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A Generalization of the Regularization Method to the Singularly Perturbed Integro-Differential Equations With Partial Derivatives

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Abstract—We generalize the Lomov's regularization method to partial integro-differential equations. It turns out that the procedure for regularization and the construction of a regularized asymptotic solution essentially depend on the type of the integral operator. The most difficult is the case, when the upper limit of the integral is not a variable of differentiation. In this paper, we consider its scalar option. For the integral operator with the upper limit coinciding with the variable of differentiation, we investigate the vector case. In both cases, we develop an algorithm for constructing a regularized asymptotic solution and carry out its full substantiation. Based on the analysis of the principal term of the asymptotic solution, we study the limit in solution of the original problem (with the small parameter tending to zero) and solve the so-called *initialization problem* about allocation of a class of input data, in which the passage to the limit takes place on the whole considered period of time, including the area of boundary layer.

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The Lomov regularization method ([1], pp. 126–129) was mainly applied for ordinary singularly disturbed integro-differential equations (see the detailed bibliography in [2, 3]). The systematic elaboration of this method for systems of integro-differential equations with partial derivatives has not been carried out earlier. In the present paper, such an attempt is made for the first time. The type of the integral operator plays an important role in the propagation of the regularization method to partial differential equations. It is necessary to distinguish two fundamentally different cases:

$$\varepsilon\frac{\partial y\left(t,x,\varepsilon\right)}{\partial t}=A\left(t,x\right)y\left(t,x,\varepsilon\right)+\int_0^t K\left(t,x,s\right)y\left(s,x,\varepsilon\right)ds+h\left(t,x\right),\quad y\left(0,x,\varepsilon\right)=y^0\left(x\right),\ \ (t,x)\in[0,T]\times[0,X],\tag{1}$$

$$\varepsilon\frac{\partial y\left(t,x,\varepsilon\right)}{\partial t}=A\left(t,x\right)y\left(t,x,\varepsilon\right)+\int_0^x K\left(t,x,s\right)y\left(s,x,\varepsilon\right)ds+h\left(t,x\right),\quad y\left(0,x,\varepsilon\right)=y^0\left(x\right),\ \ (t,x)\in[0,T]\times[0,X].\tag{2}$$

First we consider the first case. In the second case, there arises the problem of describing the space of solutions of iterative problems and the conditions for their solvability in the indicated space. The second part of our work is devoted to solving this problem. We now turn to the presentation of the results for problem (1).

1. Regularization of problem (1). Without loss of generality, we can assume that $T=X=1$ and, since the dependence of the matrix A on x does not have a significant effect on the algorithm developed below, we consider the case $A=A(t)$. Assume that

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1) functions $A(t) \in C^{\infty}([0,1], C^{n^2})$, $h(t,x) \in C^{\infty}([0,1] \times [0,1], \mathbb{C}^n)$, the kernel

$$K(t,x,s) \in C^{\infty} \ \ (0 \le t \le 1, \ \ 0 \le s \le t \le 1, \ \ \mathbb{C}^{n^2});$$

2) the spectrum $\sigma(A(t)) = \{\lambda_j(t)\}$ of the matrix $A(t)$ satisfies the following requirements:

a) $\lambda_i(t) \neq \lambda_j(t)$, $i \neq j$, $i,j = \overline{1,n}$, $\forall t \in [0,1]$,

b) $\operatorname{Re}\lambda_j(t) \le 0$, $\lambda_j(t) \neq 0 \ \ \forall t \in [0,1]$.

We introduce the regularizing variables

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(s)\,ds = \frac{\psi_j(t)}{\varepsilon}, \ \ j = \overline{1,n}, \qquad \tau = (\tau_1, \dots, \tau_n), \psi(t) = (\psi_1(t), \dots, \psi_n(t)), \tag{3}$$

and for the function $\widetilde{y}(t,x,\tau,\varepsilon)$ we formulate the problem

$$\varepsilon\frac{\partial \widetilde{y}}{\partial t} + \sum_{j=1}^{n}\lambda_j(t)\frac{\partial \widetilde{y}}{\partial \tau_j} - A(t)\widetilde{y} - \int_0^t K(t,x,s)\,\widetilde{y}\Big(s,x,\frac{\psi(s)}{\varepsilon}m\varepsilon\Big)ds = h(t,x), \ \ \widetilde{y}(0,x,0,\varepsilon) = y^0(x). \tag{4}$$

Therefore, in problem (1) the spectrum of matrix $A(t)$ can contain purely imagine values $\lambda_j(t) \equiv \pm i\omega_j(t)$, $\omega_j(t) > 0$, which were not possible in earlier considered papers (for example, [4], where the ordinary nonlinear integro-differential equation of the Volterra type was investigated).

The connection of problem (4) with initial problem (1) is the following: If $\widetilde{y} = \widetilde{y}(t,x,\tau,\varepsilon)$ is a solution to problem (4), then its narrowing $y(t,x,\varepsilon) \equiv \widetilde{y}\big(t,x,\frac{\psi(t)}{\varepsilon}\big)$ on regularizing functions (3) will be, evidently, the solution to initial problem (1). However, one cannot treat problem (4) as completely regularized, because in it the integral operator

$$J\widetilde{y} = \int_0^t K(t,x,s)\,\widetilde{y}\Big(s,x,\frac{\psi(s)}{\varepsilon}\Big)ds$$

is not regularized. For its regularization it is necessary to introduce ([1], P. 62) the space M_ε, which is asymptotically invariant with respect to the operator J. It is constructed as follows. One introduces the class U of anticipated solutions to iterative problems (see below):

$$U = \Big\{ y : y(t,x,\tau) = \sum_{j=1}^{n} y_j(t,x)\,e^{\tau_j} + y_0(t,x), \ \ y_j(t,x) \in C^{\infty}([0,1] \times [0,1], \mathbb{C}^n), \ \ j = \overline{0,n} \Big\},$$

and then one takes a narrowing of this class with $\tau = \psi(t)/\varepsilon$. It is a space M_ε. To substantiate this fact, it is necessary to show that the image $Jy(t,x,\tau)$ of the integral operator J on the element $y(t,x,\tau)$ of the space U can be presented in the form of power series

$$\sum_{k=0}^{\infty}\varepsilon^k\Big(\sum_{j=1}^{n} y_j^{(k)}(t,x)\,e^{\frac{\psi_j(t)}{\varepsilon}} + y_0^{(k)}(t,x)\Big),$$

which converges asymptotically with $\varepsilon \to +0$ (uniformly with respect to $(t,x) \in [0,1] \times [0,1]$). Let us consider this question. We have

$$Jy(t,x,\tau) \equiv \sum_{j=1}^{n}\int_0^t K(t,x,s)\,y_j(s,x)\,e^{\frac{\psi_j(s)}{\varepsilon}}ds + \int_0^t K(t,x,s)\,y_0(s,x)\,dx.$$

