum of $L^\sharp$ and the abelian ideal P, i.e., an extension $0 \to P \to L \to L^\sharp$ is not always splittable. This extension is with an abelian kernel, therefore it is defined uniquely up to an equivalence of extensions (but not necessarily up to an isomorphism of Lie algebras) by its characteristic class $c(L) \in H^2(L^\sharp, L^\sharp, \mathrm{ad})$, a two-dimensional cohomology class of the Lie algebra $L^\sharp$ with coefficients relative to its adjoint representation on itself. We use this characteristic class in the next section in the description of nondegenerate dual Lie algebras of small dimensions.

Another interpretation of the above defined characteristic class $c(L)$ should be noted. The space of cocycles $Z^2(L^\sharp, L^\sharp, \mathrm{ad})$ can be considered as the tangent space in the sense of algebraic geometry to the space of all n-dimensional Lie algebras at the point corresponding to the Lie algebra $L^\sharp$. Elements of $Z^2(L^\sharp, L^\sharp, \mathrm{ad})$ are called infinitesimal deformations of $L^\sharp$. They not always correspond to smooth deformations of this Lie algebra since vectors from this cohomology space not always are tangent vectors to curves lying in the space of Lie algebras. The cohomology space $H^2(L^\sharp, L^\sharp, \mathrm{ad})$ can be considered as parameterization of nontrivial deformations. Consequently, as the above indicated construction shows, elements of this space always correspond to deformations of dual Lie algebras. It is natural to expect that methods of dual Lie algebras can be useful in the study of deformations of arbitrary real Lie algebras.

Below we obtain assertions on $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebras. A nondegenerate Lie $\mathbf{D}$-algebra L is called $\mathbf{D}$-semisimple if its radical has minimal possible dimension. Since the ideal P has dimension which equals one-half of the dimension of L, this Lie algebra is $\mathbf{D}$-semisimple if and only if $L^\sharp = L/P$ is semisimple and has dimension which equals one-half of the dimension of L. It turns out that many assertions known for usual semisimple Lie algebras remain valid for nondegenerate $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebras. Below we formulate some of such assertions. Note that a nondegenerate $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebra will be called $\mathbf{D}$-simple if the corresponding Lie algebra $L^\sharp$ is simple.

Proposition 10. *A nondegenerate* $\mathbf{D}$*-semisimple Lie* $\mathbf{D}$*-algebra L is of the form $S \otimes \mathbf{D}$ for some semisimple Lie algebra S, in fact, for L/P. In other words, Lie* $\mathbf{D}$*-algebra L is isomorphic to the tangent Lie* $\mathbf{D}$*-algebra of the corresponding semisimple Lie algebra $L^\sharp$.*

Proof. The Lie algebra $S = L/P$ is semisimple, therefore, as is known, the extension $0 \to P \to L \to S \to 0$ splits and gives the Levi decomposition. Thus, the semisimple Lie algebra S is realized as the Levi subalgebra in L. By Corollary 3, its actions on P and on itself are equivalent. This means that L is isomorphic to $L \otimes \mathbf{D}$. □

It is not difficult to deduce from Proposition 10 that, for nondegenerate $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebras, which are not semisimple in the usual sense, many usual properties of semisimple Lie algebras hold. For example, for a nondegenerate $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebra L, we always have $[L, L] = L$, and the center $Z(L)$ is always trivial.

Proposition 11. *A nondegenerate* $\mathbf{D}$*-semisimple Lie* $\mathbf{D}$*-algebra is decomposed into the direct sum of nondegenerate* $\mathbf{D}$*-simple Lie* $\mathbf{D}$*-algebras, and this decomposition is unique up to permutation of direct summands.*

Proof. A $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebra L has the Levi decomposition $L_{\mathbf{R}} = S + R$ (see Section 1). For a nondegenerate Lie algebra L, the radical R is abelian and has dimension which equals one-half of the dimension of L. The semisimple part S is isomorphic to $L^\sharp$. By statement iv) of Proposition 9, the actions of S on S and on R are isomorphic.