Let us integrate by parts:

$$\int_0^t K(t,x,s)\,y_j(s,x)\,e^{\frac{1}{\varepsilon}\int_0^s\lambda_j(\theta)d\theta}ds = \varepsilon\int_0^t \frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)}d_s\Big(e^{\frac{1}{\varepsilon}\int_0^s\lambda_j(\theta)d\theta}\Big)$$

$$= \varepsilon\left[\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)} e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta}\Big|_{s=0}^{s=t} - \int_0^t \frac{\partial}{\partial s}\left(\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)}\right) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds\right]$$

$$= \varepsilon\Big[\left(I_j^0\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta} - \left(I_j^0\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=0}\Big]$$

$$-\varepsilon \int_0^t \frac{\partial}{\partial s}\left(I_j^0\left(K(t,x,s)\,y_j(s,x)\right)\right) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds,$$

where $I_j^0\left(K(t,x,s)\,y_j(s,x)\right) \equiv \frac{K(t,x,s)y_j(s,x)}{\lambda_j(s)}$, $j=\overline{1,n}$. Continuing this process, we obtain the series

$$\int_0^t K(t,x,s)\,y_j(s,x)\,e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} d =$$

$$= \sum_{k=0}^{\infty} (-1)^k \varepsilon^{k+1} \sum_{j=1}^{n} \Big[\left(I_j^k\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=t}$$

$$\times e^{\tau} - \left(I_j^k\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=0}\Big], \quad (5)$$

in which operators I_j^k have the form

$$I_j^0\left(K(t,x,s)\,y_j(s,x)\right) \equiv \frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)},$$

$$I_j^1\left(K(t,x,s)\,y_j(s,x)\right) = \frac{1}{\lambda_j(s)}\frac{\partial}{\partial s} I_j^0\left(K(t,x,s)\,y_j(s,x)\right), \ldots, \quad (6)$$

$$I_j^m\left(K(t,x,s)\,y_j(s,x)\right) = \frac{1}{\lambda_j(s)} I_j^{m-1}\left(K(t,x,s)\,y_j(s,x)\right), \;\; m \geq 1.$$

The asymptotic convergence of series (5) can be proved by the same way as the analogous assertion in [2] (Chap. 8).

2. Construction of a regularized asymptotic solution to problem (1). Let $\widetilde{y}(x,t,\tau,\varepsilon)$ be an arbitrary function, which is continuous with respect to $(t,x,\tau) \in [0,1]\times[0,1]\times\{\tau : \mathrm{Re}\,\tau_j \leq 0, j=\overline{1,n}\}$ and has the asymptotic decomposition

$$\widetilde{y}(t,x,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t,x,\tau), \;\; y_k(t,x,\tau) \in U, \quad (7)$$

converging as $\varepsilon \to +0$ (uniformly with respect to $(t,x,\tau) \in [0,1]\times[0,1]\times\{\tau : \mathrm{Re}\,\tau_j \leq 0, j=\overline{1,n}\}$). Let us introduce operators $R_m : U \to U$, acting on each element $y(t,x,\tau)$ of the space U by the rule

$$R_0 y(t,x,\tau) \equiv R_0\left(\sum_{j=1}^{n} y_j(t,x)\,e^{\tau_j} + y_0(t,x)\right) = \int_0^t K(t,x,s)\,y_0(s,x)\,ds,$$

$$R_{k+1} y(x,t,\tau) = (-1)^k \sum_{j=1}^{n} \Big[\left(I_j^k\left(K(t,x,s)\,y_1(s,x)\right)\right)_{s=t} e^{\tau_j}$$

$$-\left(I_j^k\left(K(t,x,s)\,y_1(s,x)\right)\right)_{s=0}\Big], \quad (8)$$

where operators I_j^k have the form (6), $k \geq 0$. Operators R_m are said to be *operators of order* (by ε), because being applied to the function $y(t,x,\tau)$ they generate members of order ε^m. An extended operator for the integral operator J can be naturally defined as follows.

Definition. *A formal extension* of the operator J is an operator $\widetilde{J}$, which acts on each function

$\widetilde{y}(t,x,\tau,\varepsilon)$ of form (7) by the rule

$$\widetilde{J}\widetilde{y} \equiv \widetilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k y_k(t,x,\tau)\right) \stackrel{\text{def}}{=} \sum_{r=0}^{\infty} \varepsilon^r \left(\sum_{k=0}^{r} R_{r-k} y_k(t,x,\tau)\right).$$

Now we can write the completely regularized (with respect to the initial problem (1)) the problem

$$\mathscr{L}_\varepsilon \widetilde{y}(t,x,\tau,\varepsilon) \equiv \varepsilon \frac{\partial \widetilde{y}}{\partial t} + \sum_{j=1}^{n} \lambda_j(t) \frac{\partial \widetilde{y}}{\partial \tau_j} - A(t)\widetilde{y} - \widetilde{J}\widetilde{y} = h(t,x), \quad \widetilde{y}(0,x,0,\varepsilon) = y^0(x), \tag{9}$$

where $\widetilde{y}(t,x,\tau,\varepsilon)$ is series (7). Substituting this series into (9) and equating coefficients at the same powers of ε, we obtain the following iterative problems:

$$\mathscr{L} y_0 \equiv \sum_{j=1}^{n} \lambda_j(t) \frac{\partial y_0}{\partial \tau_j} - A(t) y_0 - R_0 y_0 = h(t,x), \;\; y_0(0,x,0) = y^0(x); \tag{9_0}$$

$$\mathscr{L} y_1 = -\frac{\partial y_0}{\partial t} + R_1 y_0, y_1(0,x,0) = 0; \tag{9_1}$$

$$\ldots$$

$$\mathscr{L} y_k = -\frac{\partial y_{k-1}}{\partial t} + R_1 y_{k-1} + \cdots + R_k y_0, \;\; y_k(0,x,0) = 0, \;\; k \geq 1. \tag{9_k}$$