Consider the decomposition of a semisimple Lie algebra S into the direct sum $\oplus S_i$ of simple ideals. It is unique up to permutation of direct summands. There arises a direct decomposition of the radical R as an S-module into the direct sum $\oplus R_i$ of submodules which are simple and defined uniquely up to permutation. Let $L_i = S_i + R_i$ be $\mathbf{D}$-simple Lie $\mathbf{D}$-algebras, and let $L = \oplus L_i$. It remains to prove that Lie $\mathbf{D}$-algebras L_i are nondegenerate. But, if at least one of them were degenerate, then the operator $\mathcal{E}$ in its action on L would be degenerate. □

Note that Proposition 11 can be obtained in a different way, using Proposition 10.

A nondegenerate $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebra is isomorphic to the tangent Lie algebra uniquely defined by a usual semisimple Lie algebra. Therefore, classification of nondegenerate $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebras presents no problems. In the degenerate case, there are possible peculiarities which complicate the theory of $\mathbf{D}$-semisimple Lie $\mathbf{D}$-algebras.

The following statement is similar to a statement of the classical theory of Lie algebras, though Lie algebras in question are not semisimple in the usual sense.

Proposition 12. *Any derivation of a nondegenerate* $\mathbf{D}$*-semisimple Lie* $\mathbf{D}$*-algebra is inner. In other words, the Lie algebra of outer derivations is trivial.*

Below we obtain results concerning nondegenerate Lie $\mathbf{D}$-algebras which are not $\mathbf{D}$-semisimple. In fact, of interest are the cases when the Lie algebra $L^\sharp$ is abelian or nilpotent.

Assume first that $L^\sharp$ is nilpotent. Then, by Proposition 10, its action on the ideal $P = \operatorname{Im} \mathcal{E}$ is also nilpotent and, therefore, the Lie algebra L is also nilpotent. Thus, for nondegenerate Lie $\mathbf{D}$-algebras, L is nilpotent if and only if $L^\sharp$ is nilpotent.

The following questions arise. Given a nilpotent Lie algebra N, does there exists a nondegenerate Lie $\mathbf{D}$-algebra L such that $N = L^\sharp$, and how one can describe all such algebras L?

For each N, one of such Lie $\mathbf{D}$-algebras L always exists, this is the tangent Lie algebra $N \otimes \mathbf{D}$. For the description of all nondegenerate Lie $\mathbf{D}$-algebras such that $L^\sharp = N$, we use the characteristic class $c(L)$ mentioned above.

As is known [4], for an arbitrary nilpotent Lie algebra N, the cohomology space $H^2(N, N, \mathrm{ad})$ has dimension ≥ 2. Therefore, there are many possibilities for the characteristic class $c(L)$, it can be any element from the nontrivial cohomology space $H^2(N, N, \mathrm{ad})$. This gives a partial solution to the problem stated above, a description of all L up to equivalence of extensions of the form $0 \to P \to L \to N \to 0$, but not up to an isomorphism. The fact that the space $H^2(N, N, \mathrm{ad})$ is nontrivial guarantees existence of Lie algebras for which this extension does not split.

Consider absolutely particular case when $N = L^\sharp$ is abelian. Since L is nondegenerate, the actions of N on itself by the adjoint representation and on the abelian ideal P are trivial. But this means that the ideal P is not central, and the Lie algebra L itself is nilpotent of nilpotency class 2 (or metabelian). The cohomology space $H^2(N, N, \mathrm{ad})$, in this case, is isomorphic to $\wedge^2 N \otimes N^*$. Consider, in this case, how a nondegenerate Lie $\mathbf{D}$-algebra L is constructed by given an abelian Lie algebra A and a characteristic class $c \in \wedge^2 A \otimes A^*$. A characteristic class c can be regarded as a linear mapping $c : A \wedge A \to A$. But, as has been shown above, such a mapping defines the desired structure on $L = A \oplus A$.

3. ON CLASSIFICATION OF SOME CLASSES OF DUAL LIE ALGEBRAS

Classification can be made by the number $d(L)$ of essential parameters. For $d(L) = 1$, there exist only two (up to $\mathbf{D}$-isomorphism) Lie $\mathbf{D}$-algebras, and they both are abelian. These are the one-dimensional Lie algebra $\mathbf{R}$ with trivial $\mathcal{E}$ and the two-dimensional Lie algebra $\mathbf{D}$ isomorphic to $\mathbf{R}^2$ as a real Lie algebra.

There are classifications of Lie $\mathbf{D}$-algebras for the cases of two and three essential parameters [2, 5]. In fact, in [2, 5] exactly Lie $\mathbf{D}$-algebras in the sense of our definition are classified with the number of essential parameters 2 or 3 (thou the notion of a Lie $\mathbf{D}$-algebra is not used). We will proceed from just such understanding of papers [2, 5].

It turns out that unsolvable Lie $\mathbf{D}$-algebras exist only beginning with dimension 6 (and three essential parameters). These are the two Lie algebras $su(2) \otimes \mathbf{D}$ and $sl(2, \mathbf{R}) \otimes \mathbf{D}$ (each of the form $S \otimes \mathbf{D}$, where S is one of two simple three-dimensional Lie algebras); these Lie algebras are tangent Lie algebras in the sense of the above introduced definition. For a greater number of essential parameters and for greater dimension, classifications are not available because, for this, we need to have classifications of real Lie algebras of dimension ≥ 8, which is at this time unsolvable problem. The top of today achievements in this direction is the classification of all Lie algebras of dimension ≤ 6 and all nilpotent Lie algebras of dimension 7. Therefore, the result of paper [2] at present is the maximal possible one. The list of nilpotent Lie $\mathbf{D}$-algebras of real dimension ≤ 3 (considered up to $\mathbf{D}$-isomorphism) consists of the abelian Lie algebras $\mathbf{R}$, $\mathbf{R}^2$ (with trivial $\mathcal{E}$), $\mathbf{D}$, $\mathbf{D} \oplus \mathbf{R}$, and the nonabelian Lie algebra $N(3, \mathbf{R})$. Note that the two Lie algebras $\mathbf{D}$, $\mathbf{D} \oplus \mathbf{D}$ reflect the specific of dual Lie algebras. For the cases 4, 5, and 6, see [2].

After citing known classification results, we add new ones concerning classification of nondegenerate dual Lie algebras. We restrict ourselves to consideration of Lie algebras of small dimension. Our purpose is not to present a complete (and certainly awkward) list of such Lie algebras up to isomorphism, but to describe a general cohomological approach which will give in particular possibilities to deal with eight-dimensional nilpotent Lie algebras, which, in the case of usual Lie algebras, lie outside the present classification possibilities.

For a Lie $\mathbf{D}$-algebra L, consider the corresponding ideal P, Lie algebra $L^\sharp = L/P$, and characteristic class $c(L)$. Since the dimensions of Lie algebras P and $L^\sharp$ equal one-half of that of L, it is easy to classify them than the initial Lie algebras L. The calculation of the cohomology space $H^2(L^\sharp, L^\sharp, \mathrm{ad})$ containing $c(L)$ usually is very simple. The dimensions of this space and of the space of the corresponding cocycles for Lie algebras of not so large dimension were calculated with the use of computer (see, for example, [6]).

Consider in detail the case when a Lie $\mathbf{D}$-algebra L is nondegenerate and nilpotent. For $\dim L = 2, 4, 6, 8$, the dimension of $L^\sharp$ equals, respectively, to 1, 2, 3, 4.

If $\dim L^\sharp = 1, 2$, the Lie algebra $L^\sharp$ is abelian (recall that it is assumed to be nilpotent). Therefore, as it was indicated above, L is metabelian. What is more, for $\dim L^\sharp = 1$, it is abelian since L is a two-dimensional nilpotent Lie algebra, and such an algebra is always abelian. For $\dim L^\sharp = 2$, L is nilpotent (of nilpotency class 2) four-dimensional Lie algebra. There are two such algebras: The abelian one and the Lie algebra $N(3, \mathbf{R}) \oplus \mathbf{R}$.

If $\dim L^\sharp = 3$ and $L^\sharp$ is nonabelian, it is isomorphic to $N(3, \mathbf{R})$ and $\dim H^2(L^\sharp, L^\sharp, \mathrm{ad}) = 5$. Thus, all possible corresponding Lie $\mathbf{D}$-algebras are parameterized by five parameters. As a matter of fact, they are not independent since nonisomorphic Lie algebras correspond to orbits of the automorphism group acting on the cohomology space.