Each of iterative problems (9_k) has the form

$$\mathscr{L} y(t,x,\tau) \equiv \sum_{j=1}^{n} \lambda_j(t) \frac{\partial y}{\partial \tau_j} - A(t) y - R_0 y = H(t,x,\tau), \quad y(0,x,0) = y_*(x), \tag{10}$$

where $H(t,x,\tau) = \sum\limits_{j=1}^{n} H_j(t,x) e^{\tau_j} + H_0(t,x) \in U$, $y_*(x) \in C^\infty[0,1]$ are known functions, and

$$R_0 y(x,t,\tau) \equiv R_0\left(\sum_{j=1}^{n} y_j(t,x) e^{\tau_j} + y_0(t,x)\right) = \int_0^t K(t,x,s) y_0(s,x)\, ds$$

is an operator. Let us try to solve problem (10). Substituting the element $y = \sum\limits_{j=1}^{n} y_j(t,x) e^{\tau_j} + y_0(t,x)$ of the space U into (10), we have

$$\sum_{j=1}^{n} \lambda_j(t) y_j(t,x) e^{\tau_j} - \sum_{j=1}^{n} A(t) y_j(t,x) e^{\tau_j} - A(t) y_0(t,x) - \int_0^t K(t,x,s) y_0(s,x)\, ds = \sum_{j=1}^{n} H_j(t,x) e^{\tau_j} + H_0(t,x).$$

Equating here separately the free members and coefficients with different exponents, we obtain the systems of equations

$$[\lambda_j(t) I - A(t)] y_j(t,x) = H_j(t,x), \;\; j = \overline{1,n}, \quad y_0(t,x) = \int_0^t \left(-A^{-1}(t) K(t,x,s)\right) y_0(s,x)\, ds - A^{-1}(t) H_0(t,x). \tag{11}$$

For resolving the first n systems of Eqs. (11) in the space $C^{\infty}([0,1]\times[0,1],\mathbb{C}^n)$, it is necessary and sufficient that [2]

$$(H_j(t,x),\chi_j(t))\equiv 0,\ \ j=\overline{1,n},\ \ \forall (x,t)\in[0,1]\times[0,1],$$

where $\chi_j(t)-\overline{\lambda}_j(t)$ is the proper vector of the matrix $A^*(t)$, $(\ ,\ )$ is the scalar product in the space $\mathbb{C}^n$. The last equation in (11) is the Volterra equation of the second kind with the smooth kernel $G(t,x,s)=-A^{-1}(t)K(t,x,s)$ (the variable x plays the role of a parameter), therefore it has a unique solution in the class $C^{\infty}([0,1]\times[0,1],\mathbb{C}^n)$. If in the space U we introduce the scalar (with each $(t,x)\in[0,1]\times[0,1]$) product

$$\langle y(t,x,\tau),z(t,x,\tau)\rangle\equiv\Big\langle\sum_{j=1}^{n}y_j(t,x)e^{\tau_j}+y_0(t,x),\sum_{j=1}^{n}z_j(t,x)e^{\tau_j}+z_0(t,x)\Big\rangle \stackrel{\mathrm{def}}{=}\sum_{j=0}^{n}(y_j(t,x),z_j(t,x)),$$

and denote by $\varphi_j(t)-\lambda_j(t)$ the proper vectors of matrix $A(t)$, where $(\varphi_j(t),\chi_i(t)\equiv\delta_{ji}$, then from the previous reasoning the next theorem follows.

Theorem 1. *Let in system* (10) *the right-hand side $H(t,x,\tau)\in U$ and conditions* 1) *and* 2) *be fulfilled. Then for the resolvability of system* (10) *in the space U it is necessary and sufficient that*

$$\langle H(t,x,\tau),\chi_j(t)e^{\tau_j}\rangle\equiv 0,\ \ j=\overline{1,n},\ \ \forall (t,x)\in[0,1]\times[0,1]. \tag{12}$$

Providing this condition, system (10) *in the space U has the solution*

$$y(t,x,\tau)=\sum_{j=1}^{n}\alpha_j(t,x)\varphi_j(t)e^{\tau_j}+\sum_{j=1}^{n}\Bigg(\sum_{s=1,s\neq j}^{n}\frac{H_j(t,x),\chi_s(t)}{\lambda_j(t)-\lambda_s(t)}\varphi_s(t)\Bigg)e^{\tau_j} -\int_0^t\mathscr{R}(t,x,s)A^{-1}(s)H_0(s,x)\,ds-A^{-1}(t)H_0(t,x), \tag{13}$$

where $\mathscr{R}(t,x,s)$ is the resolvent of the kernel $G(t,x,s)=-A^{-1}(t)K(t,x,s)$, $\alpha_j(t,x)$ are arbitrary functions of the class $C^{\infty}([0,1]\times[0,1],\mathbb{C}^1)$, $j=\overline{0,n}$.