Consider in a similar way the case when $\dim L^\sharp = 4$. In this case, the nonabelian Lie algebra $L^\sharp$ is isomorphic, as it follows from the classification of four-dimensional nilpotent Lie algebras, either to the Lie algebra $N(3, \mathbf{R}) \oplus \mathbf{R}$, or to the Lie algebra N_4, which, in terms of an appropriate basis X_1, X_2, X_3, X_4, is given by the relations (we indicate only nontrivial ones) $[X_1, X_i] = X_{i+1}$ for $i = 2, 3$. For these two algebras, the dimensions of the cohomology spaces $H^2(L^\sharp, L^\sharp, ad)$ equal, respectively, 13 and 6. By comparison, note that, for the abelian four-dimensional Lie algebra $L^\sharp$, this dimension equals 24. One can see that the numbers of parameters which describe nondegenerate nilpotent Lie $\mathbf{D}$-algebras are rather large. That is why there is no sense to present the list of all nilpotent Lie $\mathbf{D}$-algebras of dimension 6 up to isomorphism (the more so for dimension 8).

As for real Lie algebras, we can consider the spaces of all Lie $\mathbf{D}$-algebras of a fixed dimension. One can easily understand that it is an algebraic subset in $\mathbf{D}^m = \mathbf{R}^{2m}$ for some natural (and rather large) m. Nondegenerate Lie $\mathbf{D}$-algebras also form an algebraic subset, the same concerns the spaces of solvable and nilpotent Lie $\mathbf{D}$-algebras, and all these spaces are, generally speaking, irreducible. As a result of the above discussion, we have the following proposition.

Proposition 13. *The spaces of nondegenerate nilpotent Lie $\mathbf{D}$-algebras of dimension ≤ 12 are irreducible algebraic sets.*

4. APPENDIX (ON APPROXIMATE LIE ALGEBRAS)

The author came to the notion of a dual Lie algebra when he studied papers devoted to approximate symmetries of differential equations depending on small parameter. For the exposition of the theory of such symmetries we refer to the fundamental paper [7], where the notion of an approximate Lie algebra was introduced. It turns out natural to pass to more general notion of a dual Lie algebra or a Lie $\mathbf{D}$-algebra, which allows one to present many results in more convenient form. Below we describe the way which has brought us to the use of the notion of a dual Lie algebra.

The notion of an approximate Lie algebra was introduces long ago. First it was used only for Lie algebras of vector fields (or the corresponding differential operators), and as an object of study it was singled out not long ago (see, for example, [2, 7, 8]) and then was subjected to detailed investigation. The main initial idea was to consider functions and vector fields approximately, up to $o(\varepsilon)$, which was equivalent to their linearization. For example, a smooth (or analytical) function $F(x, \varepsilon)$ is replaced by the function of the form $f_0(x) + \varepsilon f_1(x)$, where f_0 and f_1 are smooth (or analytical) functions. Such replacement will be denoted by the symbol $\approx$. Similarly, for a smooth (or analytical) vector field $X(x, \varepsilon)$ depending on a parameter ε, we have an approximate equality of the form $X(x, \varepsilon) \approx X_0(x) + \varepsilon X_1(x)$, where $X_0(x)$ and $X_1(x)$ are vector fields. The commutator of such approximate vector fields has the form $[X, Y] \approx [X_0 + \varepsilon X_1, Y_0 + \varepsilon Y_1] \approx [X_0, Y_0] + \varepsilon([X_0, Y_1] + [X_1, Y_0])$.

The commutator operation thus defined remains bilinear and skew-symmetric, the Jacobi identity also holds. As a result, we obtain a Lie algebra. The usual commutator of linearized vector fields, generally speaking, is not linear in ε. Therefore the indicated commutator is not the usual commutator of vector fields, it is different. Thus, we arrive at a new notion which unites all Lie algebras depending on a parameter ε. This object is called an approximate Lie algebra. Now we describe several different approaches to consideration of such objects.

Construction 1. Let Φ be a Lie algebra of vector fields (or their germs) in a neighborhood of $x \in \mathbf{R}^n$. Consider the set of vector fields of the form $X + \varepsilon Y$, where $X, Y \in \Phi$ and ε is a parameter. Denote this set by $\Phi(\varepsilon)$ or represent it in the form $\Phi + \varepsilon\Phi$. Generally speaking, this set (of germs) of vector fields does not form a Lie algebra since the operation of commuting may lead to a vector field not lying in $\Phi(\varepsilon)$. But if we define the commutator operation on $\Phi(\varepsilon)$ in another way (as it has been done above for approximate vector fields), we obtain a Lie algebra, but it is not a Lie algebra of vector fields. Such Lie algebras are called in the above cited papers approximate Lie algebras.