Proof. We subdue the solution to (13) to the initial condition $y(0,x,0)=y_*(x)$:

$$\sum_{j=1}^{n}\alpha_j(0,x)\varphi_j(0)+\sum_{j=1}^{n}\Bigg(\sum_{s=1,s\neq j}^{n}\frac{H_j(0,x),\chi_s(0)}{\lambda_j(0)-\lambda_s(0)}\varphi_s(0)\Bigg)=y_*(x)+A^{-1}(0)H_0(0,x).$$

By scalar multiplying this equality by $\chi_k(0)$ and taking into account the bi-orthonormality condition of systems of proper vectors $\{\varphi_j(t)\}$ and $\{\chi_k(t)\}$, we obtain equalities

$$\alpha_k(0,x)+\sum_{j=1}^{n}\sum_{s\neq j,\,s=1}^{n}\frac{(H_j(0,x),\chi_s(0))}{\lambda_j(0)-\lambda_s(0)}\delta_{sk}=(y_*(x)+A^{-1}H(0,x),\chi_k(0)),$$

hence we find values

$$\alpha_k(0,x)=(y_*(x)+A^{-1}H(0,x),\chi_k(0))+\sum_{j=1,\,j\neq k}^{n}\frac{(H_j(0,x),\chi_k(0))}{\lambda_j(0)-\lambda_k(0)},\ \ k=\overline{1,n}. \tag{14}$$

However, the functions $\alpha_j(t,x)$ were not completely found. An additional requirement on the solution to problem (10) is necessary. Such a requirement is dictated by iterative problems (9_k), from which one can see that the natural additional constraint is the condition

$$\Big\langle-\frac{\partial y}{\partial t}+R_1y+P(t,x,\tau),\chi_j(t)e^{\tau_j}\Big\rangle\equiv 0,\ \ j=\overline{1,n},\ \ \forall (t,x)\in[0,1]\times[0,1], \tag{15}$$

where $P(t,x,\tau) = \sum_{j=1}^{n} P_j(t,x)\, e^{\tau_j} + P_0(t,x) \in U$ is a known function. Let us show that under condition (15) problem (10) has a unique solution in the space U.

Theorem 2. *Let conditions* 1), 2) *be fulfilled and the right-hand side of* $H{=}H(t,x,\tau) \equiv \sum_{j=1}^{n} H_j(t,x) e^{\tau_j} \in U$ *satisfy the condition of orthogonality of* (12). *Then problem* (10) *under additional condition* (15) *is uniquely resolvable in the space* U.

Proof. In order to use condition (15), we calculate the expression $-\frac{\partial y}{\partial t} + R_1 y + P(t,x,\tau)$. By denoting

$$H_{js}(t,x) \equiv (\lambda_j(t) - \lambda_s(t))^{-1}(H_j(t,x), \chi_s(t)),$$
$$y_0(x,t) \equiv \int_0^t \mathscr{R}(t,x,s)\, A^{-1}(s)\, H_0(s,x)\, ds - A^{-1}(t)\, H_0(t,x),$$

we have

$$-\frac{\partial y}{\partial t} + R_1 y + Q(t,\tau) = -\sum_{j=1}^{n}\left(\frac{\partial \alpha_j}{\partial t}\varphi_j + \alpha_j \dot{\varphi}_j\right)e^{\tau_j} - \sum_{j=1}^{n}\sum_{s\neq j,\, s=1}^{n}\left(\frac{\partial H_{js}}{\partial t}\varphi_s\right.$$
$$\left. + H_{js}\dot{\varphi}_s\right)e^{\tau_j} - \frac{\partial y_0(t,x)}{\partial t} + \sum_{j=1}^{n}\left[\frac{K(t,t)\varphi_j(t)}{\lambda_j(t)}\alpha_j(t)e^{\tau_j} - \frac{K(t,0)\varphi_j(0)}{\lambda_j(0)}\alpha_j(0)\right]$$
$$+ \sum_{j=0}^{n}\left[\frac{K(t,t)}{\lambda_j(t)}\sum_{s\neq j,\, s=1}^{n} H_{js}(t,x)\varphi_s(t)e^{\tau_j} - \frac{K(t,0)}{\lambda_j(0)}\sum_{s\neq j,\, s=1}^{n} H_{js}(0,x)\varphi_s(0)\right]$$
$$+ \sum_{j=1}^{n} P_j(t,x)e^{\tau_j} + P_0(t,x).$$

Here we used (see (8)) the expression

$$R_1 y \equiv R_1\left(\sum_{j=1}^{n} y_j(t,x)\, e^{\tau_j} + y_0(t,x)\right) = \sum_{j=1}^{n}\left(\frac{K(t,t)y_j(t,x)}{\lambda_j(t)}e^{\tau_j} - \frac{K(t,0)y_j(0,x)}{\lambda_j(0)}\right),$$

$$y_j(t,x) = \alpha_j(t,x)\varphi_j(t) + \sum_{s\neq j, s=1}^{n} H_{js}(t,x)\varphi_s(t), \quad j = \overline{1,n}.$$

Now conditions (15) can be written in the form

$$-\frac{\partial \alpha_j}{\partial t} - (\dot{\varphi}_j(t), \chi_j(t))\alpha_j - \sum_{s\neq j,\, s=1}^{n} H_{js}(t,x)(\dot{\varphi}_s(t), \chi_j(t)) + \left(\frac{K(t,t)}{\lambda_j(t)}\varphi_j(t), \chi_j(t)\right)\alpha_j$$
$$+ \sum_{s\neq j,\, s=1}^{n} \frac{H_{js}(t,x)}{\lambda_j(t)}\,(K(t,t)\varphi_s(t), \chi_j(t)) + (P_j(t,x), \chi_j(t)) = 0$$

or

$$\frac{\partial \alpha_j}{\partial t} = \left(\frac{K(t,t)\varphi_j(t)}{\lambda_j(t)} - \dot{\varphi}_j(t), \chi_j(t)\right)\alpha_j + l_j(t,x), \quad j = \overline{1,n},$$

where by $l_j(t,x)$ we denoted the known function in the previous equality, independent of α_j. Combining these equations and initial conditions (14), we find

$$\alpha_j(t,x) = e^{\int_0^t p_j(\theta)d\theta}\left(\alpha_j(0,x) + \int_0^t e^{-\int_0^s p_j(\theta)d\theta} l_j(s,x)ds\right),$$
$$p_j(t) \equiv (\lambda_j^{-1}(t)K(t,t)\varphi_j(t) - \dot{\varphi}_j(t), \chi_j(t)), \quad j=\overline{1,n}. \tag{16}$$

Therefore solution (13) to problem (10) is uniquely defined in the space U. □

Applying Theorems 1 and 2 to iterative problems (9_k), we construct a series (7) with coefficients from the class U. Let $y_{\varepsilon N}(t,x)$ be a narrowing of the Nth partial sum of this series with $\tau = \frac{\psi(t)}{\varepsilon}$: $y_{\varepsilon N} = \sum\limits_{k=0}^{N} \varepsilon^k y_k\left(t, x, \frac{\psi(t)}{\varepsilon}\right)$. As in [2] (Chap. 8), it is not difficult to prove the following theorem.