In addition to approximate Lie algebras constructed in the above described way, one should consider some their subalgebras, those which are compatible with the multiplication by ε. Exactly such Lie subalgebras correspond to Lie groups of approximate symmetries of differential equations, the study of which initiated the notion of an approximate Lie algebra.

Construction 2. Let Φ be a Lie algebra, and let $\widetilde{\Phi} = \Phi \otimes \mathbf{D}$, the tensor product of Φ and $\mathbf{D}$. Elements of $\widetilde{\Phi}$ can be written in the form $a + \varepsilon b$, where $a, b \in \Phi$. The commutator operation in $\widetilde{\Phi}$ is defined as follows: The commutator operation with respect to the first tensor factor, and the multiplication with respect to the second tensor factor. As a result, we obtain a structure of Lie algebra on $\widetilde{\Phi}$. Up to isomorphism, the Lie algebra $\widetilde{\Phi}$ can be considered as the semidirect sum $\Phi +_{\mathrm{ad}} \Phi$ of the subalgebra isomorphic to Φ and the abelian ideal corresponding to the adjoint action ad of the Lie algebra Φ on Φ regarded as a vector space. It is clear that if Φ is a Lie algebra of vector fields, then we obtain the approximate Lie algebra from Construction 1.

This construction can be generalized. For example, consider the algebra of the form $D_p = \mathbf{R}[\varepsilon]/\langle\varepsilon^p\rangle$, where $\mathbf{R}(\varepsilon)$ is the algebra of polynomials in ε and $\langle\varepsilon^p\rangle$ is the ideal generated by the element ε^p. As a result, we obtain a p-dimensional associative and commutative algebra with unity, which is called sometimes the algebra of plural numbers. For $p = 2$, this is the algebra $\mathbf{D}$ of dual numbers. Consider the tensor product $\Phi \otimes D_p$ with natural commutator operation, the obtained Lie algebra can be regarded as a approximate Lie algebra for approximations of order p with the use of $o(\varepsilon^p)$. This algebra is the semidirect sum of the subalgebra isomorphic to Φ and the nilpotent ideal of the form $\varepsilon\Phi + \varepsilon^2 \cdot \Phi \cdots + \varepsilon^{p-1} \cdot \Phi$. This opens possibilities for the study of plural Lie algebras.

One can first use the term approximate Lie algebras for Lie algebras of the form $L \otimes \mathbf{D}$ (not only for algebras related to vector fields), tensor products of usual Lie algebras L and the two-dimensional algebra $\mathbf{D} = \langle 1, \varepsilon\rangle$ of dual numbers. Such algebras can also be considered as $\mathbf{D}$-modules. What is more, approximate Lie algebras of symmetries of differential equations also have structures of $\mathbf{D}$-modules. This follows from the following simple fact: If a vector field X is an infinitesimal symmetry of a differential equation, then the vector field εX is also an infinitesimal symmetry of the same equation. Thus, the Lie algebra of infinitesimal symmetries is invariant with respect to multiplications by elements of $\mathbf{D}$.

Bearing in mind applications, one should consider not only Lie algebras of the form $L \otimes \mathbf{D}$, but their subalgebras. Generally speaking, an arbitrary Lie subalgebra $L \otimes \mathbf{D}$ is not closed with respect to the multiplication by ε and therefore is not a Lie $\mathbf{D}$-subalgebra. For example, the one-dimensional abelian subalgebra in $L \otimes \mathbf{D}$ spanned by an arbitrary element $X \in L$ is not invariant with respect to $\mathbf{D}$. For Lie algebras of approximate symmetries, it is natural to consider only subalgebras which are $\mathbf{D}$-modules. Just they can be called approximate Lie algebras in the study of approximate differential equations and their symmetries. Thus, we arrive at the necessity to consider some class of Lie algebras defined over $\mathbf{D}$. Namely, as the next approximation to the final definition, it makes sense to use the term "approximate Lie algebra" for arbitrary Lie subalgebras in Lie algebras of the form $L \otimes \mathbf{D}$ which are invariant with respect to the multiplication by ε. In the theory of approximate symmetries of differential equations, the Lie algebra of smooth or analytic vector fields plays role of L. In order to separate this general notion from connection with vector fields, we suggest to call Lie algebras which are $\mathbf{D}$-modules Lie $\mathbf{D}$-algebras (dual Lie algebras). These are exactly dual Lie algebras or Lie $\mathbf{D}$-algebras considered in this paper.

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