Theorem 3. *Let conditions* 1), 2) *be fulfilled. Then with* $\varepsilon \in (0,\varepsilon_0]$, *where* $\varepsilon_0 > 0$ *is sufficiently small, problem* (1) *has a unique solution* $y(t,x,\varepsilon)$ *in the class* $C^1([0,1]\times[0,1],\mathbb{C}^n)$. *Here the vector-function* $y_{\varepsilon N}(t,x)$ *satisfies the estimate*

$$\|y(t,x,\varepsilon) - y_{\varepsilon N}(t,x)\|_{C([0,1]\times[0,1])} \le C_N \varepsilon^{N+1}, \quad N=0,1,2,\ldots,$$

where the constant $C_N > 0$ *is independent of* $\varepsilon \in (0,\varepsilon_0]$.

3. Solution to the first iterative problem. Since in system (9_0) the vector-function $H(t,x,\tau) \equiv h(t,x)$ is independent of τ, conditions (12) are automatically fulfilled for it. Therefore in the space U the system (9_0) has a solution, which can be written in the form (see (13))

$$y_0(t,x,\tau) = \sum_{j=1}^{n} \alpha_j^{(0)}(t,x)\varphi_j(t)e^{\tau_j} + y_0^{(0)}(t,x), \tag{17}$$

where $y_0^{(0)}(t,x) = -A^{-1}(t)h(t,x) - \int\limits_0^t \mathscr{R}(t,s)A^{-1}(s)h(s,x)ds$, and $\alpha_j^{(0)}(t,x) \in C^\infty([0,1]\times[0,1],\mathbb{C}^1)$, $j=\overline{1,n}$, are arbitrary functions. For calculating these functions we write the right-hand side of the following iterative system (9_1):

$$H(t,x,\tau) \equiv \sum_{j=1}^{n}\left[\frac{\partial\alpha_j^{(0)}(t,x)}{\partial t}\varphi_j(t) + \alpha_j^{(0)}(t,x)\dot{\varphi}_j(t)\right]e^{\tau_j} - \frac{\partial y_0^{(0)}(t,x)}{\partial t}$$
$$+ \sum_{j=1}^{n}\left(\frac{K(t,t)\varphi_j(t)}{\lambda_j(t)}\alpha_j^{(0)}(t,x)e^{\tau_j} - \frac{K(t,0)\varphi_j(0)}{\lambda_j(0)}\alpha_j^{(0)}(0,x)\right).$$

Subordinating $H(t,x,\tau)$ to the condition of orthogonality (12), we have

$$-\frac{\partial\alpha_j^{(0)}}{\partial t} - (\dot{\varphi}_j(t),\chi_j(t))\alpha_j^{(0)} + \frac{(K(t,t)\varphi_j(t),\chi_j(t))}{\lambda_j(t)}\alpha_j^{(0)} = 0, \quad j=\overline{1,n}. \tag{18}$$

We find initial conditions for the functions $\alpha_j^{(0)}(t,x)$ from the equality $y_0(0,x,0)=y^0(x)$:

$$\sum_{j=1}^{n}\alpha_j^{(0)}(0,x)\varphi_j(0) - A^{-1}(0)h(0,x) = y^0(x) \Leftrightarrow \sum_{j=1}^{n}\alpha_j^{(0)}(0,x)\varphi_j(0) = y^0(x) + A^{-1}(0)h(0,x).$$

Multiplying this equality by $\chi_k(0)$, we obtain

$$\alpha_k^{(0)}(0,x) = (y^0(x) + A^{-1}(0)h(0,x), \chi_k(0)) = (y^0(x), \chi_k(0))$$

$$+\left(h(0,x),A^{-1*}(0)\chi_k(0)\right)=\left(y^0(x),\chi_k(0)\right)+\frac{1}{\lambda_k(0)}\left(h(0,x),\chi_k(0)\right),\ \ k=\overline{1,n}. \quad (19)$$

System (18) is homogeneous, therefore, taking into account initial conditions (19), it has a solution

$$\alpha_j^{(0)}(t,x)=e^{\int_0^t p_j(\theta)d\theta}\left(y^0(x)+\frac{h(0,x)}{\lambda_j(0)},\chi_j(0)\right),\ \ j=\overline{1,n}, \quad (20)$$

where $p_j(t)$ is the function, written in formula (16). Therefore, solution (17) to problem (9_0) is uniquely found in the space U. This solution is used further for the investigation of the limit proceeding in system (1) (as $\varepsilon\to+0$).

4. Limit transition in problem (1). Problem of initialization and its solution. Let $\operatorname{Re}\lambda_j(t)<0$, $j=\overline{1,n}$, $\forall t\in[0,1]$. Then by Theorem 3 we have

$$\|y(t,x,\varepsilon)-y_{\varepsilon 0}(t,x)\|_{C([0,1]\times[0,1])}\le c_0\varepsilon$$
$$\Leftrightarrow\left\|y(t,\varepsilon)-\left(\sum_{j=1}^{n}\alpha_j^{(0)}(t,x)\varphi_j(t)e^{\frac{1}{\varepsilon}\int_0^t\lambda_j d\theta}+y_0^{(0)}(t,x)\right)\right\|_{C([0,1]\times[0,1])}\le c_0\varepsilon.$$

So, with any $\delta\in(0,1]$ we obtain

$$c_0\varepsilon\ge\left\|y(t,x,\varepsilon)-\sum_{j=1}^{n}\alpha_j^{(0)}(t,x)\varphi_j(t)e^{\frac{1}{\varepsilon}\int_0^t\lambda_j(\theta)d\theta}-y_0^{(0)}(t,x)\right\|_{C([0,1]\times[0,1])}$$
$$\ge\left\|y(t,x,\varepsilon)-\sum_{j=1}^{n}\alpha_j^{(0)}(t,x)\varphi_j(t)e^{\frac{1}{\varepsilon}\int_0^t\lambda_j(\theta)d\theta}-y_0^{(0)}(t,x)\right\|_{C([\delta,1]\times[0,1])}$$
$$\ge\left\|y(t,x,\varepsilon)-y_0^{(0)}(t,x)\right\|_{C([\delta,1]\times[0,1])}-\left\|\sum_{j=1}^{n}\alpha_j^{(0)}(t,x)\varphi_j(t)e^{\frac{1}{\varepsilon}\int_0^t\lambda_j(\theta)d\theta}\right\|_{C([\delta,1]\times[0,1])},$$

hence we deduce

$$\|y(t,x,\varepsilon)-y_0^{(0)}(t,x)\|_{C([\delta,1]\times[0,1])}\le c_0\varepsilon$$
$$+\left\|\sum_{j=1}^{n}\alpha_j^{(0)}(t,x)\varphi_j(t)e^{\frac{1}{\varepsilon}\int_0^t\lambda_j(\theta)d\theta}\right\|_{C([\delta,1]\times[0,1])}$$
$$\le c_0\varepsilon+\sum_{j=1}^{n}||\alpha_j^{(0)}(t,x)\varphi_j(t)||_{C([\delta,1]\times[0,1])}e^{-\frac{\varkappa_j\delta}{\varepsilon}},$$

where $\varkappa_j=\min\limits_{t\in[0,T]}(-\operatorname{Re}\lambda_j(t))>0$. Therefore,

$$\|y(t,x,\varepsilon)-y_0^{(0)}(t,x)\|_{C([\delta,1]\times[0,1])}\to 0,\ \ \varepsilon\to+0. \quad (21)$$

Theorem 4. *If conditions* 1) *and* 2) *are fulfilled, and* $\operatorname{Re}\lambda_j(t)<0$, $j=\overline{1,n}$, $\forall t\in[0,1]$, *then the limit transition* (21) *takes place, where* $y=y(t,x,\varepsilon)$ *is the exact solution to problem* (1), *and the function* $y_0^{(0)}(t,x)$ *is a solution to the integral system*

$$-A(t)y_0^{(0)}(t,x)=\int_0^t K(t,x,s)y_0^{(0)}(s,x)ds+h(t,x).$$

This system is degenerate with respect to the initial system (1).

However, in our case purely imagine proper values $\lambda_j(t)$ are allowed, therefore the limit transition in the metric of space $C([0,1]\times[0,1])$ becomes impossible. In this connection, the following *initialization problem* arises: What should be the initial data of problem (1) so that the uniform limit transition

$y(t,x,\varepsilon) \to y_0^{(0)}(t,x)$ (as $\varepsilon \to +0$) was possible on the set $[0,1]\times[0,1]$, including the boundary layer zone with respect to t? Initial data of problem (1), which satisfy this requirement, form the *class of initialization.*

Let, for example, proper values $\lambda_j(t)$ be such that

3) $\operatorname{Re}\lambda_{j_1}(t) \equiv \operatorname{Re}\lambda_{j_2}(t) \equiv \cdots \equiv \operatorname{Re}\lambda_{j_k}(t) \equiv 0$, $\operatorname{Re}\lambda_j(t) < 0$, $j=\overline{1,n}$, $j \neq j_s$, $s=\overline{1,k}$ $(k \leq n)$.

In this case in the sum $\sum\limits_{j=1}^{n} \alpha_j^{(0)}(t,x)\varphi_j(t)e^{\frac{1}{\varepsilon}\int\limits_0^t \lambda_j d\theta} + y_0^{(0)}(t,x)$ addends with numbers $j=j_s$, $s=\overline{1,k}$, rapidly oscillate and prevent the existence of a limit transition $y(t,x,\varepsilon)\to y_0^{(0)}(t,x)$ on the set $[0,1]\times[0,1]$, therefore it is necessary to delete them, i.e., to set $\alpha_{j_s}^{(0)}(t,x)\equiv 0$ $\left(s=\overline{1,k}, (t,x)\in[0,1]\times[0,1]\right)$. Since (see (20))

$$\alpha_j^{(0)}(t,x) = e^{\int_0^t p_j(\theta)d\theta}\left(y^0(x) + \frac{h(0,x)}{\lambda_j(0)}, \chi_j(0)\right), \ \ j=\overline{1,n},$$

we have

$$\alpha_{j_s}^{(0)}(t,x) \equiv 0 \Leftrightarrow \left(y^0(x) + \frac{h(0,x)}{\lambda_{j_s}(0)}, \chi_{j_s}(0)\right) \equiv 0, \ \ s=\overline{1,k}, \ \ x\in[0,1].$$

Hence, the class of initialization $\Sigma = \left\{h(t,x), y^0(x), A(x)\right\}$ is independent of the kernel $K(t,x,s)$ and has the form

$$\Sigma = \left\{\left(h, y^0, A\right) : \left(y^0(x) + \frac{h(0,x)}{\lambda_{j_s}(0)}, \chi_{j_s}(0)\right) \equiv 0, \ \ s=\overline{1,k}, \ \ x\in[0,1]\right\}.$$

The following theorem was proved.

Theorem 5. *Let for problem* (1) *conditions* 1)–3) *be fulfilled. Then for the passage to the limit* $\|y(t,x,\varepsilon) - y_0^{(0)}(t,x)\|_{C([0,1]\times[0,1])} \to 0$ $(\varepsilon\to+0)$, *it is necessary and sufficient that* $\left(h(t,x), y^0(x), A(x)\right)\in\Sigma$.

5. The case of integral operator with upper limit, independent of differentiation variable. We now turn to the presentation of the results for the singularly perturbed integro-differential equation:

$$\begin{gathered}\varepsilon\frac{\partial y(t,x,\varepsilon)}{\partial t} = a(x)\,y(t,x,\varepsilon) + \int_0^x K(t,x,s)\,y(s,x,\varepsilon)\,ds + h(t,x),\\ y(0,x,\varepsilon) = y^0(x), \ \ (t,x)\in[0,T]\times[0,X].\end{gathered} \tag{22}$$

Recall that earlier we considered the case, when these variables coincide. Such a seemingly small modification of the problem leads to the fact that in the neighborhood of the point $t=0$ the boundary layer will depend not only on t, but also on x. For the sake of simplicity we will consider the scalar case. As earlier, without loss of generality, we can assume that $T=1$. We assume that the following conditions are fulfilled:

1*) functions $a(x)\in C^\infty([0,1],\mathbb{C})$, $h(t,x)\in C^\infty([0,1]\times[0,1],\mathbb{C})$, the kernel

$$K(t,x,s)\in C^\infty\ \ (\{0\leq t, \ \ x\leq 1, \ \ 0\leq s\leq x\leq 1\},\mathbb{C});$$

2*) $\operatorname{Re} a(x)\leq 0$, $a(x)\neq 0$, $\forall x\in[0,1]$.

As in the case of ordinary differential operator, we introduce the regularizing variable

$$\tau = \frac{1}{\varepsilon}\int_0^t a(x)\,ds = \frac{a(x)\,t}{\varepsilon} \equiv \frac{\psi(t,x)}{\varepsilon}.$$

Here elements of the space of solutions V to the below arising iterative problems will depend on the variable $\sigma\equiv\sigma(x) = e^{\frac{a(x)x}{\varepsilon}}$ as a parameter. This case is typical for singularly disturbed integro-differential equations of the Fredholm type with one independent variable (for example, [2]). For the extension $\widetilde{y} = \widetilde{y}(t, x, \tau, \sigma, \varepsilon)$ of solution $y(t, x, \varepsilon)$ to problem (22) we obtain the partially regularized problem

$$\varepsilon \frac{\partial \widetilde{y}}{\partial t} + a(x) \frac{\partial \widetilde{y}}{\partial \tau} - a(x)\widetilde{y} - \int_0^x K(t, x, s)\, \widetilde{y}\left(s, x, \frac{\psi(s, x)}{\varepsilon}, \sigma, \varepsilon\right) ds$$
$$= h(t, x), \;\; \widetilde{y}(0, x, 0, \sigma, \varepsilon) = y^0(x).$$

For the complete regularization it is necessary to introduce the space of solutions to iterative problems:

$$V = \left\{ y(t, x, \tau, \sigma) : y = y_1(t, x, \sigma)\, e^{\tau} + y_0(t, x, \sigma) \right\},$$

where coefficients $y_j(t, x, \sigma)$ are polynomials with respect to σ:

$$y_j(t, x, \sigma) = \sum_{k=0}^{N_j} y_j^{(k)}(t, x)\, \sigma^k, \;\; N_j < \infty, \;\; j = 1, 2,$$

with coefficients $y_j^{(k)}(t, x) \in C^{\infty}([0, 1] \times [0, 1])$, $k = \overline{0, N_j}$, $j = 1, 2$, and realize the regularization of the integral

$$Jy(t, x, \tau, \varepsilon) = \int_0^x K(t, x, s)\, y\left(s, x, \frac{\psi(s, x)}{\varepsilon}, \varepsilon\right) ds.$$

As in the previous case, applying the integration by parts, we construct the problem

$$\varepsilon \frac{\partial \widetilde{y}}{\partial t} + a(x) \frac{\partial \widetilde{y}}{\partial \tau} - a(x)\widetilde{y} - \widetilde{J}\widetilde{y} = h(t, x), \;\; \widetilde{y}(0, x, 0, \sigma, \varepsilon) = y^0(x), \tag{23}$$

which is completely regularized with respect to the initial problem (22). Here operators of order have the form

$$R_0 y(t, x, \tau, \sigma) \equiv R_0\left(y_1(t, x, \sigma)\, e^{\tau} + y_0(t, x, \sigma)\right) = \int_0^x K(t, x, s)\, y_0(s, x, \sigma) ds,$$

$$R_{k+1} y(x, t, \tau, \sigma) = (-1)^k \big[(I_x^{\nu}(K(t, x, s)\, y_1(s, x, \sigma))_{s=x}$$
$$\times\, \sigma - I_x^{\nu}(K(t, x, s)\, y_1(s, x, \sigma))_{s=0}) \big], \;\; k \geq 0,$$

where

$$I_x^0(K(t, x, s)\, q(s, x, \sigma)) \equiv \frac{K(t, x, s)\, q(s, x, \sigma)}{a(x)}, \ldots,$$
$$I_x^{\nu}(K(t, x, s)\, q(s, x, \sigma)) = \frac{1}{a(x)} \frac{\partial}{\partial s} I_x^{\nu-1}(K(t, x, s)\, q(s, x, \sigma)), \;\; \nu \geq 1,$$

and the extension $\widetilde{J}$ of integral operator is

$$\widetilde{J}\widetilde{y} \equiv \widetilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k y_k(t, x, \tau, \sigma)\right) \triangleq \sum_{r=0}^{\infty} \varepsilon^r \left(\sum_{k=0}^{r} R_{r-k} y_k(t, x, \tau, \sigma)\right).$$

Substituting the series

$$\widetilde{y}(t, x, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t, x, \tau, \sigma), \;\; y_k(t, x, \tau, \sigma) \in V, \tag{24}$$

in (23) and equating coefficients at the same powers ε, we obtain the following iterative problems:

$$\mathscr{A} y_0 \equiv a(x) \frac{\partial y_0}{\partial \tau} - a(x)\, y_0 - R_0 y_0 = h(t, x), \;\; y_0(t, x, \tau, \sigma)\,|_{t=\tau=0} = y^0(x);$$

$$\mathscr{A} y_1 = -\frac{\partial y_0}{\partial t} + R_1 y_0, y_1(0, x, 0, \sigma) = 0;$$

$$\dots$$

$$\mathscr{A} y_k = -\frac{\partial y_{k-1}}{\partial t} + R_1 y_{k-1} + \dots + R_k y_0, \;\; y_k(0, x, 0, \sigma) = 0, \;\; k \geq 1. \tag{25}$$

Each of the iterative problems (25) has the form

$$\begin{gathered} \mathscr{A} y(t, x, \tau, \sigma) \equiv a(x) \frac{\partial y}{\partial \tau} - a(x) y - R_0 y_0 = h(t, x, \tau, \sigma), \\ y(0, x, 0, \sigma) = y^*(x, \sigma), \end{gathered} \tag{26}$$

where $y^*(x, \sigma) \in C^\infty [0, 1] \times [0, 1]$, $h(t, x, \tau, \sigma) \in V$ are known functions,

$$R_0 y \equiv R_0 (y_1(t, x, \sigma) e^\tau + y_0(t, x, \sigma)) = \int_0^x K(t, x, s) y_0(s, x, \sigma) ds.$$

Let $h(t, x, \tau, \sigma) = h_1(t, x, \sigma) e^\tau + h_0(t, x, \sigma)$. Defining the solution to problem (26) in the form of the function

$$y(t, x, \tau, \sigma) = y_1(t, x, \sigma) e^\tau + y_0(t, x, \sigma),$$

we obtain equations

$$\begin{gathered} 0 \cdot y_1(t, x, \sigma) = h_1(t, x, \sigma), \\ -a(x) y_0(t, x, \sigma) - \int_0^x K(t, x, s) y_0(s, x, \sigma) ds = h_0(t, x, \sigma). \end{gathered} \tag{27}$$

The first of these equations is resolvable if and only if

$$h_1(t, x, \sigma) \equiv 0 \Leftrightarrow < h(t, x, \tau, \sigma), e^\tau > \equiv 0$$

(here we introduced the ordinary scalar product with each (t, x, σ) in V, see [2]). The second equation in (27) with each fixed $x \in [0, 1]$ is the Fredholm equation of the second kind. Let us consider the simplest case

3*) with $x \in [0, 1]$ the unity is not a characteristic value of the kernel $\frac{K(t,x,s)}{-a(x)}$ of integral operator.

This condition means that the homogeneous equation

$$-a(x) y_0(t, x, \sigma) - \int_0^x K(t, x, s) y_0(s, x, \sigma) ds = 0$$

has trivial solution $y_0(t, x, \sigma) \equiv 0$ only. In this case, heterogeneous equation (27) will have the unique solution in the form

$$y_0(t, x, \sigma) = \frac{h_0(t, x, \sigma)}{-a(x)} + \int_0^x \mathscr{R}(t, x, s) \frac{h_0(s, x, \sigma)}{-a(x)} ds,$$

where $\mathscr{R}(t, x, s)$ is the resolvent of kernel $\frac{K(t,x,s)}{-a(x)}$, and Eq. (26) in the space V has the family of solutions $y(t, x, \tau, \sigma) = y_0(t, x, \sigma) + \alpha(t, x) e^\tau$, where $\alpha(t, x) \in C^\infty \left([0, 1] \times [0, 1], C^1\right)$ is an arbitrary function. The initial condition $y(0, x, 0, \sigma) = y^*(x, \sigma)$ allows one to find the function

$$\alpha(0, x) = y^*(x, \sigma) + \frac{h_0(0, x, \sigma)}{a(x)} + \int_0^x \mathscr{R}(0, x, s) \frac{h_0(s, x, \sigma)}{a(x)} ds. \tag{28}$$

For complete calculation of the function $\alpha(t, x)$ it is necessary to define an additional condition. Taking into account the structure of iterative problems (25), we conclude that a natural additional condition for Eq. (26) is the requirement

$$\left\langle -\frac{\partial y}{\partial t} + R_1 y, e^\tau \right\rangle \equiv 0 \;\; \forall (t, x, \sigma) \in [0, 1] \times [0, 1] \times [0, 1]. \tag{29}$$

Since in $R_1(y)$ there is no exponent, subordinating the solution $y(t,x,\tau,\sigma)=y_0(t,x,\sigma)+\alpha(t,x)e^{\tau}$ to condition (29), we will have $\frac{\partial\alpha(t,x)}{\partial t}\equiv 0$, hence we deduce $\alpha(t,x)\equiv\alpha(0,x)$, where $\alpha(0,x)$ is function (28). Therefore, the solution to problem (27) with the additional condition (29) is uniquely defined in the space V:

$$y(t,x,\tau,\sigma)=\left(y^*(x,\sigma)+\frac{h_0(0,x,\sigma)}{a(x)}+\int_0^x\mathscr{R}(0,x,s)\frac{h_0(s,x,\sigma)}{a(x)}ds\right)e^{\tau}+\frac{h_0(t,x,\sigma)}{-a(x)}+\int_0^x\mathscr{R}(t,x,s)\frac{h_0(s,x,\sigma)}{-a(x)}ds.$$

By applying the obtained result to iterative problems (25), we construct series (24) an by the method of differential inequalities (for example, [5]) we prove that the solution $y(t,x,\varepsilon)$ to problem (22) uniquely exists and for all $N=0,1,2,\ldots$ the estimate holds

$$\left\|y(t,x,\varepsilon)-\sum_{k=0}^{N}\varepsilon^k y_k\left(t,x,\varepsilon^{-1}\psi(t,x)\right)\right\|_{C([0,1]\times[0,1]\times[0,1])}\le C_N\varepsilon^{N+1},$$

where the constant $C_N>0$ is independent of ε if $\varepsilon\in(0,\varepsilon_0]$, $\varepsilon_0>0$ and is sufficiently small. The case, when condition 3*) is not fulfilled, is more difficult to investigate. Here different cases are possible. However, the case when the unity for all $x\in[0,1]$ presents a characteristic value of rank $r<\infty$ of the kernel $\frac{K(t,x,s)}{-a(x)}$ of the integral operator, can be completely studied with the help of the above stated technique.

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